This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

A TREATISE ON SOME

NEW GEOMETRICAL METHODS,

CONTAINING ESSAYS ON

TANGENTIAL COORDINATES,
PEDAL COORDINATES, RECIPROCAL POLARS,
THE TRIGONOMETRY OF THE PARABOLA,
THE GEOMETRICAL ORIGIN OF LOGARITHMS,
THE
GEOMETRICAL PROPERTIES OF ELLIPTIC INTEGRALS,
AND OTHER KINDRED SUBJECTS.

Nova methodus, nova seges.

IN TWO VOLUMES.—VOL. I.

BY

JAMES BOOTH, LL.D., F.R.S., F.R.A.S., &c. &c.,
VICAR OF STONE, BUCKINGHAMSHIRE.

LONDON:
LONGMANS, GREEN, READER, AND DYER,
PATERNOSTER ROW.
MDCCCLXXIII.

[*All rights reserved.*]

183. e. 1

BODLEIAN LIBRARY
OXFORD.

PRINTED BY TAYLOR AND FRANCIS,
RED LION COURT, FLEET STREET.

Inscribed

TO THE

PRESIDENT AND COUNCIL

OF THE

ROYAL ASTRONOMICAL SOCIETY

OF

LONDON.

INTRODUCTION.

THE discoveries and inventions which have enriched and systematized every department of Natural knowledge within the last half century, are justly looked upon as the great intellectual conquests of our time. Indeed the rapid succession of brilliant discoveries, and their important applications, would seem at first sight to justify the opinion of many that it is only in the regions of physical inquiry, in the extension of man's dominion over nature, that our knowledge can receive much accession. They believe that mere intellectual speculation, in which the mind is as well the instrument as the subject, is barren of useful results, and that but little remains to reward the labours of inquirers in that field of investigation.

Now, the reality of this distinction cannot justly be admitted. Man's triumphs in the sphere of intellect are as important in our own age as at any former period of this world's history. Let us limit our view to the progress of mathematical science alone; and it will suffice merely to mention the names of Euler, Lagrange, Laplace, Legendre, Jacobi, Abel, and a host of others to show the advance which this department of human knowledge has made in modern times, depending, as it does, neither on experiment nor observation, neither on the use of ingenious machinery nor of instruments of precision.

But it may be replied, those discoveries are only to be found in the higher analysis, a comparatively modern invention, while geometry remains much as it was when it left the hands of Euclid, Archimedes, and Apollonius*. But surely there will occur to us the names of Descartes and Pascal, and Leibnitz, but above all that of Newton, as original inquirers, and as inventors of methods of investigation of the utmost value and importance. It has well been observed by an English mathematician of some celebrity, that "He who invents a new method, or who renders a mode of investigation intelligible with whatever paucity of mere illustration, does more for the interests of science than he who collects all the illustrative examples of such methods that could be given."

---

* "Quelle est la cause de cette direction exclusive dans les études mathématiques? Quelle sera son influence sur le caractère et les progrès de la science? Nous n'essayerons pas de répondre à ces questions, sur lesquelles on serait peut-être difficilement d'accord. Mais quelles que soient les opinions à leur égard, on ne disconviendra pas du moins, qu'il serait utile que la méthode ancienne, suivie jusqu'au siècle dernier, continuât d'être encouragée et cultivée concurremment avec la nouvelle."—CHASLES, *Aperçu Historique*, p. 551.

The addition of a new method of investigation to those already in use, the development of its principles, with illustrations of the mode of its application, are surely not of less value to a philosophical appreciation of what that is in which mathematical knowledge truly consists, than the invention of problems, which, while they embody no general principle, are yet often difficult to solve; and when solved, frequently afford no clue by which the solution may be rendered available in other cases.

Again, it often happens that an investigation which, if pursued by one method, would prove barren of results or altogether impracticable, when followed out from a different point of view and by the help of another method, not unfrequently leads by a few easy steps to the discovery of important truths, or to the consideration of others under a novel aspect. Hence the multiplication of methods of investigation tends widely to enlarge the boundaries of science.

It must, however, be acknowledged that for more than a century and a half geometry made but little progress, especially in this country. The new geometry originated with Monge and his disciples. It may safely be asserted, there is scarcely any branch of knowledge, certainly none of abstract speculation, that has put forth a more vigorous growth, and received a wider cultivation, than the science which, once alone called learning, was the subject matter of contemplation of the exalted genius of a Plato or an Archimedes.

To hold that no discoveries are to be made in a field which has been repeatedly searched, is an opinion than which there are few more erroneous. To have stumbled upon a new theorem or two was once looked upon as a proof of rare mathematical genius. In the geometry of the ancients the discovery of a new theorem suggested no correlative; it terminated in itself and continued sterile; while in the modern geometry a new theorem often becomes the prolific source of others that may be derived from it either by the method of reciprocal polars or central projection, or pedal coordinates or other modes of transformation. The discovery of new theorems is reduced to a mere mechanical operation. We have only to take any property of a sphere, suppose. By a single reciprocating or polar transformation, as shown by M. Chasles, we may obtain the correlative property of a surface of revolution of the second order, and then by a second polarization we may derive the corresponding property for a surface having three unequal axes. We have only to make a judicious selection of the pole, and then new theorems may be evolved in a continuous stream so long as we can supply theorems derived from the original figure. A man of some celebrity in his day, as a geometer and natural philosopher, Sir John Leslie, could write some fifty years ago, before the date, or at least before the development of the modern geometry, in the preface to his very elegant work on Geometrical Analysis, published at Edinburgh in 1821*, "The multifarious labours of former writers have left scarcely any room for invention. I have therefore occupied myself in improving the simplicity, clearness, and elegance of the demonstrations." What would be his surprise if it had been given to him to forecast the wonderful progress which the science of pure geometry has made in the interval between that time and the present? Geometry is stationary no longer. The modern methods of investigation are the most powerful solvents of problems the most complicated and abstruse. Crowds of theorems spring up and pour in upon us. The names of those men of original minds who have shed the light of their genius, within the present century, on this the oldest and most certain of all the sciences, constitute a bright galaxy on the sphere of human intelligence. In proof of this assertion do we need to say more than to mention the names of Monge and Dupin, of Poncelet and Chasles, of Gergonne and Steiner, and a host of others scarcely less eminent who have enriched the Transactions of Continental Societies and the pages of foreign periodicals with their contributions? They will be found in the twenty volumes of the 'Annales Mathématiques' of Gergonne, a work unrivalled for the originality of matter it contains. Its continuation will be found in the pages of Liouville and Crelle.

Among the most important methods of the modern geometry are the theory of transversals, the various kinds of projection, particularly that of central projection, the theory of reciprocal polars, that of anharmonic ratio, the elliptic coordinates of Lamè, the method of "inverse curves," and, if one might be permitted to add, the theory of tangential coordinates developed in the following pages. There is a method called by its inventors "trilinear coordinates," as also another named "tangential coordinates." These methods have been extensively used by Mr. Whitworth, Mr. Ferrers, Mr. Routh, and other English mathematicians. Though not familiar with these methods, I understand them sufficiently to know that they have nothing in common with the method to which more than thirty years ago I gave the name "tangential coordinates."

To Carnot, a Member of the French Directory, is due the development of the theory of transversals; and it may not be out of place to state here, as an instance, if any were wanted, that mathematical genius is not incompatible with the possession of practical common sense, business habits, and untiring energy, how the same Carnot was proverbially known as the organizer of victory for the armies of the Republic.

---

* Leslie, 'Geometrical Analysis,' p. vii.

To Poncelet, an officer of Engineers of the French army which invaded Russia in 1812, and who was made a prisoner in that campaign, so disastrous to the arms of France, we must assign the high distinction of being the inventor of the methods of "reciprocal polars" and "central projection." While detained a prisoner at Saratoff, without the use of books and depressed by captivity, he thought out those methods which in the history of the progress of mathematical science will ever render his name illustrious. To the student who has a true taste for excellence and elegance combined in mathematical investigation, and who is tired of the endless manipulation of quadratic equations, there is no book I can more earnestly recommend than the great work of Poncelet, published at Paris in 1822, on the projective properties of figures. It is only in such works that the student will obtain a broad view and comprehensive grasp of the principles and methods of the science. Nimble dexterity in the management of algebraical symbols in accordance with prescribed mechanical rules is too often taken (mistaken one should say, in this country at least) for a knowledge of the science.

In pursuing mathematical investigations we must guard against any bias that would lead us to prefer geometrical to algebraical methods, or reciprocally. They have each their proper sphere of action, so that an investigation which is both elegant and simple by the one method may be operose and complicated by the other. Thus to attempt to discuss the theory of Elliptic Integrals by the aid of pure geometry would result in nothing but labour lost and ingenuity thrown away. This is a very instructive illustration; for elliptic integrals of the three orders are nothing but formulæ of rectification for those curves of double curvature in which cones of the second degree are intersected by concentric spheres or paraboloids. Thus it often happens that we cannot treat a question of pure geometry by methods purely geometrical. The essential difference between the two appears to consist in this: that metrical properties are best treated by algebraical methods, and graphical properties by geometrical constructions.

A wide induction would, I believe, show that all algebraical equations do in fact represent geometrical truths. Thus trigonometry treats of the relations of circular arcs, parabolic trigonometry of the arcs of a parabola; logarithms are nothing more than the relations between the arcs of a parabola, and the corresponding vectors of a logocyclic curve; elliptic integrals are the algebraical expressions for the arcs of those symmetrical curves in which spheres and paraboloids are intersected by concentric cones, of which arcs the circle and the parabola afford the extreme cases. It is in geometry alone that we can find any intelligible explanation of imaginary quantities; and I doubt not that most definite integrals might be shown to be the representatives of certain geometrical magnitudes.

The system of coordinates invented by Descartes may be designated as *projective coordinates,* since they are the projections of a point in a plane on two right lines drawn in the same plane, or the projections of a point in space on three coordinate planes. This system is primarily applicable to the investigation of the locus of a point*, whether it be constrained to move along a straight line or a curve; it is only indirectly, and by the help of principles borrowed from a higher calculus, that the system can be applied to a moving straight line or a moving plane. Hence in many cases simple solutions may be found for whole classes of questions, which present almost insuperable difficulties when treated by projective coordinates.

The great advantage which the tangential system of coordinates exhibits over the Cartesian consists in this, that in the transformation of coordinates, whether by rotation or translation, the absolute term continues unchanged in the tangential system, while in the projective system the absolute term changes with every transformation. One other advantage this method possesses, the facility which it affords for the graphic description of curves. We have only to assume a set of values along one of the axes of coordinates; the equation of the curve will enable us to set off a series of corresponding values on the other axis of coordinates. A straight line may be drawn from one of those points to the other. This will be a limiting tangent to the curve.

Many years ago, after I had taken my Degree, I was much interested in the study of the original memoirs on reciprocal curves and curved surfaces, published in the 'Annales Mathématiques' of Gergonne, and in the works of such accomplished geometers as Monge, Dupin, Poncelet, and Chasles. In the course of my own researches, it occurred to me that there ought to be some way of expressing by common algebra the properties of such reciprocal curves and surfaces, some method which would, on inspection, show the relations existing between the original and derived surfaces. I was then led to the discovery of a simple method and compact notation from the following considerations.

---

* "En effet l'élément primitif des corps auquel on applique d'abord les premiers principes de cette science est comme dans la géométrie ancienne, le *point* mathématique. Ne sommes-nous pas autorisés à penser maintenant, qu'en prenant le *plan* pour l'élément de l'étendue, et non plus le *point,* on sera conduit à d'autres doctrines, faisant pour ainsi dire une nouvelle science."—CHASLES, *Aperçu Historique,* p. 408.

When two figures in the same plane, or more generally in space, are so related that one is the *reciprocal polar* of the other, then to every point in the one corresponds a plane in the other; to every straight line in the one a straight line also in the other; to any number of points in the same straight line in the one, as many planes all intersecting in the same straight line in the other; to any number of points in the same plane in the one, as many planes all meeting in the same point in the other. They may be called correlative figures. Now we know that in the application of algebra to geometry by the method of coordinates, a point is determined in position by its projections on three coordinate planes, or by three equations—that is, by three conditions. A straight line may in like manner be determined when we are given the positions of two points in it; and a plane is determined by one condition, which is called its equation. But in the inverse method, a point should be determined by one condition, a straight line by two, and a plane by three. Again, a straight line may be determined by considering it as joining two fixed points, or as the common intersection of two fixed planes. Now all these conditions may be expressed by taking as a new system of coordinates the segments of the common axes of coordinates between the origin and the points in which they are met by a movable plane. Thus, if these segments be designated by the symbols X, Y, Z, the three equations which determine a plane are

$$X=\text{constant}, \quad Y=\text{constant}, \quad Z=\text{constant}.$$

Again, the equation in (x, y, z) of a plane passing through a point of which the coordinates are x, y, z, and which cuts off from the axes of coordinates the segments X, Y, Z, is $\frac{x}{X}+\frac{y}{Y}+\frac{z}{Z}=1$. Now this is the *projective*, or common equation of the plane, if we make x, y, and z vary, and consider X, Y, Z constant. But we may invert these conditions, and consider x, y, z constant, while X, Y, and Z vary. The equation now, instead of being the projective equation of a fixed plane, becomes the tangential equation of a fixed point. In this latter case let α, β, and γ be put for x, y, z, and $\frac{1}{\xi}, \frac{1}{v}, \frac{1}{\zeta}$ for X, Y, Z; then the equation may be written

$$\alpha\xi+\beta v+\gamma\zeta=1,$$

which may be called the tangential equation of a point.

Moreover, as the continuous motion of a point, in a plane suppose, subjected to move in accordance with certain fixed conditions expressed by a certain relation between x and y may be conceived to describe a curve, so the successive positions of a straight line cutting off segments from the axes of coordinates having a certain relation to each other may be imagined to wrap round or envelop a certain curve, just as we may see a curve described on paper by the successive intersections of a series of straight lines. Hence there are two distinct modes according to which we may conceive all curves to be generated, namely by the motion of a tracing point, or the successive intersections of straight lines—by a pencil or straight edge, as a joiner would say. These conceptions are the logical basis of the methods by which the principles and notation of common algebra are generalized from the discussion of the properties of abstract number to those of pure space. The former view gave rise to the method of *projective* coordinates; the latter suggests the method of *tangential* coordinates.

It is sometimes very easy to express both the projection and tangential equations of the same curve or curved surface; it is frequently a matter of extreme difficulty.

Thus, if the projective equation of an ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

its tangential equation will be

$$a^2\xi^2+b^2\upsilon^2+c^2\zeta^2=1,$$

a, b, c being, as in the preceding equation, the semiaxes.

The chief object kept in view in the following pages is to develop the principle of *Geometrical Duality*, a principle apparently one of the most obvious, and singularly fruitful in results, beyond any other geometrical method hitherto discovered.

That the principle of Duality should not have been discovered by the great geometers of Ancient Greece is the more remarkable, as the five regular solids, the Platonic[*] bodies as they were called, were with them a favourite subject of speculation. In a later age Kepler believed them to be the archetypes of the planetary motions[†].

---

[*] "Il n'est pas étonnant qu'Aristée ait écrit sur les cinq corps réguliers; car cette théorie à été fort cultivée, et en grand honneur dès la plus haute antiquité des sciences chez les Grecs. Pythagore en avait fait le principe de sa cosmogonie, dans laquelle les cinq corps réguliers répondaient aux quatre élémens et à l'univers, ce qui a fait qu'on les appelait les cinq figures *mondaines* (*figuræ mundanæ*) Platon adoptait ces idées, et avait aussi cultivé cette théorie, sur laquelle Theatète, l'un de ses disciples, passe pour avoir écrit le premier. Ensuite, on trouve donc Aristée, puis Euclide, Apollonius et Hypsicle. Ces cinq corps réguliers ont joué un si grand rôle dans l'antiquité, par suite des idées pythagoriciennes et platoniciennes, qu'on les regardait comme étant le but final auquel étaient destinées et l'étude et la science des géomètres."—CHASLES, *Aperçu Historique*, p. 514.

[†] "Repetat lector ex Mysterio meo Cosmographico, quod edidi ante 22 annos: numerum planetarum seu curriculorum circa solem desumptum esse a sapientissimo Conditore ex quinque figuris regularibus solidis."—KEPLER, *Harmonices Mundi*, lib. v. p. 276.

But the five regular solids afford a prominent illustration of this principle of geometrical duality. The cube and the octahedron have each the same number of edges, while they interchange their faces and solid angles. Their edges are *conjugate polars,* and are consequently the same in number. In the same way the dodecahedron and icosahedron are correlative solids; they have the same number of edges, while they interchange their faces and solid angles; and, finally, the tetrahedron is its own correlative solid. When Euler, after many bootless efforts, had succeeded at last in establishing by elaborate proof[*] the singularly simple relation which connects the number of faces, corners, and edges of any polyhedron, he did not perceive how by the application of this principle of duality the number of his conclusions might have been doubled. Had Euler known this universal relation, he would have seen that, whatever be the formula connecting the solid angles, faces, and edges of any polyhedron, it must be satisfied by the interchange of the solid angles and faces, while the number of edges continues the same. Had Legendre been acquainted with this principle he would have known that when any polyhedron, regular or irregular, has any given number of edges, there must be possible another polyhedron having the same number of edges, but interchanging the numbers of its faces and solid angles with those of the correlative polyhedron.

To restrict the application of this principle to geometrical curves, surfaces, or solids would be to mask *the exceptionless universality* of this all-pervading principle. It not only holds good as regards curve lines and surfaces, whether they be algebraical or transcendental, whether they be continuous or discontinuous, whether they be described in accordance with some geometrical law or capriciously (*libera manu,* so to speak), but is true for every form of bounded space. The rounded pebble on the beach, the angular fragment in the quarry, each has its own strictly defined correlative polar figure, just as well as the most symmetrical solid of the ancient geometry. So universal is the principle of duality that to some it appears as a general law of nature[†]. In Physics we find it in attraction and repulsion, in the motions of rotation and translation; in the emotions of the mind, good and evil, virtue and vice; in the double origin of our ideas; in the sensations of the body, pleasure and pain. The principle of duality meets us everywhere. But this is not the place to follow out such a train of inquiry.

---

[*] “On a vu dans le IIIme volume du présent recueil (p. 169) que ce n'est qu'après des tentatives réitérés qu'Euler est parvenu à établir d'une manière à la fois complète et générale, son curieux théorème sur la relation constante entre le nombre des faces, celui des sommets, et celui des arêtes d'un polyèdre quelconque.

“On sait que, dans ces derniers temps M. Cauchy a démontré d'une manière beaucoup plus simple un autre théorème dont celui d'Euler n'est qu'un cas particulier.”—GERGONNE, *Annales Mathématiques,* vol. xix. p. 333.

A very elegant and simple demonstration of this curious theorem which had so long baffled that illustrious geometer Euler, will be found at page 333 of the XIX.th volume of the ‘Annales Mathématiques’ of Gergonne, based on the relations of a group of reticulated polygons.

[†] “On peut croire pourtant qu'une unité absolue n'est pas le principe de la nature. Les dualismes nombreux qui se remarquent dans les phénomènes natu-

The idea or conception of duality or correlation had long been admitted into and developed in spherical trigonometry, and the properties of polar triangles investigated by geometers with abundant labour and much acuteness. But it would seem never to have occurred to those accomplished mathematicians that the correlation which may be developed in spherical triangles and polygons was only a particular case of a general principle which binds together, so to speak, all the properties of space without exception.

For those who have not yet mastered the great principle of duality and the results which flow from it, one or two simple illustrations of it may not be here out of place. The well-known theorem of Pascal on the hexagon inscribed in a conic section, and its dual the hardly less celebrated theorem of Brianchon, are subjoined side by side. This latter is otherwise remarkable, as being the first result of the application of the principle of duality. See XIII. *Cahier du Journal de l'Ecole Polytechnique*, p. 301.

Pascal's Theorem.	*Brianchon's Theorem.*
The opposite *sides* of a hexagon *inscribed* in a conic section being *produced* to *meet*, two by two, *in* three *points*; these three *points range along* the same *straight line*.	The opposite *angles* of a hexagon *circumscribed* to a conic section being *joined* two by two *by* three *straight lines*, these three *lines meet in* the same *point*.
Maclaurin's Theorem for the organic description of conic sections.	*Its Reciprocal Polar.*
The three *sides* of a triangle *pass through* three *fixed points*; and two of its *angles move along* fixed *straight lines*; the third *angle* will *describe* a conic section.	The three *angles* of a triangle *rest on* three *fixed lines*; two of its *sides pass through fixed points*; the third *side* will *envelop* a conic section.

No reader, who is capable of understanding the subject, will imagine that this new system of coordinates is proposed with a view to supersede the old. The Cartesian system cannot be super-

---

rels, comme dans les différentes parties des connaissances humaines, tendent au contraire à nous faire supposer qu'une *dualité* constante, ou double unité est le vrai principe de la nature. Cette dualité, nous la trouvons dans l'objet même de la géométrie, ainsi que nous venons de le dire; dans la nature des propriétés de l'étendue; dans le double mouvement des corps célestes, où sa constance reconnue la fait admettre comme principe; et dans mille autres phénomènes; et l'on sera conduit, je crois, à regarder qu'un *dualisme universel* est la grande loi de la nature, et règne dans toutes les parties des connaissances de l'esprit humain." —CHASLES, *Aperçu Historique*, p. 290.

seded. As the properties of space are dual, so must the systems of investigation be dual also. The one method supplements the other[*]. "Ita utrumque, per se indigens, alterum alterius auxilio eget." Where the one may with ease be applied the other will be found to fail[†]. Thus, while in the solution of problems of rectification the tangential method holds out many advantages on the score of facility and simplicity, in quadratures it will afford but little help, and recourse must be had to the familiar integral $\int y\,dx$.

It would be a very inadequate conception of the power of this method to suppose that the systems of tangential and projective coordinates are merely supplemental one to the other. From their combination whole classes of problems may be evolved and determined without difficulty. For example, it will be shown that, the projective and tangential equations of the same curve being given, we may at once write down the equation of the locus of the vertex of a given angle, one of whose sides passes through a fixed point while the other side envelops the given curve, as also the equation of the locus of the foot of the perpendicular let fall from the origin on the tangent. As an instance, if

$$\mathrm{A}x^2+\mathrm{A}_{,}y^2+2\mathrm{B}xy+2\mathrm{C}x+2\mathrm{C}_{,}y=1, \quad . \quad . \quad . \quad . \quad \text{(a)}$$

$$a\xi^2+a_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{,}v=1, \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

[*] "On se tromperait en croyant que la géométrie dans ses moyens de procéder à la recherche de la vérité, doit avoir des bornes posées seulement par la nature de cette science, et non par la nature même des choses. On se tromperait également en croyant ces bornes moins reculées que celles où l'analyse peut atteindre en marchant vers le même but. Ces deux méthodes sous des formes différentes, sont les développements identiques d'une seule et même science qui soumet a la fois toutes les grandeurs à ses combinaisons, à ses rapprochements. L'une, sans jamais perdre de vue les choses mêmes qu'elle doit considerer, porte partout l'évidence avec elle; elle rend sensibles toutes ses conceptions, toutes ses opérations, et les grandeurs graphiques sont pour elle un moyen de peindre dans l'espace, et sa marche et ses résultats. L'autre substitue aux quantités dont elle s'occupe des signes purement abstraites, elle dépouille les grandeurs de tout ce qui n'est pas inhérent aux relations qu'elle envisage; elle ramène tout à des lois générales; elle représente les objets par des symboles qui les remplacent; elle est une langue; elle parle, qu'on me passe l'expression, elle parle et elle exprime tout ce que pense ou conçoit la première; et ces deux marches si différentes ont l'une par rapport à l'autre, les mêmes avantages et les mêmes désavantages que la pensée relativement à la parole. . . . . En considérant ainsi l'analyse et la géométrie dans leurs rapports, ces deux sciences s'éclaireront mutuellement, et chacune d'elles s'acroîtrai de tous les progrès et l'autre. Ne rejetons donc aucun de ces moyens pour procéder à la recherche de la vérité," &c.—DUPIN, *Développements de Géométrie*, p. 236.

[†] "The exercise of the mind in understanding a series of propositions, where the last conclusion is geometrically in close connexion with the first cause, is very different from that which it receives from putting in play the long train of machinery in a profound analytical process. The degrees of conviction in the two cases are very different. To the greater number of students, therefore, I conceive a popular geometrical explanation is more useful than an algebraical investigation."—Sir G. B. AIRY, the Astronomer Royal, *On Gravitation*, p. 7.

be the projective and tangential equations of a curve of the second degree,

$$A\xi^2 + A_{,}\upsilon^2 + 2B\xi\upsilon + 2(C\xi + C_{,}\upsilon)(\xi^2 + \upsilon^2) = (\xi^2 + \upsilon^2)^2\,;$$

will be the *tangential* equation of the curve enveloped by one side of a right angle which moves along the curve (a) while the other side always passes through a fixed point, and

$$ax^2 + a_{,}y^2 + 2\beta xy + 2(\gamma x + \gamma_{,}y)(x^2 + y^2) = (x^2 + y^2)^2$$

will be the *projective* equation of the point in which a perpendicular from the origin meets a tangent to the curve whose tangential equation is (b).

It may appear to some that examples are needlessly multiplied in the following pages; but it may be replied that in the treatment of a very abstract subject, especially if it be a novel theory, examples, if judiciously selected, may throw light upon the obscurities of imperfect explanations and defective discussions. It is granted but to few to grasp a theory as its development proceeds, even in the hands of a master. Examples are required *pour fixer les idées.* They assist us to incorporate new truths with that older knowledge which we have made our own. In many cases the author has been satisfied with laying down the principles of the method as applied to a few particular instances, without following up the investigations into all their details. To have done this would have swelled the bulk of the volume without any equivalent advantage. Something must be left to the ingenuity and industry of the reader to develop and to amplify. The book is intended to be suggestive, not exhaustive. Where the claims of so many new and important theorems to recognition are so continuous and pressing, one cannot stop to draw out the deductions that follow from each into a cluster of corollaries. So far from any exercise of ingenuity being necessary, the discovery of new theorems by the help of this and other kindred methods does not call for the exercise of much patient thinking. They spring up so spontaneously, as it were, that the difficulty is to keep under restraint the imagination as it courses along those trains of thought sure to end in the discovery of some new, and it may be unexpected, geometrical truth. It would have been an easy task, a fascinating labour, to have swelled the pages of this volume with diversified researches; but I have kept within the limits which at the outset I prescribed to myself, and have been satisfied to point out the way to others.

A somewhat detailed account of the subjects investigated in this volume will be expected by the reader. In the first place I have endeavoured as far as practicable to adopt a uniform and consistent notation. I have striven to use the same symbols in the same sense. Wherever possible I have made the absolute terms of the equations equal to unity. In this way the constants are reduced to the smallest number, and the orders of the constants, whether they be lines, surfaces, or solids, may be inferred from inspection, being always of the *same* or the *inverse* order of the variables of which they are the coefficients.

Hence the projective equation of the surface of the second order will be written

$$Ax^2 + A_{\prime}y^2 + A_{\prime\prime}z^2 + 2Byz + 2B_{\prime}xz + 2B_{\prime\prime}xy + 2Cx + 2C_{\prime}y + 2C_{\prime\prime}z = 1,$$

in which A, $A_{\prime}$, $A_{\prime\prime}$, B, $B_{\prime}$, $B_{\prime\prime}$ are inverse rectangles, and C, $C_{\prime}$, $C_{\prime\prime}$ inverse straight lines.

In like manner the tangential equation of the surface of the second order may be written

$$a\xi^2 + a_{\prime}\upsilon^2 + a_{\prime\prime}\zeta^2 + 2\beta\upsilon\zeta + 2\beta_{\prime}\xi\zeta + 2\beta_{\prime\prime}\xi\upsilon + 2\gamma\xi + 2\gamma_{\prime}\upsilon + 2\gamma_{\prime\prime}\zeta = 1,$$

in which a, $a_{\prime}$, $a_{\prime\prime}$, β, $\beta_{\prime}$, $\beta_{\prime\prime}$ are rectangles, and γ, $\gamma_{\prime}$, $\gamma_{\prime\prime}$ straight lines.

The first twenty-four Chapters of this volume treat of the transformations of tangential coordinates, and of the application of this system, by one uniform method* to the discussion of theorems and problems. The ever-recurring analogies between the Cartesian and the tangential system of coordinates are continuously indicated as the work proceeds. The method is first applied to develop the properties of curves and curved surfaces of the second order, next to the genesis and rectification of curves of higher orders, their evolutes and involutes, &c.

In Chapter XXI. another system of coordinates is established, which I have ventured to call *pedal tangential coordinates,* and their properties developed.

In Chapter XXV. the peculiar notation of tangential coordinates is laid aside, and the principles of geometrical duality are established on the simplest elementary conceptions of pure geometry, and then applied to the investigation of the properties of surfaces of the second order having three unequal axes. It is shown that every such surface, with two exceptions, has four directrix planes parallel to the circular sections of the surface, and four corresponding foci, which directrix planes and foci coalesce when the surface becomes one of revolution. It is moreover shown that for every property of a sphere there exists its correlative on a surface having three unequal axes. These theorems will be found in Chapter XXVI.

In Chapter XXIX. metrical methods are applied to the theory of reciprocal polars, and new classes of properties of the conic sections established, more particularly those which are connected with the lines called the *minor directrices* of these curves.

---

* "Le vrai secret d'un système est dans sa méthode. Mettez une méthode dans le monde, vous y mettez un système que l'avenir se chargera de développer." —VICTOR COUSIN, *Cours de l'histoire de la philosophie*, vol. i. p. 84.

In Chapter XXX. the properties of the curve which I have named the Logocyclic Curve are discussed, a curve which has singular analogies with the circle. If the vectors of this curve be drawn from the origin and be taken to represent all the natural numbers from 0 to ∞, the corresponding arcs of a conjugate parabola will represent the logarithms of the numbers.

In the same Chapter the geometrical origin of logarithms is established; and it is shown that every system of logarithms may be represented by the arcs of a corresponding parabola, the peculiarity of the Napierian parabola being that the distance from its focus to its vertex must be assumed as equal to unity.

In Chapter XXXI. the principles of the trigonometry of the parabola are investigated. It is universally true to state that there is and can be no relation established between the arcs of a circle for which we cannot find a correlative for the arcs of a parabola. The transition may easily be made by changing $\cos\theta$ into $\sec\vartheta$, $\sqrt{-1}\sin\theta$ into $\tan\vartheta$, $+$ into $\perp$, and $-$ into $\top$.

It is not a little remarkable that while these mysterious imaginary expressions connected with the circle have been thoroughly investigated, the corresponding reciprocal theorem has entirely eluded discovery. Volumes have been written on the development of the imaginary theorem

$$(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos n\theta + \sqrt{-1}\sin n\theta,$$

while nothing has been known of the real theorem

$$(\sec\theta + \tan\theta)^n = \sec(\theta \perp \theta \perp \theta \text{ \&c.}) + \tan(\theta \perp \theta \perp \theta \text{ \&c.}).$$

In the same Chapter the principles of parabolic trigonometry are applied to the investigation of the properties of the Catenary and the Tractrix. Curious relations are established between the arcs of a catenary whose abscissæ are in arithmetical progression. It is also shown by the same method that the catenary is the evolute of the tractrix.

The last Chapter is devoted to the investigation of the projective equation of a cone whose vertex is at the intersection of three confocal surfaces, and which touches a fourth confocal surface.

I propose, if declining years and failing strength permit me, to complete this work, and to embody in a second volume my researches on the geometrical origin and properties of Elliptic Integrals, and to apply them to the investigation of the free motion of a rigid body round a fixed point, together with other collateral inquiries.

The tone of thought demanded by subjects such as these falls dull upon the public ear, and excites no responsive sympathy. It may therefore be proper to say that this volume is sent forth to the world with the anticipation of a very limited circulation, because it must be admitted that a cultivator of abstract science, without any view to practical results or profitable returns, has no reason-able ground of expectation that his labours will be recognized or appreciated in this country. With us the pursuit of knowledge for its own sake or indulgence in scientific research, unless it may be made to minister to some practical result (that is, to some paying result), is looked upon as little better than intellectual trifling*. Will it pay? is the test of all mental labour. It was very different in the schools and agoræ of that nation we are so prone to hold up for admiration as exhibiting models of intellectual greatness hitherto unequalled. Nor is this exclusive devotion to the adaptation of science to money-making so universal in other countries as amongst ourselves. Yet it was not always so. One might appeal to the age of Newton and Locke, the age of deep thinking and profound learning, in proof of this position. The causes of this degradation in the objects of intellectual pursuit are many, and some of them deeply seated. Not the least of these is the influence which the philosophy of Bacon has exerted on the tone and tendency of public opinion in this country. No doubt the author of the 'Novum Organon' conferred great benefits on mankind by laying down so clearly the true principles of physical investigation. He has marred his philosophy, however, by the motives he presents to us for its cultivation. He who could propound the maxim, worthy of Epicurus, that the true object of science is† to make men comfortable, had no very exalted conception of the dignity of man's understanding.

It is plain from his tone of thought that the philosophical Chancellor had a very clear prænotion, to use his own phraseology, of that emphatically English idea, comfort. There is little doubt that he would have valued more the invention of an efficient kitchen-range, or an ingenious corkscrew, than the Ideas of Plato or the discoveries of Archimedes‡.

This may to some appear a philosophical heresy; but yet it is quite certain that Lord Bacon's powerful influence, based on the soundness of his notions as to the true mode of procedure in conducting experimental inquiries, has had a depressing effect on the views of his countrymen, whose highest intellects are now devoted to the production of sensation novels, or to the discovery of ingenious contrivances to subserve the unbounded luxury and promote the material enjoyments of a self-indulgent people.

And if we turn aside from the paths of commerce and the busy haunts of men to the quiet cloisters of academical retirement, we shall find the same motives equally powerful and all but universal.

---

* "In primis, hominis est propria veri inquisitio atque investigatio."—*Cicero* de Officiis, lib. i. c. 13.

† "Meta autem scientiarum vera et legitima non alia est quam ut dotetur vita humana novis inventis et copiis."—BACON, *Novum Organon*, lib. i. aph. 81.

‡ Lord Bacon, like Hobbes, knew but little even of the elements of Mathematics.

Mathematical studies, and indeed one might say all studies, are pursued not for their own intrinsic worth, but with reference to the universal competitive examination. The inquiry is not, is it true? but will it tell? not, is it important as a principle? but is it likely to be asked as a question? In what profession but the law are profound research and extensive learning valued? and these are so because they pay.

Our universities are admirable institutions for the development of the intellect and the formation of habits; but they are not equally adapted to enlarge the boundaries of knowledge[*]. It is with regret that one is compelled to admit the fact that the great discoveries of modern times have been made by men who conducted their researches and worked out their discoveries far away from the theatres and libraries of our great centres of learning.

There is no reason in the nature of things why it should be so. Is the water that is drawn from the stagnant pool or spreading lake fresher or purer than that which rises from the gushing spring? Who ought to be so ready to draw from his stores of knowledge as he who spends his life in acquiring them? The communication of knowledge to youth is an important function of a university; but it is only one, an important or most important one if you will; but there are others hardly less important. Were Eton and Harrow and Rugby and the other great public schools of this country to be translated in all their integrity to Oxford or Cambridge, they would not constitute a university, in the highest and best sense of the word, because while teaching is the exclusive province of the public schools, our universities have, or ought to have, a higher function to discharge[†]. We ought to have not only the standing army but the pioneers of knowledge. Our universities do not encourage the Livingstones of science. Here, too, the same blighting influences prevail[‡]. It is useless to deny it. In this country profound acquirements in literature or science are not held in the same estimation they once commanded even amongst ourselves. In times gone by, for those denied the

---

[*] "Rursus in Moribus et institutis Scholarum Academiarum, Collegiorum et similium Conventuum, quæ doctorum hominum sedibus et eruditionis culturæ destinatæ sunt, omnia progressui scientiarum adversa inveniuntur. Studia enim hominum in ejusmodi locis in quorundam auctorum scripta, veluti in carceres, conclusa sunt; a quibus si quis dissentiat, continuo ut homo turbidus et rerum novarum cupidus corripitur."—BACON, *Novum Organon*, lib. i. aph. 90.

[†] "... Sit denique alia scientias colendi, alia inveniendi ratio. Atque quibus prima potior et acceptior est, ob festinationem, vel vitæ civilis rationes, vel quod illam alteram ob mentis infirmitatem capere et complecti non possint, optamus ut quod sequuntur teneant."—BACON, *Preface to the Novum Organon*.

[‡] The decline of science in England was long ago commented upon by one who was himself a brilliant luminary in his day. . . . "Here whole branches of continental discovery are unstudied, and indeed almost unknown even by name. It is vain to conceal the melancholy truth. We are fast dropping behind. In Mathematics we have long since drawn the rein, and given over a hopeless race; in Chemistry the case is not much better. Nor need we stop here. There

---

are indeed few sciences which would not furnish matter for similar remark."—Sir J. F. W. HERSCHEL, *Article on Sound*, *Encyclopædia Metropolitana*, p. 810.

And to the same effect, another high and still more recent authority:—

"Now, as to this important subject, the spirit in which we pursue education, the degree in which we turn our advantages to account, I must say of us here in England that we do not stand well. Our old Universities, and the schools above the rank of primary, have as a class the most magnificent endowments in the world. It may, however, be doubted whether the amount of these endowments, in England alone, is not equal to their amount on the whole continent of Europe taken together. Matters have mended, and are, I hope, mending. We have good and thorough workers, but not enough of them. The results may be good as far as they go; but they do not go far. But in truth this 'beggarly return,' not of empty but of ill-filled boxes, is but one among many indications of a wide-spread vice—a scepticism in the public mind, of old as well as young, respecting the value of learning and of culture, and a consequent slackness in seeking their attainment. We seem to be spoiled by the very facility and abundance of the opportunities around us. We do not in this matter stand well, as compared with men of the middle ages, on whom we are too ready to look down. For then, when scholarships and exhibitions, and fellowships and headships, were few, and even before they were known, and long centuries before triposes and classes had been invented, the beauty and the power of Knowledge filled the hearts of men with love, and they went in quest of her, even from distant lands, with ardent devotion, like pilgrims to a favoured shrine.

"Again, we do not stand well as compared with Scotland, where, at least, the advantages of education are well understood, and, though its honours and rewards are much fewer, yet self-denying labour, and unsparing energy in pursuit of knowledge, are far more common than with us. And once more, we do not stand well as compared with Germany, where, with means so much more slender as to be quite out of comparison with ours, the results are so much more abundant, that, in the ulterior prosecution of almost every branch of inquiry, it is to Germany, and the works of the Germans, that the British student must look for assistance. Yet I doubt if it can be said with truth that the German is superior to the Englishman in natural gifts, or that he has greater or even equal perseverance, provided only the Englishman had his heart in the matter. But Germany has two marked advantages: a far greater number of her educated class are really in earnest about their education; and they have not yet learned, as we, I fear, have learned, to undervalue, or even in a great measure to despise, simplicity of life.

"Our honours, and our prizes, and our competitive examinations, what for the most part are they, but palliatives applied to neutralize a degenerate indifference, to the existence of which they have been the most conclusive witness? Far be it from me to decry them, or to seek to do away with them. In my own sphere, I have laboured to extend them. They are, however, the medicines of our infirmity, not the ornaments of our health. They supply from without inducements to seek knowledge, which ought to be its own reward. They do something to expel the corroding pest of idleness, that special temptation to a wealthy country, that deadly enemy in all countries to the body and the soul of man. They get us over the first and most difficult stages in the formation of habits, which, in a proportion of cases, at least, we may hope will endure, and become in course of time self-acting.

"One other claim I must make on behalf of examinations. It is easy to point gifts of fortune, the only access to the Temple of Fame was through the portals of the learned professions, as they were called, or the services of their country, whether military or naval. The essential elements of success in these were high intellectual and moral endowments, unflagging labour and enduring perseverance.

Qui studet optatam cursu contingere metam
Multa tulit fecitque puer, sudavit et alsit.

But all this is changed. There are so many ways now, and some of them very questionable, of attaining to high social position, and to the possession of enormous wealth with a very attenuated garb of shreds and patches of trite information picked up anyhow, inaccurate and vague, that men do not care to undergo the study and the toil required to "plate themselves in the habiliments" of knowledge. With many, the maxim, to sell in the dearest and to buy in the cheapest market, would seem to comprise the whole duty of man.

It is no satisfactory answer to these remarks to say that there are exceptions to be found. No doubt there are, and brilliant exceptions. It might seem invidious to particularize, where varied excellence is so abundant. But this only adds strength to the argument; for it might with truth be said, what development of genius might we not witness were not its genial current frozen by the cold and chilling maxims of a spurious political economy? It would be blame misplaced to find fault with the Government for not encouraging profound learning or scientific research. Did the country desire (which it does not) that such should be honoured and rewarded, there is little doubt that the Government of the day, consisting mostly of men of high attainments themselves, would be glad to foster and advance the cultivators of that learning which is popularly considered no better than pedantry, and of science which is held to be of scant utility.

It would not be hard to prove that views such as these are short-

---

out their inherent imperfections. Plenty of critics are ready to do this; for in the case of first employment under the State, they are the only tolerably efficient safeguard against gross abuses, and such abuses are never without friends. But from really searching and strong examinations, such as the best of those in our Universities and schools, there arises at least one great mental benefit, difficult of attainment by any other means. In early youth, while the mind is still naturally supple and elastic, they teach the practice, and they give the power, of concentrating all its force, all its resources, at a given time, upon a given point. What a pitched battle is to the commander of an army, a strong examination is to an earnest student. All his faculties, all his attainments must be on the alert, and wait the word of command; method is tested at the same time with strength; and over the whole movement presence of mind must preside. If, in the course of his after life, he chances to be called to great and concentrated efforts, he will look back with gratitude to those examinations, which more perhaps than any other instrument may have taught him how to make them."—*Address delivered at the Liverpool College by the Right Hon.* W. E. GLADSTONE, *Dec.* 21, 1872.

sighted and low. It has been somewhere observed that had not the ancient Greek geometers investigated the properties of the plane sections of a cone, Kepler would have had no curve to fall back upon when he was obliged to abandon the circle as an accurate type of the orbits of the planets, and Newton would have lacked the profound and laborious arithmetical calculations of Kepler, the work almost of a lifetime[*], on which to base the connexion of the three great laws of planetary motion with the principle of universal gravitation. The true mechanism of the heavens would still be to us an inscrutable problem, and navigation would even now be guided by the stars, as in the time of Palinurus.

But for the wonderful discoveries of Newton in the science of Astronomy, and the consequent improvements in the art of Navigation, with the influence of this latter on the extension of commerce, it might have been that these transcendent truths would in our time have come to be looked upon as no better than *Verba otiosorum senum*, as Dionysius the tyrant of Syracuse sneeringly said of the speculations of Plato.

No man can forecast the consequences that may follow from any discovery in abstract science or physical research. What important results in aërial locomotion were at one time anticipated from the invention of the balloon! How little was once thought of the experiments on steam by the Marquis of Worcester! or who attached any importance to the investigation of the nature of those feeble forces elicited by the attrition of a bit of sealing-wax or a lump of amber? The consequences of any truth so discovered may well be held, using the language of Bacon, to be "partus temporis, non partus ingenii."

---

[*] Kepler, who lived in advance of his age, who appealed to the verdict of posterity, and not in vain, in a striking passage of the *Harmonice Mundi* which breathes a tone of saddened enthusiasm, thus records the grand discovery of his life, after seventeen long years of unremitting labour, the prerequisite of a still grander discovery, the law of Universal Gravitation.

"Rursum igitur hic aliqua pars mei Mysterii Cosmographici suspensa ante 22 annos quia nondum liquebat, absolvenda et huc inferenda est. Inventis enim veris orbium intervallis per observationes Brahei plurimi temporis labore continuo, tandem, tandem, genuina proportio temporum periodicorum ad proportionem orbium—

Sera quidem respexit inertem,
Respexit tamen et longo post tempore venit;

eaque, si temporis articulos petis, 8. Mart. hujus anni millesimi sexcentesimi decimi octavi animo concepta, sed infeliciter ad calculos vocata eoque pro falsa rejecta, denique 15. Maji reversa, novo capto impetu expugnavit mentis meæ tenebras tanta comprobatione et laboris mei septendecennalis in observationibus Braheanis et meditationis hujus in unum conspirantium, ut somniare me et præsumere quæsitum inter principia primo crederem, sed res est certissima exactissimaque, quod *Proportio quæ est inter binorum quorumcunque planetarum tempora periodica, sit præcise sesquialtera proportionis mediarum distantiarum, id est orbium ipsorum.*"—KEPLER, *Harmonices Mundi* lib. v. cap. iii. p. 279.

I know not that apologies disarm criticism; it is said they more frequently provoke it. But still it may be proper to say that this book is the result of the meditations of the better part of a lifetime. Desultory and rare they have been, intermitted for years, drawn away to other subjects, taken up again at different times and lengthened intervals. It has been to me a heavy drawback and deep discouragement, that I have had no fellow-workers to share in these researches. Neither have I entered into the labours of any. Without sympathy and without help I have worked upon those monographs now presented to the public. Nor let any one imagine that this isolation of the understanding is but a little loss or trivial hindrance. From the sympathetic contact of mind with mind truth is elicited. The electric spark of thought in its passage often flashes light on that which was clouded or obscure before. That which to one intelligence may appear clear as crystal, to another intelligence, nowise inferior, may seem distorted or confused by the media through which it is transmitted. Thus the concentrated and patient thinking of many intellects, some of a high order perhaps, combined and cooperating to one end, develops and expands a principle or a theory which the silent efforts and unaided labours of the solitary worker would have failed to accomplish. Nor have I had the help of those who have gone before me; for these researches, as here presented to the reader, are entirely original. I may add, that this work was not written to improve the text-books in use; nor is it published now to lighten the labours of tuition, or to supply the requirements of official competitive examinations. While this renunciation of scholastic utility may contract its circulation, on the other hand it has left me at liberty to follow out any train of thought to whatsoever conclusions it might carry me. I have waited long in the expectation (or shall I say hope?) that some of the many accomplished mathematicians of the present day would take up those subjects and expand them (for they admit of great development), and so produce a treatise from which any student of moderate ability might have gleaned enough to enable him to extend those researches still further. But I have waited in vain.

Had I seen any likelihood that those results, of which from time to time I have given abstracts in the proceedings of learned Societies, would be developed and published in a connected form, I should have shrunk from the toil of compilation, and left it to fresher and more elastic minds to expand and deliver to the world those discoveries which I would not willingly leave to dull forgetfulness or "to lie in cold obstruction."

J. B.

Stone Vicarage,
June 10, 1873.

ADVERTISEMENT

TO THE

FIRST EDITION OF THE

ESSAY ON TANGENTIAL COORDINATES.

I FEAR that brevity and compression have been but too much studied in the following essay; but the necessity of comprising the whole matter in a small compass, and the pressure of other avocations, will plead, I hope, a sufficient apology.

From the same cause I have been obliged to omit altogether subjects which might have been with propriety introduced, for example the general theory of *shadows*, and have only touched upon others which would require perhaps further development.

Among other applications of the method, that to the theory of *reciprocal polars* will, I trust, be found simple and satisfactory.

My attention has just been directed by a friend to a letter from M. Chasles, dated December 10, 1829, published in the 'Correspondence Mathématique' of M. Quetelet, tom. vi. p. 81, in which the writer asserts his claim to the invention of a system of coordinates, noticed by M. Plücker in one of the livraisons of Crelle's Journal, to which work I have never had an opportunity of referring. After some preliminary observations, he states his system as follows:—"Pour cela, par trois points fixes A, B, C, je mène trois axes parallèles entre eux, un plan quelconque rencontre ces axes en trois points dont les distances aux points A, B, C, respectivement, sont les coordonnées x, y, z, du plan," &c.; and then goes on to apply his system to a few examples, using the principles and notation of the differential calculus. To any one consulting the letter from which the above extract is taken, it will be apparent that the method there proposed, however excellent and ingenious it may be, bears not the least resemblance to the one developed in the following pages.

It must have often appeared an anomalous fact in the application of algebraic analysis to geometrical investigations, that while the locus of a point could be found from the simplest and most elementary considerations, the envelope of a right line or plane could be determined only by the aid of principles, artificial and obscure, derived from a higher department of analysis.

But this is not the only or the greatest objection to the method at present universally followed; it is in most cases operose, and in some impracticable, to reduce the equation $V=0$ to the form $\frac{dV}{d\alpha}=0$, and then eliminate the auxiliary variable α between these equations—a difficulty which becomes far more formidable in problems of three dimensions, where we are obliged to eliminate the auxiliary variables α and β between the three equations

$$V=0, \quad \frac{dV}{d\alpha}=0, \quad \frac{dV}{d\beta}=0.$$

As it follows *à priori* from the principle of *duality**, that for every locus of a point there exists a corresponding envelope of a right line or plane, it would seem that the comparative paucity of theorems of the latter species generally known can be owing to nothing but the want of a simple and direct mode of investigation.

From these considerations I have been led to the discovery of a method simple in principle, and easy of application, analogous to, but different from, that of rectilinear or *projective* coordinates—as for distinction they may be called—in which the reciprocals of the distances of the origin from the points where the axes of coordinates are met by a right line, or plane, touching a curve or curved surface, are denoted by the letters ξ, υ, ζ; an equation established between them may be called the *tangential equation* of the curve or curved surface.

By the help of this equation we may elude the necessity of differentiating the equation $V=0$, and discover the envelopes of right lines and planes with the same facility as the locus of a point by projective coordinates.

But it is not alone in inquiries of this nature that the method is chiefly valuable; there is a large class of theorems relating to curves touching given right lines, and surfaces in contact with given planes, which may be treated by the method proposed with the greatest facility, whose solution by projective coordinates would lead to exceedingly complicated and unmanageable expressions.

J. B.

Trinity College,
March 25th, 1840.

---

* See various memoirs on this subject by MM. Gergonne, Poncelet, and others, dispersed through the volumes of the 'Annales de Mathématiques.'

TABLE OF CONTENTS.

[The numbers on the left hand denote the sections, the numbers on the right hand the pages.]

CHAPTER I.

ON THE TANGENTIAL EQUATIONS OF A POINT AND A STRAIGHT LINE IN A PLANE.

1.] Tangential equations of a point and a plane 1
2.] On the transformation of coordinates 2
3.] On the translation of coordinates 2
4.] When the angle of ordination is changed............................ 3
5.] Length of perpendicular from a given point on a given line in terms of its tangential coordinates and the projective coordinates of the point.. 3
6.] Expression for the value of the angle between two given straight lines whose tangential coordinates are given............................ 3
7.] The tangential coordinates of a straight line passing through two fixed points whose projective coordinates $(\alpha\beta)$ and $(\alpha_,\beta_,)$ are given 4
8.] to 10.] Discussion of the forms which the expressions for the perpendicular assume in projective and tangential coordinates 4

CHAPTER II.

ON THE TANGENTIAL EQUATIONS OF THE CONIC SECTIONS.

12.] Tangential equations of the central conic sections referred to centre and axes 7
Formulæ of transformation when the rectangular axes pass through any point in the plane of the section, when the section is an ellipse.. 8
13.] When it is an hyperbola 9
14.] When the origin is at a focus 9
15.] When the origin of coordinates is on the curve 9
16.] Expression for the angle which the asymptote of the hyperbola makes with the axis of X 10
17.] Relation between the constants when the tangential equation of the curve breaks up into two linear equations......................... 11
18.] Values of the constants when the curve touches the axes of X or Y .. 11
19.] To find the values of ξ and υ when they are tangents to the curve 12

20.] Tangential equations of the poles of the axes of X and Y ... 12
21.] To find the species of the conic section when the reciprocal of the perpendicular from the origin on the tangent is a rational function of the tangential variables ... 13
22.] General expressions for the values of ξ and v in terms of x and y and conversely ... 14
23.] Relations between the partial differentials of the projective and tangential equations of a given curve ... 14
24.] Application of these formulæ ... 15
25.] On Asymptotes ... 16
26.] On the tangential equation of the curve referred to oblique axes of coordinates ... 17
27.] General expression for the distance between the point of contact of a tangent and the foot of the perpendicular let fall on that tangent from the origin ... 17
28.] Value of a semidiameter in terms of the tangential coordinates of the tangent through the extremity of the diameter ... 18

CHAPTER III.

THE PRINCIPLES ESTABLISHED IN THE PRECEDING SECTIONS APPLIED TO EXAMPLES.

29.] The product of pairs of perpendiculars let fall from two points on a straight line is constant; the line envelopes a conic section ... 19
30.] The vertex of a right angle moves along the circumference of a circle; one side passes through a fixed point; the other envelopes a conic section ... 19
31.] Tangents are drawn to an ellipse from any point of a concentric circle whose radius is $\sqrt{a^2+b^2}$; the line joining the points of contact envelopes a confocal conic ... 20
32.] The sum of the perpendiculars let fall from n given points in a plane to a straight line in the same plane is constant. The straight line envelopes a circle ... 21
33.] Perpendiculars are drawn to the extremities of the diameters of an ellipse. They envelope a curve; to determine its tangential equation 22
34.] Two semidiameters of a conic section and the chord joining their extremities contain a given area. The curve enveloped by this chord is a similar conic section ... 22
If a polygon of n sides be inscribed in a conic section, the sides being inversely as the perpendiculars let fall upon them from the centre, the polygon will circumscribe a conic section similar to the given one .. 23
35.] Envelope of the chord joining the extremities of two focal chords of a conic section ... 23
36.] A straight line revolves in a conic section, having always a constant ratio to the parallel diameter; it will envelope a similar conic section 25
37.] The product of the sides of a right-angled triangle diminished by fixed quantities, is constant; the hypotenuse will envelope a conic section.. 26
38.] Let the sides of a rectangle be produced, and cut by a transversal; to find the tangential equation of the curve to which this line is always a tangent, under certain conditions ... 26

39.] The vertex of an angle of constant magnitude moves along the circumference of a circle; one side passes through a fixed point; to determine the curve that will be enveloped by the other 28
40.] An angle of given magnitude revolves round a fixed point, intersecting by its sides two given straight lines; the line which joins the points of intersection envelopes a conic section 29
41.] An angle of given magnitude θ revolves round a point in the plane of a conic section, cutting the curve in two points; the line joining these points will envelope a curve whose tangential equation is of the fourth order 30
42.] A right-angled triangle has its right angle at a focus of a conic section, while the hypotenuse envelopes the curve; one acute angle of the triangle moves along a given straight line, the other will describe a conic section 32
43.] Form of the tangential equation of a conic section when the origin is anywhere in a concentric circle passing through the foci 33
44.] A triangle is inscribed in a conic section; two of its sides always pass through two fixed points, the third side envelopes a conic section .. 34
45.] A series of central conic sections have the same centre, and their axes in the same direction, but such that the difference of the reciprocals of the squares of their axes is constant; the tangents drawn to a point on each, their intersection with a common diameter, envelope a concentric hyperbola, if the intersected curves be a series of ellipses, and an ellipse if the intersected curves be hyperbolas 37

On Polygons inscribed and circumscribed to Conic Sections.

46.] Conic sections are inscribed in the same quadrilateral, the polar of any point in their plane envelopes a conic section 38
47.] A series of conic sections are inscribed in the same quadrilateral, their centres range on the same straight line......................... 39
48.] The centres of conic sections inscribed in the same quadrilateral all range on the straight line which joins the points of bisection of the two diagonals of the quadrilateral. (Newton's theorem) 40

CHAPTER IV.

ON THE TANGENTIAL EQUATIONS OF THE PARABOLA.

49.] Tangential equation of the parabola derived from its projective equation 41
50.] Relation between the constants when the curve touches the axes of coordinates 43
This theory applied to examples 43
51.] The sum of the sides of a right angle is constant. The hypotenuse envelopes a parabola 43
52.] An angle of given magnitude moves along a fixed straight line, one side always passes through a fixed point, the other side will envelope a parabola 44
53.] Parabolas are inscribed in a triangle; the locus of their foci is the circumscribing circle 44
54.] An angle of given magnitude revolves round the focus of a parabola; to determine the curve enveloped by the chord which joins the points in which the parabola is intersected by the sides of the angle 45

CHAPTER V.

ON THE TANGENTIAL EQUATIONS OF THE POINT, THE PLANE, AND THE STRAIGHT LINE IN SPACE.

56.] On the tangential equations of a point and a plane in space 49

On the transformations of tangential coordinates in space............ 51

57.] Expression for the perpendicular from the origin on a plane whose tangential coordinates are given 51

58.] On the translation of the axes of coordinates in parallel directions 52

59.] On the tangential equations of a plane passing through the origin of coordinates.. 53

CHAPTER VI.

ON THE TANGENTIAL EQUATIONS OF THE STRAIGHT LINE IN SPACE.

60.] Two methods of defining the position of a straight line in space 54

61.] Tangential method 55

62.] To express the cosines of the angles which a straight line makes with the axes of coordinates in terms of the constants μ, ν, a, β of the tangential equations of the given straight line 56

63.] To determine the conditions that two straight lines may meet in space 56

65.] To investigate the conditions that a given line may be found in a given plane 57

66.] To find the tangential coordinates of a plane which shall pass through a given point and a given straight line 57

67.] To determine the angle between two given planes.................. 58

68.] A straight line is perpendicular to a given plane, to determine the relations between the coefficients of the given straight line and the given plane 58

69.] To determine the angles which the straight line, in which two given planes intersect, makes with the axes of coordinates 59

70.] To determine the conditions in order that a given straight line and a given plane may be parallel.................................... 60

71.] A straight line is parallel to a given straight line, to determine the relation between the constants 60

72.] To investigate an expression for the angle between two given straight lines whose equations are given 60

73.] To find an expression for the angle between a given plane and a given straight line 61

CHAPTER VII.

ON THE TANGENTIAL EQUATIONS OF SURFACES OF THE SECOND ORDER.

74.] On the tangential equations of surfaces of the second order referred to their centres and axes, as axes of coordinates 62

75.] Translation of the origin to any other point in space, the axes of coordinates continuing parallel 63

76.] Values of the tangential ordinates ξ, v, ζ in terms of the partial differential coefficients of the projective equation of the given surface 64
77.] Tangential equations of the poles of the three coordinate planes 66
Tangential equations of the polar plane of the origin................ 66
Condition in order that the origin of coordinates may be on the given surface 67
78.] Values of certain coefficients when the surface touches one or more of the coordinate planes 67
79.] Determination of the point of contact 67
80.] *Formulæ of transition* from the projective to the tangential equation of any surface, and reciprocally 68

CHAPTER VIII.

ON THE MAGNITUDE AND POSITION OF THE AXES OF A SURFACE OF THE SECOND ORDER.

81.] Transformation of the tangential equation of a surface of the second order 69
82.] A general expression for the distance between the point of contact of a tangent plane to a surface, and the foot of the perpendicular let fall from the origin on this tangent plane 70
83.] Definition of axis of central surface 71
Tangential *cubic equation of axes* 71
84.] To determine the angles which one of the axes of the surface makes with the axes of coordinates.. 72
85.] On the particular case when the surface is one of revolution 73

CHAPTER IX.

ON THE TANGENTIAL EQUATIONS OF THE PARABOLOIDS.

86.] The tangential equation derived from the projective equation of the paraboloids. 75
87.] On the transformation of the axes of coordinates in the case of the paraboloids 76
88.] Given the general equation of the paraboloid to any set of rectangular axes passing through the vertex, to determine the magnitude and position of the parameters of the principal sections................ 77
89.] On the hyperbolic paraboloid.. 78
90.] Limits of the surface 80
91.] Tangential equation of this surface derived from its projective equation 80
92.] To ascertain whether in a tangent plane to this surface there can exist any linear generatrices.. 80
93.] No two successive generatrices of this surface can meet, or be in the same plane.. 82
94.] To determine the sections of this surface made by tangent planes in certain positions 82

CHAPTER X.

ON THE APPLICATION OF ALGEBRA TO THE THEORY OF RECIPROCAL POLARS.

95.] Line of contact of a cone touching a surface of the second order is a plane curve 83

96.] Case when the surface is a central ellipsoid 84

97.] *Polar equations* of surfaces of the second order 84

Illustrations and instances of the principle of *Duality* 85

98.] Application of the principles established in the foregoing sections to the investigation of certain theorems 85

The sum of the perpendiculars let fall from n given points on a plane is constant. To determine the envelope of the plane 89

99.] The sum of the squares of the perpendiculars let fall from n given points on a plane whose tangential coordinates are ξ, v, ζ is constant, and equal to nk^2; the plane envelopes a surface of the second order 89

100.] A series of surfaces of the second order touch seven fixed planes; the poles of any given plane relative to these surfaces are also on a fixed plane 90

101.] A surface of the second order touches seven given planes, to find the locus of its centre 91

102.] If two surfaces of the second order are enveloped by a cone, they may also be enveloped by a second cone 91

103.] Let a plane cut off from three fixed rectangular axes, segments the sum of which, multiplied by a constant area, shall be equal to the tetrahedron whose faces are the three coordinate planes and the limiting tangent plane; to determine the surface enveloped by this latter plane 92

104.] If a series of planes retrench from a cone of the second degree a constant volume, they will envelope a discontinuous hyperboloid, or one of two sheets 94

105.] On the cubic equation of axes, when the surface is one of revolution, and the origin at a focus 96

106.] Three straight lines, constituting a right-angled trihedral angle, revolve round a fixed point in space, meeting a surface of the second order (S) in three points. The plane which passes through these three points envelopes a surface of revolution (Σ) of the second order, whose focus is at the given point, and whose directrix plane relative to this focus is the polar plane of the fixed point relative to the given surface (S) 97

107.] To show that the continuous hyperboloid admits of linear generatrices 99

108.] A surface of the second order is cut by a given secant plane; to determine the tangential equation of the section of the surface made by this plane 100

109.] The reciprocal polar of any surface of the second order, the centre of the directrix surface being on the given surface, is a paraboloid .. 101

CHAPTER XI.

ON CONCYCLIC SURFACES OF THE SECOND ORDER.

110.] Concyclic surfaces are the reciprocal polars of confocal surfaces 102

111.] Examples of the analogies between concyclic and confocal surfaces—dual relations 103

112.] Through a given point three central confocal surfaces may be described—an ellipsoid, a continuous and discontinuous hyperboloid 104

113.] A series of concyclic surfaces of the second order touch a given plane whose tangential coordinates are ξ, v, ζ. To determine the equation of the axes of these surfaces 105

114.] A common tangent plane is drawn to three concyclic surfaces of the second order, the three points of contact two by two subtend right angles at the centre 105

115.] Let there be two concyclic ellipsoids, and any point on the external one be assumed as the vertex of a cone enveloping the other, the plane of contact will meet the tangent plane to the first surface through the vertex of the cone in a straight line, such that the diametral plane passing through this line will be at right angles to the diameter which passes through the vertex of the cone 106

116.] Let a cone envelope an ellipsoid, so that the plane of contact shall touch a second surface confocal with the former. The line drawn from the vertex of the cone to the point of contact of this tangent plane will be at right angles to it 107

117.] Parallel planes are drawn to a series of confocal ellipsoids; to determine the locus of the points of contact 109

118.] To a series of concyclic surfaces tangent planes are drawn touching the surfaces in the points where they are pierced by a common diameter; to find the surface enveloped by these tangent planes 110

CHAPTER XII.

ON THE SURFACE OF THE CENTRES OF CURVATURE OF AN ELLIPSOID.

119.] Tangential equation of *the surface of centres of an ellipsoid* 112

120.] Investigation of the shape of this surface 113

Any two parallel tangents being drawn to the surface of centres and to the ellipsoid, the difference of the squares of the coincident perpendiculars let fall upon them from the centre is always equal to the square of the coinciding semidiameter of the ellipsoid 114

121.] On the umbilical lines of curvature of an ellipsoid 115

123.] The areas of the umbilical parallelogram, of the ellipse, and of the evolute circumscribed by and inscribed in the four umbilical normals, have certain reciprocal relations which are independent of the axes of the ellipsoid 118

124.] On the projective equation of the surface of centres of an ellipsoid .. 120

125.] Three *concyclic* surfaces of the second order are touched by a common tangent plane in three points; these points, two by two, will subtend right angles at the centre, and the locus of all the points of contact with the two variable hyperboloids will be a surface which may be called the "surface of contacts" 122

CHAPTER XIII.

ON THE APPLICATION OF THE METHOD OF TANGENTIAL COORDINATES TO THE INVESTIGATION OF THE PROPERTIES OF TRANSCENDENTAL AND OTHER CURVES OF A HIGHER ORDER THAN THE SECOND.

126.] On the tangential equation of the caustic by reflection of the circle .. 124
127.] Two cases of the general theorem which may repay discussion 127
128.] Determination of the points in which the limiting tangent, when vertical, cuts the axis of X 128

CHAPTER XIV.

ON EPICYCLOIDS AND HYPOCYCLOIDS.

129.] The theory of tangential coordinates applied to investigate the properties of epicycloids and hypocycloids generally 129
130.] to 132.] The theory applied to selected cases 130
133.] On hypocycloids 133
To determine the equation of the hypocycloid, when the radius of the rolling circle is one half that of the base circle 133
134.] On the hypocycloid whose radius is one fourth that of the base circle 133
135.] To determine the involute of the quadrantal hypocycloid 134
136.] On the projective equation of the involute of the quadrantal hypocycloid 135
137.] On the tangential equation of the trigonal hypocycloid 137
138.] On the tangential equation of the hexagonal hypocycloid 138
139.] On the tangential equation of the cycloid 139
141.] On the tangential equation of the logarithmic curve 140
142.] On the tangential equation of the cissoid 141
143.] On the tangential equation of the lemniscate 141
144.] On the tangential equation of the cardioid 142
145.] On the projective equation of the curve whose tangential equation is $a^2v^2+b^2\xi^2=a^2b^2(\xi^2+v^2)^2$ 143
146.] On the tangential equation of the curve inverse to the central ellipse, or the curve whose equation is $a^2x^2+b^2y^2=(x^2+y^2)^2$ 144
147.] On the reciprocal polar of the evolute of an ellipse 145
148.] On the tangential equation of the semicubical parabola 147
149.] On the tangential equation of the cubical parabola 147
150.] On the involute of the cycloid 147

CHAPTER XV.

ON PARALLEL CURVES.

151.] Definition of parallel curves 149
152.] Tangential equation of the curve parallel to the ellipse 150
153.] The normals of a parabola are increased by the constant line h; to find the tangential equation of the curve 151

Tangential equation of the curve parallel to the evolute of an ellipse 152
To find the parallel curve of the quadrantal hypocycloid 152
154.] To find the equations of the parallel curves of the involute of the quadrantal hypocycloid 152
Tangential equation of the curve parallel to the semicubical parabola 152
Tangential equation of the curve parallel to the cycloid 153

CHAPTER XVI.

ON THE TANGENTIAL EQUATIONS OF EVOLUTES.

155.] Values of the variables in the tangential equation of the evolute in terms of the partial differential coefficients of the projective equation of the curve whose evolute is required 153
156.] To determine the tangential equation of the evolute of the ellipse .. 153
157.] Tangential equation of the evolute of the parabola................ 154
158.] On the evolute of the semicubical parabola 154
159.] On the evolute of the cubical parabola.......................... 154
160.] On the evolute of the quadrantal hypocycloid 155
161.] On the tangential equation of the evolute of the lemniscate 156
162.] On the evolute of the curve whose equation is $a^2x^2+b^2y^2=(x^2+y^2)^2$ 157

CHAPTER XVII.

ON REVOLVING ANGLES, PROJECTIVE AND TANGENTIAL.

163.] Definitions of the curves termed tangential and projective pedals .. 158
164.] Application of this theory to examples.......................... 159
  (α) Tangential pedal of a straight line.
  (β) Tangential pedal of a point.
  (γ) Projective pedal of a conic section.
  (δ) Becomes a cardioid, when the origin is on the curve.
165.] Tangential pedal of a circle is a conic section 160
166.] Projective and tangential pedals of the semicubical parabola 161
167.] Projective pedal of the cycloid 161
Simple geometrical proof of this theorem........................ 161
168.] A right angle moves along a cycloid; one side passes through the centre of the base, the other side will envelope a curve whose tangential equation may be found 161
169.] Projective pedal of the cardioid................................ 162
170.] Projective pedal of the quadrantal hypocycloid 162
171.] Tangential pedal of the logocyclic curve 163
172.] Tangential pedal of the evolute of an ellipse 163
Projective pedal of the evolute of an ellipse....................... 163
Tangential pedal of a cissoid is a parabola 163
Projective pedal of a cissoid 163
173.] Projective pedal of the curve parallel to an ellipse................ 163
174.] Projective pedal of the curve parallel to a parabola is a conchoid.... 164

CHAPTER XVIII.

ON RIGHT ANGLES REVOLVING ROUND FIXED POINTS IN THE PLANE OF A CURVE.

175.] General projective equation of the locus in terms of the partial differential coefficients of the tangential equation of the given curve.. 165
176.] Projective equation of the locus when the given curve is an ellipse.. 166
177.] When the given curve is a cubical parabola 166
178.] When the given curve is a semicubical parabola.................. 166
179.] When the given curve is a parabola 167
180.] When the revolving angle is no longer a right angle. General formulæ in this case for the locus 168

CHAPTER XIX.

TO INVESTIGATE THE LOCI WHEN THE REVOLVING ANGLES IN THE TWO PRECEDING CHAPTERS ARE OTHER THAN RIGHT ANGLES.

181.] Transformation of formulæ on this hypothesis.................... 169
182.] Application of modified formulæ to the ellipse 170
183.] To the trigonal hypocycloid 171

CHAPTER XX.

ON THE LOCI OF RIGHT ANGLES REVOLVING AROUND GIVEN CURVES.

184.] A right angle revolves so as to touch with its sides a given curve, to determine the projective equation of the vertex of the right angle 171
185.] Application of this method to examples 172
When the given curve is an ellipse 172
186.] When the curve is a parabola 172
187.] To investigate the equation of the vertex of the right angle which envelopes the involute of the quadrantal hypocycloid............ 173
188.] A right angle revolves, touching with its sides the curve parallel to the parabola; to determine the locus of the vertex.............. 173
189.] A right angle revolves, touching with its sides a cycloid; to determine the locus of the vertex.. 174
190.] A right angle revolves round a fixed point, whose sides meet a curve in certain points; to determine the tangential equation of the curve which is enveloped by the line joining these points 175
191.] Illustration of this method.................................... 175
192.] Another illustration.. 176
193.] Recapitulation of the principles established in Chapters XV., XVI., XVII., XVIII., XIX., and XX....... : 177

CHAPTER XXI.

ON PEDAL TANGENTIAL COORDINATES.

194.] Definition of this system of coordinates 178
195.] Illustrations of this method 179
Pedal tangential of a straight line 179
Pedal tangential of a circle 179
Pedal tangential of an ellipse 179
196.], 197.] Properties of the evolute of an ellipse analogous to those of an ellipse 180
198.] On the pedal tangential curve of a semicubical parabola 181
199.] The pedal tangential of the curve whose equation is $a^2y^2+b^2x^2=x^2y^2$ is an ellipse 181

CHAPTER XXII.

ON THE RADIUS OF CURVATURE AND THE RECTIFICATION OF PLANE CURVES.

200.] Definitions of the rectification of plane curves and of the radius of curvature 181
Primary expression for the radius of curvature 182
201.] Primary expression for the arc of a plane curve 183
How the ambiguity of sign is to be determined 184
202.] Explanation of the *critical point* in Fagnani's theorem on the arcs of an ellipse 184
203.] Expression for the radius of curvature in terms of the partial differential coefficients of the tangential equation of a plane curve 185
Expression for the arc of curve in terms of the same coefficients 185
204.] On the rectification of the circle by the method of tangential coordinates, the origin being taken any where in the plane of the circle.. 185
205.] On the radius of curvature of the ellipse 186
206.] On the radius of curvature of the parabola 186
207.] On the rectification of the ellipse 187
208.] On the rectification of the parabola 187
209.] On the radius of curvature and the rectification of the cycloid 187
The square of the radius of curvature at any point of a cycloid, together with the square of the arc measured from that point to the vertex, are equal to the square of the diameter of the generating circle 188
210.] On the rectification of the evolute of the ellipse 188
211.] On the rectification of the semicubical parabola, and its radius of curvature 189
212.] On the rectification of parallel curves 190
213.] On the radius of curvature, and the rectification of epicycloids and hypocycloids 190

214.] Application of these formulæ to examples 193
- (α) The cardioid.
- (β) The semicircular epicycloid.
- (γ) The quadrantal epicycloid.
- (δ) The trigonal epicycloid.

215.] On the curvature and rectification of hypocycloids................ 193

216.] Relation between the radius of curvature and the arc of any epicycloid 194

217.] On the radius of curvature of the cubical parabola................ 194

218.] On the rectification of the involute of the quadrantal hypocycloid .. 195

219.] On the rectification of the curve whose tangential equation is $b^2\xi^2+a^2v^2=a^2b^2(\xi^2+v^2)^2$ 196

220.] On the rectification of the inverse curve of the central ellipse 196

CHAPTER XXIII.

ON THE RELATION BETWEEN TANGENTIAL EQUATIONS, AND THE SINGULAR SOLUTIONS OF THE DIFFERENTIAL EQUATIONS OF PLANE CURVES.

221.] Relation between the tangential and projective equations of the same curve 200

(α), (β) Illustrations of this principle 201

CHAPTER XXIV.

ON THE GEOMETRICAL THEORY OF RECIPROCAL POLARS.

222.] Elementary principles on which the geometrical theory of reciprocal polars is established 202

223.] Definition of polarizing sphere 202

224.] If any point $\pi_{,}$ be assumed in a plane (Π), the polar plane (Π$_{,}$) of this point will pass through π the pole of (Π) 204

When any number of straight lines are parallel to one another, their conjugate polars will all lie in the same plane 204

The polar plane of any point assumed in a straight line will pass through the conjugate polar of this straight line; and the polar plane of any point assumed in a plane will pass through the pole of this plane.. 204

225.] The reciprocal polar of any plane curve is a cone whose vertex is in the pole of the plane of the curve 204

226.] From the centre ω of the polarizing circle (Ω) a perpendicular ωP is let fall on a tangent drawn to the curve (S) at T. The radius ωT produced will be perpendicular to the corresponding tangent drawn to the reciprocal polar curve (Σ); and the perpendicular ωP on the tangent to the curve (S) at T will pass through the point of contact τ of the corresponding tangent to the reciprocal polar (Σ) 205

227.] To find the conjugate polar of the normal to a tangent plane applied to the primitive surface (S) 206

CHAPTER XXV.

THE GEOMETRICAL THEORY OF RECIPROCAL POLARS APPLIED TO THE DEVELOPMENT OF A NEW METHOD OF DERIVING THE PROPERTIES OF SURFACES OF THE SECOND ORDER WITH THREE UNEQUAL AXES FROM THOSE OF THE SPHERE.

231.] Fundamental theorem.—If a point taken in the plane of a circle (S) be made the centre of a polarizing circle (Ω), the reciprocal polar (Σ) of the circle (S) will be a conic section having a focus at the centre of (Ω) 208

232.] The polar of the centre of the circle (S) is the directrix of the conic section (Σ) 209

233.] If in the transverse axis of a surface of revolution (Σ) we assume a point V as the common vertex of two cones of revolution circumscribing the sphere (S) and the polarizing sphere (Ω) whose centre is at ω the focus of (Σ), if the base of the former cone pass through the focus of (Σ), the base of the latter will pass through the centre of (Σ) 209

234.] The properties of surfaces of the second order having three unequal axes derived from those of surfaces of revolution of the second order 211

235.] The properties of two surfaces of the second order may not be so derived 212

236.] Relations established between *a*, *b*, and *c* the semiaxes of the reciprocal surface (Σ), A and C the semiaxes of the primitive surface (S), and R and D, the former being the radius of the polarizing sphere (Ω), while D is the distance between the centres of (S) and (Ω) 212

237.] Conversely, the semiaxes of the primitive surface (S) and D may be expressed in terms of *a*, *b*, and *c*, the semiaxes of (Σ). Eccentricities of the three principal sections of (Σ), expressed in terms of A, C, and D 215

238.] Definition of the *polar focus* 215

239.] Definition of *the conjugate umbilical directrix planes* 216

240.] Position and inclination of the conjugate umbilical directrix planes.. 216

241.] Angle between an umbilical diameter and an umbilical directrix plane 217

242.] In every central surface of the second order having three unequal axes, there are *four* foci. Position of these umbilical foci defined...... 218

243.] Definition of the *Principal Parameter* of a surface of the second order having three unequal axes 219

244.] The umbilical focus is the pole of the corresponding umbilical directrix plane with respect to the reciprocal surface (Σ) 220

245.] In the axis of the surface (S) there are seven remarkable points. Positions of their correlative seven polar planes 220

246.] The principal semiparameter of the surface (Σ) expressed in terms of A, C, D and R, found to be independent of A and D. Remarkable results which follow from the independence of L the principal semiparameter of (Σ), a surface having three unequal axes on A, the radius of the principal circular section of the primitive surface (S) and D the distance between the centre of (S) and the polar focus of (Σ) 221

Principal parameter of the elliptic paraboloid 222

247.] Position of the polar plane of the extremity of the major axis of (S) 222

248.] A Table of the values of the magnitudes and positions of the several planes, lines, and points connected with a central surface (Σ) of the second order having three unequal axes 223

249.] On the hyperboloid of two sheets and its asymptotic cone.......... 225

250.] The polar plane of ω the centre of (Ω) with respect to (S) is the polar plane of C, the centre of (Σ) with respect to (Ω)................ 226

251.] The reciprocal polar of a continuous hyperboloid is also a continuous hyperboloid 226

252.] On the oblate spheroid 227

CHAPTER XXVI.

ON THE APPLICATION OF THE THEORY DEVELOPED IN THE PRECEDING CHAPTER TO THE DISCUSSION OF SOME THEOREMS AND PROBLEMS.

253.] LEMMA I.—To express the base A B of a triangle A B C in terms of the perpendiculars on the polars of the points A, B, and the distance of the pole of A B from C. Poncelet's theorem 228

LEMMA II.—From a given point R let a perpendicular R P be let fall on the given straight line C V, and let the point O be assumed as the centre of the polarizing circle whose radius is R; let S be the pole of C V, and B T the polar of the point R. The perpendiculars from the points R and S on the straight lines C V and B T will have the same ratio to each other as the lines O R and O S 228

254.]*If from any point on an umbilical surface of the second order perpendiculars be let fall on the two conjugate umbilical directrix planes, the rectangle under these perpendiculars will have to the square of the distance of this point from the polar focus a constant ratio 229

(α), (β), (γ), (δ), (ε). Particular cases of this theorem 231

Algebraical proof of this theorem ....(note) 231

255.] When the surface becomes an oblate spheroid.................... 232

256.] The sum of the products, taken two by two, of the perpendiculars let fall from the four umbilical foci on a tangent plane to (Σ) is equal to $2b^2 \sin^2 \nu + \frac{2c^2u^2}{a^2} \cos^2 \nu$ 233

257.] Through the polar focus of (Σ), a surface of the second order, a plane and a straight line being drawn at right angles to each other, the line meeting the surface in the point τ, and the umbilical directrix plane (Δ) in the straight line (δ), the plane which passes through the point τ and the straight line (δ) will envelop a surface of revolution (Σ,) whose focus will coincide with the polar focus of (Σ), and whose directrix plane will pass through the polar directrix of the polar focus ω in the principal section in the plane of XY 234

---

[*] This theorem, founded on Lemma II., was published in the Philosophical Magazine for the year 1840, p. 432.

"Il est une propriété principale des coniques, qui se retrouve dans les cônes, et dont nous n'avons point encore fait mention relativement aux surfaces du second degré; c'est que: 'la somme ou la différence des rayons vecteurs menés d'un point d'une conique aux deux foyers est constante.' Nous avons fait, pendant longtemps, des tentatives pour trouver quelque chose d'analogue dans les surfaces; mais sans obtenir aucun succès."—CHASLES, *Aperçu Historique,* p. 391.

258.] Algebraical proof of this theorem 235

259.] If in any surface (Σ) having three unequal axes a straight line be drawn meeting the surface in two points τ and $\tau_{,}$ and the umbilical directrix planes (Δ) and $(\Delta_{,})$ in two points also, δ and $\delta_{,}$, the angles which these pairs of points subtend respectively at the polar focus ω will be equal to each other, or the angle $\tau\omega\delta$ will be equal to the angle $\tau_{,}\omega\delta_{,}$ 237

A straight line touches the surface (Σ) and meets the umbilical directrix planes (Δ) and $(\Delta_{,})$ in two points δ and $\delta_{,}$; the segments of this line between the point of contact τ and the umbilical directrix planes will subtend equal angles at the polar focus ω 238

260.] If through any point δ assumed on one of the directrix planes $(\Delta_{,})$ a secant be drawn parallel to the other directrix plane (Δ), and meeting the surface in the points τ and $\tau_{,}$, the rectangle under these segments will be equal to the square of the distance of δ from the polar focus ω 238

261.] Let the principal parameter C $C_{,}$ of a surface of the second order (Σ) having three unequal axes be the base of a triangle whose vertex is the point G anywhere on the surface. If the sides of this triangle be produced to meet the umbilical directrix plane (Δ) in two points δ and $\delta_{,}$, the points δ and $\delta_{,}$ will subtend a right angle at the polar focus ω 240

If the shorter axis of an oblate spheroid be taken as the fixed base of a triangle C $C_{,}$ G inscribed in the spheroid, whose sides are produced to meet one of the directrix planes (Δ) in two points Q and $Q_{,}$, these points, Q and $Q_{,}$, will subtend a right angle at the centre of the oblate spheroid (Σ) 240

If the parameter of a surface of revolution (S) be the base of a triangle inscribed in it, and the sides be produced to meet the directrix plane (D) in d and $d_{,}$, the points d, $d_{,}$ will subtend a right angle at the focus F of (S) 240

If a fixed chord be taken in a surface (Σ), and this fixed chord $c\,c_{,}$ be made the base of a triangle whose vertex G is anywhere on this surface, the sides of this triangle Gc, G$c_{,}$ being produced to meet the umbilical directrix plane (Δ) in two points δ and $\delta_{,}$, the points δ and $\delta_{,}$ will subtend a constant angle at the polar focus ω 241

263.] If through a straight line (ϑ) in the umbilical directrix plane (Δ) of a surface (Σ) of the second order, two tangent planes (Θ) and $(\Theta_{,})$ be drawn meeting a third tangent plane $(\Theta_{,,})$ in the straight lines $(\vartheta_{,})$ and $(\vartheta_{,,})$, the planes through these two lines and the umbilical focus υ will cut the umbilical directrix plane (Δ) in two straight lines, through which and the polar focus ω if two planes be drawn, they will be at right angles to each other 242

264.] If two tangent planes (Θ) and $(\Theta_{,})$ are drawn to a surface of the second order (Σ), having three unequal axes, meeting in the straight line (ϑ), and if τ and $\tau_{,}$ be the points of contact of these tangent planes, let the chord $\tau\,\tau_{,}$ meet the umbilical directrix plane (Δ) in ν; and let the plane drawn through ν, the umbilical focus υ, and (ϑ) the intersection of the tangent planes (Θ) and $(\Theta_{,})$ meet the umbilical directrix plane (Δ) in the straight line (δ); the line $\omega\nu$ will be at right angles to the plane drawn through ω the polar focus and (δ) 242

265.] Let two tangent planes (Θ) and $(\Theta_{,})$ be drawn to an oblate spheroid, and their chord of contact be produced to meet the directrix plane in ν; and if a plane be drawn through the umbilical focus υ and the line (ϑ), in which these two tangent planes intersect, and meeting the directrix plane in the line (δ), the diametral plane C(δ) will be at right angles to the diameter Cν 243

266.] If a tangent plane (Θ) be drawn to (Σ), touching it in the point τ, and cutting the umbilical directrix plane in the line (ϑ), and if through the umbilical focus υ and the point τ a straight line be drawn meeting the directrix (Δ) in the point δ, the line $\omega\delta$ drawn from the polar focus ω to δ will be at right angles to the plane drawn from ω through the straight line (ϑ) 243

If a tangent plane be drawn to an oblate spheroid cutting one of the directrix planes (Δ) in a straight line (ϑ), the diametral plane drawn through (ϑ) will be at right angles to the diameter $O\tau$ drawn through the point of contact τ 243

267.] Two tangent planes, (Θ) and $(\Theta_{,})$, are drawn to a surface (Σ) having three unequal axes, cutting the umbilical planes in two straight lines (δ) and $(\delta_{,})$. A straight line drawn through the points of contact meets the directrix plane (Δ) in λ. The line drawn from the polar focus ω to λ is equally inclined to the planes $\omega(\delta)$ and $\omega(\delta_{,})$ 244

Two tangent planes are drawn to an oblate spheroid, cutting the directrix plane (Δ) in two straight lines (δ) and $(\delta_{,})$, while the chord through the points of contact meets it in λ, the diametral planes $O(\delta)$ and $O(\delta_{,})$ are equally inclined to the diameter $O\lambda$ 244

268.] Two tangent planes, (Θ) and $(\Theta_{,})$, are drawn to a surface (Σ) meeting the directrix plane (Δ) in two straight lines (δ) and $(\delta_{,})$; and if through the intersection of these tangent planes (ϑ) and the umbilical focus υ a plane (U) be drawn cutting the directrix plane (Δ) in a straight line (χ), the planes $\omega(\delta)$ and $\omega(\delta_{,})$ will be equally inclined to the plane $\omega(\chi)$ 244

269.] If two tangent planes, (Θ) and $(\Theta_{,})$, to a surface of the second order be cut by another plane (Π) passing through the points of contact $\tau\tau_{,}$, and cutting the tangent planes in two straight lines $\tau\pi$ and $\tau_{,}\pi$, if the three sides of the triangle $\tau\tau_{,}$, $\tau\pi$, and $\tau_{,}\pi$ be produced to meet the directrix plane in the points λ, δ, and $\delta_{,}$, the angles $\lambda\omega\delta$ and $\lambda\omega\delta_{,}$ will be equal 245

270.] Two tangent planes, (Θ) and $(\Theta_{,})$, are drawn to a surface having three unequal axes. A third plane (Π) is drawn through the chord of contact cutting the tangent planes in the lines (ϖ) and $(\varpi_{,})$. These lines and the line (ϑ), in which the tangent planes intersect, are produced to meet the directrix plane (Δ) in the three points π, $\pi_{,}$, and δ. The angle $\pi\omega\delta=\pi_{,}\omega\delta$ 245

271.] Two tangent planes, and a secant plane through the chord of contact, are drawn to an oblate spheroid; the diameter drawn through the point in which the common intersection of the two tangent planes meets the directrix plane (Δ) is equally inclined to the diameters which pass through the points in which the common intersections of the secant plane with the tangent planes meet the directrix plane 246

If through a surface (Σ) with three unequal axes a chord be drawn meeting this surface in the points τ and $\tau_{,}$ and the umbilical directrix plane (Δ), and if through ω the polar focus planes be drawn at right angles to $\omega\tau$ and $\omega\tau_{,}$ cutting the second directrix plane $(\Delta_{,})$ in (ϖ) and $(\varpi_{,})$, and planes be drawn through $\tau(\varpi)$ and $\tau_{,}(\varpi_{,})$ intersecting in the straight line (λ), the plane $\omega(\lambda)$ will be at right angles to the straight line $\omega\nu$ 246

272.] Let there be a triangle whose vertex is at the umbilical focus, and whose base is a chord of the surface (Σ). If the three sides be produced to meet the umbilical directrix plane (Δ) in three points ν, δ, and $\delta_{,}$, the line drawn from the polar focus ω to ν will be equally inclined to the focal lines $\omega\delta$ and $\omega\delta_{,}$ 247

273.] Any pair of conjugate tangents to the surface (Σ) being produced to meet the umbilical directrix plane (Δ) in two points α and β, the lines drawn from the polar focus ω to these points will be at right angles 248

274.] Conjugate tangents are parallel to a pair of conjugate diameters of the central plane of the surface (Σ) parallel to the tangent plane containing the conjugate tangents 248

276.] If a cone, whose vertex κ is on the umbilical directrix plane (Δ), be circumscribed to a surface (Σ), the plane of contact (G) of this cone with (Σ) will pass through υ, the umbilical focus, and will cut (Δ) in a straight line (γ). The line drawn from the polar focus ω to the vertex κ of the cone will be at right angles to the plane drawn from ω through the line (γ) in which (G), the plane of contact of the cone circumscribing (Σ), meets the umbilical directrix plane (Δ) 251

277.] If a tangent plane (Θ) be drawn to (Σ) touching it in the point τ, and cutting the conjugate umbilical directrix planes (Δ) and ($\Delta_,$) in the lines (δ) and ($\delta_,$), the planes drawn through these straight lines and the polar focus ω will be equally inclined to the line $\omega\tau$, drawn from the polar focus ω to the point of contact τ 252

When the surface becomes an oblate spheroid it will hence follow that, If a tangent plane be drawn to an oblate spheroid cutting the parallel directrix planes (Δ) and ($\Delta_,$) in the straight lines (δ) and ($\delta_,$), the diameter through the point of contact will be equally inclined to the diametral planes O(δ) and O($\delta_,$) 252

278.] If a secant plane be drawn cutting the surface (Σ) in a plane section (K), and the conjugate umbilical directrix planes (Δ) and ($\Delta_,$) in the straight lines (δ) and ($\delta_,$), the planes drawn through the polar focus ω and these straight lines (δ) and ($\delta_,$) will be parallel to the circular sections of the cone whose vertex is at ω and whose base is (K) 252

The circular sections of a cone whose vertex is at the centre of an oblate spheroid, and whose base is any plane section of this surface, are parallel to the two diametral planes which pass through the straight lines in which this base intersects the minor direcctrix planes (Δ) and ($\Delta_,$) of the oblate spheroid (Σ) 253

The cone whose vertex is at the polar focus ω of a surface (Σ) having three unequal axes, and whose base is a plane section of this surface passing through the polar directrix, is a right cone whose circular section is parallel to the plane of XY 253

279.] Algebraical proof of this theorem 253

280.] If through any straight line on the umbilical directrix plane (Δ) two tangent planes be drawn to (Σ), and if through the three perpendiculars let fall from ω on these three planes a secant plane (Π) be drawn cutting the tangent planes in lines which produced meet the plane drawn through ω parallel to the umbilical directrix plane (Δ) in the points τ and $\tau_{,}$, and if $\omega\tau=h$, $\omega\tau_,=h_,$ and $\omega\kappa=r$ in the triangle $\tau\kappa\tau_{,}$, the base $\tau\omega\tau_,$ will have to the line $\omega\kappa$ a constant ratio, or

$$\frac{h+h_,}{r}=\frac{2b^2\epsilon}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (f)$$ 255

When the surface (Σ) becomes a surface of revolution (S) round the transverse axis, $c=b$ and $\epsilon=e$, hence

$$\frac{h+h_,}{r}=2e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (g)$$ 255

Hence also this other theorem in the conic sections.

If from any point in the directrix a pair of tangents be drawn to the curve, they will cut off equal segments from the ordinate passing through the focus 255

281.] The reciprocal polar of the theorem, that in a surface of revolution the sum of the focal vectors is constant, is as follows.

To a surface (Σ), having three unequal axes, let a tangent plane (Θ) be drawn cutting the umbilical planes (Δ) and $(\Delta_{,})$ in the straight lines (δ) and $(\delta_{,})$, and the line joining the polar focus and the polar directrix into the segments h and $h_{,}$. Let the planes that are drawn from ω through the straight lines (δ) and $(\delta_{,})$ make the angles $\phi, \phi_{,}$ with the vertical ordinate passing through the polar focus ω, the resulting expression becomes

$$\frac{h}{h_{,}}(\sec\phi+\sec\phi_{,})=\frac{2b}{ce}$$ 257

CHAPTER XXVII.

282.] Some properties of surfaces of revolution of the second order, established by M. Chasles, extended to surfaces having three unequal axes 257

CHAPTER XXVIII.

ON THE RECIPROCAL POLARS OF CONFOCAL SURFACES.

284.] If there be a series of *confocal* primitive surfaces (S), $(S_{,})$, $(S_{,,})$, &c., their reciprocal surfaces (Σ), $(\Sigma_{,})$, $(\Sigma_{,,})$, &c. will be *concyclic*. They will have a common polar focus; they will have the same umbilical directrix planes; and the graphical properties of one set of surfaces may easily be transformed into their reciprocals on the polarized surfaces 265

Examples 265

CHAPTER XXIX.

ON METRICAL METHODS AS APPLIED TO THE THEORY OF RECIPROCAL POLARS.

286.] If a fixed point O be taken, and a number of fixed straight lines BT, $B_{,}T_{,,}$ &c. be drawn in the same plane, and another current point S be assumed from which and from the point O perpendiculars are let fall in pairs on these fixed lines, if the reciprocal of the vector OS, multiplied by the sum of the ratios of each pair of perpendiculars on the fixed lines, be constant, the point S will describe a conic section having its focus at O 266

287.] Geometrical proof of the theorem that, if a plane be drawn through a focus of a surface of revolution, this point will also be a focus of the section 268

289.] The common directrices and foci of the *equilateral hyperbola* possess two distinct classes of properties—those which belong to them as being the common or ordinary directrices and foci, as also that other new and equally extensive class to which they are in like manner related as being the minor directrices and foci of a central conic section 270

290.] The properties of the minor directrices and their corresponding foci derived by the help of the method of reciprocal polars, from the known properties of the common focus and directrix 270

291.] If a tangent be drawn to a conic section meeting the minor directrices in two points M and $M_{,,}$ the sum of the distances of these points from

the minor axis is to the sum of their distances from the centre in a constant ratio e, the eccentricity of the section. In the hyperbola the differences are to be taken 272
292.] The product of the focal perpendiculars let fall from the minor foci F and $F_{,}$ on a tangent to the curve is to the square of the perpendicular from the centre on the same tangent as the square of the semidiameter $a_{,}$ passing through the point of contact Q is to the square of the semi- major axis a........ 272
294.] The ratio of the focal perpendiculars on the tangent is the same as that of the distances from the centre of the points in which the tangent cuts the minor directrices 273
295.] If a tangent be drawn to a central conic section meeting the curve in the point Q, and the minor directrices in the points M and $M_{,,}$ the rectangle under the distances of these points from the centre will be to the rectangle under the segments of this tangent as the square of the perpendicular from the centre upon it is to the square of the semi- minor axis 273
296.] The ratios of the focal perpendiculars on a tangent, of the distances to the centre O of the points M and $M_{,}$ in which this tangent meets the minor directrices, of the segments into which this tangent is divided between the point of contact Q and the minor directrices, as also of the perpendiculars from this point Q upon them, are the same for all 273
297.] If from any point M in the minor directrix of a conic section we draw two tangents to it meeting the transverse axis in the points X and $X_{,,}$ the distance between these points will be to the distance of the point M from the centre as the distance between the foci is to the semiparameter 274
298.] The distances from the centre of the points in which a tangent to the curve meets the minor directrices may be expressed in rational functions of the projective coordinates of the point of contact 274
300.] If a chord be drawn through a fixed point δ in the plane of a conic section meeting the curve in the points τ and $\tau_{,,}$ the product of the tangents of half the angles $\delta F\tau$ and $\delta F\tau_{,}$ will be constant, F being a focus 276
301.] If a straight line PQ be drawn in the plane of a conic section, and from any point P in it two tangents be drawn to the curve meeting the minor directrix in two points T and $T_{,,}$ while the given straight line meets it in Q, the product of the tangents of half the angles TOQ and $T_{,}OQ$ will be constant 276
303.] If two diameters are drawn at right angles in a conic section, one meeting the curve in Q, the other meeting the minor directrix in G, the line QG envelops the circle whose diameter is the minor axis 278
306.] If a secant be drawn to a central conic section cutting the curve in the points P and $P_{,,}$ and the minor directrices in the points Q and $Q_{,,}$ and if through the points P and $P_{,}$ tangents be drawn meeting the minor directrices in the points $MM_{,}$ and $NN_{,}$, the sum of the angles which the straight lines $MM_{,}$ and $NN_{,}$ subtend at the centre will be equal to twice the angle which $QQ_{,}$ subtends at the same centre.. 280
307.] If through any two points P and $P_{,}$ on a central conic section a secant be drawn, and two tangents through the points P and $P_{,}$ meeting the minor directrices in the four points M, $M_{,,}$ N, and $N_{,}$, and perpendiculars be let fall from these four points on the secant, namely MP, $M_{,}P_{,,}$ $NP_{,,,}$ $N_{,}P_{,,,,}$ the ratios
$$\frac{MP}{MO}, \frac{M_{,}P_{,}}{M_{,}O}, \frac{NP_{,,}}{NO}, \frac{N_{,}P_{,,,}}{N_{,}O},$$
will all be equal 282
308.] Algebraical proof of this theorem 282

309.] If two fixed points A and B be assumed on a conic section, and a third point C variable in position, and if the chords CA, CB be produced to meet the minor directrices (Δ) and $(\Delta_{,})$ in the points M, $M_{,}$ and N, $N_{,}$, the line $MM_{,}$ will subtend a constant angle at the centre O; so also will the line $NN_{,}$ on the other directrix $(\Delta_{,})$; and the sum of these angles will be constant and equal to the angle AOB 285

310.] If a triangle be inscribed in a conic section whose base is the minor axis, and whose vertex is variable along the curve, the sides being produced will meet the minor directrix in two points $MM_{,}$, which will subtend a right angle at the centre 285

311.] Let the sides of a quadrilateral inscribed in a conic section meet one of the minor directrices in the points A_0, B_0, C_0, D_0; the sum of the angles which the segments A_0B_0 and C_0D_0 subtend at the centre will be equal to two right angles 286

313.] Simple method of finding the curvature of a conic section 287

314.] *On groups of conic sections having the same minor directrices.*
Let a series of concentric conic sections, all having the same minor directrices, be cut by a transversal, the portions of this line intercepted by any pair of these curves will subtend equal angles at the centre; and if, through every pair of points in which this transversal intersects the sections, tangents are drawn intercepted both ways by the directrices, the sum of the angles which any pair of these tangents subtend at the centre is constant, being equal to twice the angle which the common transversal intercepted both ways by the directrices subtends at the centre 288

315.] Two diameters at right angles revolve round the centre of two conic sections having the same minor directrices, each diameter meeting one of the curves; the line joining these points will envelop a circle 289

316.] A series of concentric conic sections, having the same minor directrices, is cut by a common diameter; the tangents drawn through the points where this diameter intersects the curves envelops a concentric conic section.. 290

317.] On certain properties of the equilateral hyperbola 291

CHAPTER XXX.

ON THE LOGOCYCLIC CURVE, THE TRIGONOMETRY OF THE PARABOLA, AND THE GEOMETRICAL ORIGIN OF LOGARITHMS.

319.] On the equations of the logocyclic curve 295

320.] to 340.] On the properties, quadrature, and rectification of the logocyclic curve 295

341.] On certain properties of inverse curves.......................... 309

342.] On the relation between the central forces by which any curve and its inverse may be described.................................... 311

CHAPTER XXXI.

ON THE TRIGONOMETRY OF THE PARABOLA.

343.] Definition of *conjugate amplitudes* 313

344.] Table of analogies between the formulæ of circular and parabolic trigonometry 315

345.] Expression for the length of an arc of a parabola 318

346.] Relations between the conjugate arcs of a parabola 319
348.] Relation of the ordinates of three conjugate parabolic arcs 322
349.] Geometrical representation of the functions $\sec(\phi \perp \chi)$ and $\tan(\phi \perp \chi)$ 323
350.] Relations between multiple parabolic arcs 323
353.] Solution of cubic equation by parabolic trigonometry 328

CHAPTER XXXII.

ON THE GEOMETRICAL ORIGIN OF LOGARITHMS.

354.] Value of e the Napierian base in terms of functions of the angle which a focal perpendicular on a tangent makes with the axis, or $\sec\theta + \tan\theta = e$ 328
356.] To represent numbers and their logarithms by the logocyclic curve and its conjugate parabola 331
362.] To represent the decimal or any other system of logarithms by a corresponding parabola 337
366.] The quadrature of the hyperbola depends on the rectification of the parabola... 343
367.] On the relations between focal perpendiculars on the n sides of a polygon circumscribing a parabola.............................. 345
370.] On some analogies between circular and parabolic trigonometry 347
371.] On the roots of $z^{2n} - 2az^n + 1 = 0$ when $a > 1$ 348
372.] On parabolic trigonometry as applied to the investigation of the catenary and the tractrix 349
373.] On some properties of the catenary 350
376.] On the tractrix... 352

CHAPTER XXXIII.

ON SOME PROPERTIES OF CONFOCAL SURFACES.

379.] Three confocal surfaces of the second order intersect in a common point Q, the vertex of a cone which envelops a fourth confocal surface; to determine the projective equation of this cone referred to the normals of the three surfaces, at the common point Q, as axes of coordinates 356
382.] Two cones having their common vertex on a surface of the second order, an ellipsoid suppose, $(a_,b_,c_,)$ envelop two confocal surfaces. The diametral plane of the surface conjugate to the diameter passing through the common vertex of the two cones will cut off from their common side a constant length, independent of the position of the common vertex of the two cones on the surface $(a_,b_,c_,)$ 363
383.] A cone whose vertex is on a surface of the second order envelops a confocal surface. To determine the length of the axis of the cone between the vertex and the plane of contact 364
385.] Along a line of curvature tangent planes are drawn to a surface of the second order. The perpendiculars from the centre on these planes generate a cone of the second order, whose focal lines coincide with the optic axes of the surface or with the perpendiculars to its circular sections.................. 368

TANGENTIAL COORDINATES.

CHAPTER I.

ON THE TANGENTIAL EQUATIONS OF A POINT AND A STRAIGHT LINE IN A PLANE.

THE projective equation of a straight line in a plane, referred to two intersecting lines in the same plane, is $\frac{x}{a}+\frac{y}{b}=1$. In this equation a and b denote the intercepts of the axes cut off by the straight line, while x and y are the variable current coordinates of any point moving along this line.

Now if we fix this point, thus making x and y constant, and suppose a and b to vary instead, we shall have the means of defining the position of this point by the help of a second equation, $\frac{x}{a_{\prime}}+\frac{y}{b_{\prime}}=1$, where a is changed into $a_{\prime}$ and b into $b_{\prime}$.

As a and b are henceforward to be assumed as variables, we must adopt some appropriate notation to designate their variable character. It will improve the symmetry of the notation if we put $a=\frac{1}{\xi}$, $b=\frac{1}{v}$; and thus the preceding equation becomes

$$x\xi+yv=1\;; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and this is the tangential equation of a point.

As this expression is of constant occurrence, being the link which unites the tangential and projective systems of coordinates, it may with propriety be called the *dual equation*.

The position of a straight line will evidently be determined, if we make $\xi=$constant, $v=$constant.

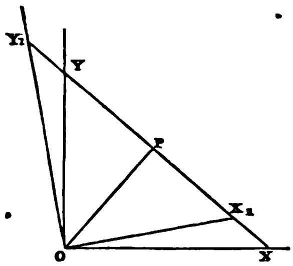
It must be borne in mind that, as ξ, v, ζ are the reciprocals of straight lines, such quantities as $a\xi$, yv, or $P\zeta$ are abstract numbers; a, y, and P being straight lines.

As x and y are the projective coordinates of a current point in a plane, so ξ and v are the tangential coordinates of a line which may be termed the *limiting tangent*.

ON THE TRANSFORMATION OF COORDINATES.

2.] Assuming the ordinary rectangular axes of coordinates, we shall imagine them first turned round through an angle θ, retaining the same origin, and afterwards suppose them displaced in parallel directions to another origin.

Fig. 1.



Let $\frac{1}{\xi}$ and $\frac{1}{v}$ denote the intercepts of two rectangular axes, OX, OY, by a fixed straight line; $\frac{1}{\xi_{,}}$ and $\frac{1}{v_{,}}$ the intercepts of the axes $OX_{,}$, $OY_{,}$ made by the same straight line, and let P, the perpendicular from the origin on the straight line, make the angles λ and $\lambda_{,}$ with the axes OX and $OX_{,}$, and let θ be the angle $XOX_{,}$.

Hence $\lambda=\lambda_{,}+\theta$,

and $$\cos\lambda=\cos\lambda_{,}\cos\theta-\sin\lambda_{,}\sin\theta;$$

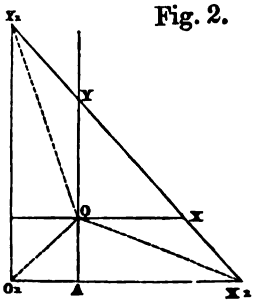
but $P\xi=\cos\lambda$, $P\xi_{,}=\cos\lambda_{,}$, $Pv=\sin\lambda$, $Pv_{,}=\sin\lambda_{,}$; substituting and dividing by P, we find $\xi=\cos\theta.\xi_{,}-\sin\theta.v_{,}$; and a like expression may be found for v. Hence, when the axes of coordinates are turned round through the angle θ, we may pass from the old system to the new, from ξ, v to $\xi_{,}, v_{,}$, by the equations

$$\left.\begin{aligned}\xi&=\cos\theta.\xi_{,}-\sin\theta.v_{,},\\ v&=\sin\theta.\xi_{,}+\cos\theta.v_{,}.\end{aligned}\right\}\quad\dots\dots\quad(2)$$

ON THE TRANSLATION OF COORDINATES.

3.] Let the axes $O_{,}X_{,}$ and $O_{,}Y_{,}$ be drawn through the point $O_{,}$ parallel to OX and OY. Let $O_{,}X_{,}=\frac{1}{\xi_{,}}$, $O_{,}Y_{,}=\frac{1}{v_{,}}$, $OX=\frac{1}{\xi}$, $OY=\frac{1}{v}$. Let p and q be the projective coordinates of the point O on the new axes $O_{,}Y_{,}$ and $O_{,}X_{,}$; join O and $X_{,}$, O and $Y_{,}$, also O and $O_{,}$.

Fig. 2.



Now the whole triangle $O_{,}X_{,}Y_{,}$ is the sum of the three component triangles $O_{,}OX_{,}$, $O_{,}OY_{,}$, and $OY_{,}X_{,}$; or, substituting their equivalent expressions, $\frac{1}{\xi_{,}v_{,}}=\frac{q}{\xi_{,}}+\frac{p}{v_{,}}+\frac{1}{\xi v_{,}}$; or multiplying by $\xi_{,}v_{,}$, we obtain $\xi=\frac{\xi_{,}}{1-p\xi_{,}-qv_{,}}$.

In like manner $v=\frac{v_{,}}{1-p\xi_{,}-qv_{,}}$.

Hence the expressions for the old coordinates, in terms of the new, translated in parallel directions, are

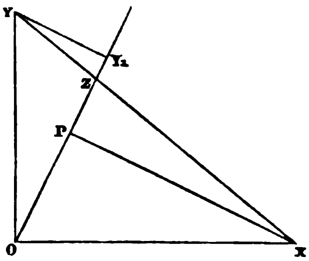
$$\xi=\frac{\xi_{,}}{1-p\xi_{,}-qv_{,}},\quad v=\frac{v_{,}}{1-p\xi_{,}-qv_{,}}; \quad . \; . \; . \; . \quad (3)$$

when the axes are first turned round through the angle θ and then translated in parallel directions,

$$\xi=\frac{\cos\theta.\xi_{,}-\sin\theta.v_{,}}{1-p\xi_{,}-qv_{,}},\quad v=\frac{\sin\theta.\xi_{,}+\cos\theta.v_{,}}{1-p\xi_{,}-qv}. \quad . \; . \quad (4)$$

4.] If we wish to change the inclination of the axes of coordinates from a right angle to an angle ω, we may easily effect this, the axis of X remaining the same, by substituting $\frac{v_{,}-\cos\omega.\xi}{\sin\omega}$ for v.

Fig. 3.



Let the new axis of $Y_{,}$ make the angle ω with the former axis of X.

Then half the area of the triangle $XOY=\frac{1}{\xi v}$; and half the area of the same triangle is $OZ=\frac{1}{v_{,}}$, multiplied by the sum of the perpendiculars let fall on it from X and Y, that is,

$$\frac{1}{\xi v}=\frac{1}{v_{,}}\left(\frac{\sin\omega}{\xi}+\frac{\cos\omega}{v}\right),\ \text{or } v=\left(\frac{v_{,}-\cos\omega.\xi}{\sin\omega}\right). \quad . \; . \quad (5)$$

5.] The perpendicular from the origin on the straight line whose tangential coordinates are ξ and v, is manifestly $\frac{1}{\sqrt{\xi^2+v^2}}$.

To determine the length of a perpendicular on the straight line whose tangential coordinates are ξ and v, from the point whose projective coordinates are p and q.

Let $O_{,}$ be the point whose projective coordinates are $O_{,}A=q$, $O_{,}B=p$; let $O_{,}Q=P$.

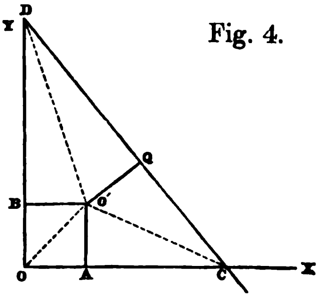


Fig. 4.

Then the area of the whole triangle $OCD = O_{,}OC + O_{,}OD + DO_{,}C$, or $\frac{1}{\xi v}=\frac{q}{\xi}+\frac{p}{v}+P\sqrt{\frac{1}{\xi^2}+\frac{1}{v^2}}$; or multiplying by ξv, and reducing,

$$P=\frac{1-p\xi-qv}{\sqrt{\xi^2+v^2}}. \quad . \; . \quad (6)$$

6.] *To find an expression for the value of the angle between two given straight lines whose tangential coordinates are given.*

Let θ be the angle between the two given straight lines whose tangential coordinates are ξ, υ and $\xi_{\prime}$, $\upsilon_{\prime}$.

Let these lines make the angles ϕ and $\phi_{\prime}$ with the axis of X; then

$$\tan\phi=\frac{\xi}{\upsilon},\quad \tan\phi_{\prime}=\frac{\xi_{\prime}}{\upsilon_{\prime}},\quad \text{and } \theta=\phi-\phi_{\prime};$$

hence
$$\tan\theta=\frac{\tan\phi-\tan\phi_{\prime}}{1+\tan\phi\tan\phi_{\prime}}=\frac{\xi\upsilon_{\prime}-\xi_{\prime}\upsilon}{\xi\xi_{\prime}+\upsilon\upsilon_{\prime}}; \quad . \quad . \quad . \quad . \quad (7)$$

$$\left.\begin{array}{ll}\text{when the lines are parallel,} & \xi\upsilon_{\prime}-\xi_{\prime}\upsilon=0,\\ \text{when the lines are at right angles,} & \xi\xi_{\prime}+\upsilon\upsilon_{\prime}=0.\end{array}\right\} \quad . \quad . \quad . \quad . \quad (8)$$

7.] *To find the tangential coordinates of a right line passing through two fixed points whose projective coordinates* $(\alpha\beta)$ *and* $(\alpha_{\prime}\beta_{\prime})$ *are given.*

Let the tangential equation of one point be $\alpha\xi+\beta\upsilon=1$, and that of the other point be $\alpha_{\prime}\xi+\beta_{\prime}\upsilon=1$.

Eliminating from these equations ξ and υ successively, we get

$$\xi=\frac{\beta_{\prime}-\beta}{\alpha\beta_{\prime}-\alpha_{\prime}\beta},\quad \upsilon=\frac{\alpha_{\prime}-\alpha}{\alpha_{\prime}\beta-\alpha\beta_{\prime}}. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The line will pass through the origin when $\dfrac{\alpha}{\alpha_{\prime}}=\dfrac{\beta}{\beta_{\prime}}$.

8.] Let us resume the tangential equation of the point (x, y), namely $x\xi+y\upsilon=1$. Multiply it by P, the perpendicular from the origin; bearing in mind that $P\xi=\cos\lambda$, $P\upsilon=\sin\lambda$, we obtain

$$\cos\lambda \,.\, x+\sin\lambda \,.\, y-P=0,$$

which is a common form for the equation of a straight line in projective coordinates.

Now if on this line we let fall a perpendicular from the point $(x_{\prime}y_{\prime})$, the length of this perpendicular will be

$$\cos\lambda \,.\, x_{\prime}+\sin\lambda \,.\, y_{\prime}-P.$$

Thus the *form* of the equation of the straight line in (x, y) becomes the *length* of the perpendicular let fall upon it from the given point $(x_{\prime}y_{\prime})$ when we substitute $x_{\prime}$ and $y_{\prime}$ for x and y.

Let $F(x,y)=0$, $F_{\prime}(x,y)=0$ be the projective equations of two straight lines meeting in a point, then the equation of any other straight line passing through the same point will be $F(x,y) \pm i\,F_{\prime}(x,y)=0$, i being any constant whatever. It is clear that the values of x and y which at the same time satisfy the equations $F(x,y)=0$, $F_{\prime}(x,y)=0$ will also satisfy the equation of any other straight line at the point of intersection, and therefore will satisfy the equation $F(x, y) \pm i\,F_{\prime}(x, y)=0$.

To find the value of i in the preceding equation,—

If in $F(x,y)$ we write $x_{\prime}, y_{\prime}$ instead of x, y, then $F(x_{\prime}, y_{\prime})$ becomes the perpendicular on $F(x,y)=0$, and $F_{\prime}(x_{\prime}, y_{\prime})$ becomes the perpendicular on $F_{\prime}(x,y)=0$; hence i denotes the ratio of the perpendiculars let fall from *any point* on the two lines $F(x,y)=0$, $F_{\prime}(x,y)=0$, and will denote the anharmonic ratio of the four straight lines $F(x,y)=0$, $F_{\prime}(x,y)=0$, $F(x,y) \pm i\,F_{\prime}(x,y)=0$.

When $i=-1$, the anharmonic ratio becomes the harmonic ratio.

9.] Now let $1-x\xi-yv=0=\mathrm{V}(\xi,v)$ be the tangential equation of the point (xy), and $1-x_{\prime}\xi-y_{\prime}v=0=\mathrm{V}_{\prime}(\xi,v)$ be the tangential equation of the point $(x_{\prime}y_{\prime})$; hence it follows that $\mathrm{V}(\xi,v)\pm j\,\mathrm{V}_{\prime}(\xi,v)=0$ will be the equation of another point on the line passing through the points (xy) and $(x_{\prime}y_{\prime})$.

On the given straight line whose tangential coordinates are $\xi_{\prime}$ and $v_{\prime}$ let fall the perpendicular z from the point (xy); the length of this perpendicular z will be $\mathrm{P}\,(1-x\xi_{\prime}-yv_{\prime})$, where P is the perpendicular from the origin on the given line $\xi_{\prime}$ and $v_{\prime}$: see (6). Hence it follows that if in the tangential equation of the point (xy), namely, $\mathrm{V}(\xi,v)=0$, we substitute $\xi_{\prime}$ and $v_{\prime}$ for ξ and v, and then multiply the expression by the perpendicular from the origin on the line $\xi_{\prime}v_{\prime}$, we shall obtain the length of the perpendicular upon it, so that if $\mathrm{V}(\xi,v)=0$ be the tangential *equation* of the point (xy), $\mathrm{P}.\mathrm{V}(\xi_{\prime}v_{\prime})$ will be the *length* of the perpendicular letfall on the line $(\xi_{\prime}v_{\prime})$ from the point (xy).

In the same way, if the perpendicular let fall from the point $(x_{\prime}y_{\prime})$ on the straight line whose tangential coordinates are $\xi_{\prime}$ and $v_{\prime}$ be $z_{\prime}$, then $z_{\prime}=\mathrm{P}\,(1-x_{\prime}\xi_{\prime}-y_{\prime}v_{\prime})$; hence

$$\frac{z}{z_{\prime}}=\frac{1-x\xi_{\prime}-yv_{\prime}}{1-x_{\prime}\xi_{\prime}-y_{\prime}v_{\prime}}.$$

If, now, in the tangential equation of a third point which lies on the straight line passing through the two points (xy) and $(x_{\prime}y_{\prime})$, namely

$$(1-x\xi-yv)\pm j\,(1-x_{\prime}\xi-y_{\prime}v)=0,$$

we substitute $\xi_{\prime}$ and $v_{\prime}$ for ξ and v, and multiply by P, we get, $z_{\prime\prime}$ being the length of this perpendicular, $z_{\prime\prime}=z\pm jz_{\prime}$.

It is obvious that the perpendicular from the *third* point is $z+jz_{\prime}$, and from a *fourth* point in the same straight line is $z-jz_{\prime}$.

When the perpendiculars z and $z_{\prime}$ are equal, it may easily be shown that the line whose tangential coordinates are $\xi_{\prime}$ and $v_{\prime}$ is parallel to that whose coordinates are ξ and v; for in this case

$$x\xi_{\prime}+yv_{\prime}=x_{\prime}\xi_{\prime}+y_{\prime}v_{\prime},$$

or

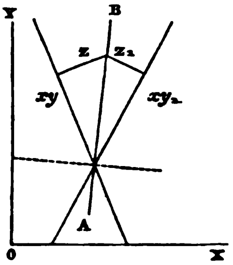
$$\frac{y-y_{\prime}}{x-x_{\prime}}=-\frac{\xi_{\prime}}{v_{\prime}};$$

hence the line which passes through the points (xy) and $(x_{\prime}y_{\prime})$ is parallel to that whose coordinates are $\xi_{\prime}$ and $v_{\prime}$.

When this third perpendicular $z_{\prime\prime}$ is 0, or when the third point is on the intersection of the lines whose tangential coordinates are ξ and v, $\xi_{\prime}$ and $v_{\prime}$, we get $0=z\pm jz_{\prime}$, or $j=\frac{z}{z_{\prime}}$, or j is the ratio of the two perpendiculars let fall from the points (xy) and $(x_{\prime}y_{\prime})$ on the straight line whose tangential coordinates are $\xi_{\prime}$ and $v_{\prime}$.

10.] We shall more clearly exhibit the duality of the relations between projective and tangential coordinates if we write down the foregoing propositions, side by side, as follows:—

Fig. 5.



Projective Coordinates.

Let $F(x,y)=0$, $F_{,}(x,y)=0$ be the *projective* equations of *two straight lines.*

Let a *third straight line* AB be assumed passing through their *intersection*, and let the *projective* equation of this line be

$$F(x,y) \pm i F_{,}(x,y)=0;$$

from a *point* in this straight line, whose coordinates are $x_{,}$ and $y_{,}$, let two perpendiculars be let fall on the given straight lines; the ratio of these perpendiculars will be i, and the four lines $F(x,y)=0$, $F_{,}(x,y)=0$, $F(x,y) \pm i F_{,}(x,y)=0$ will form an anharmonic pencil.

Let $F(x_{,}y_{,})=z$,

and $F_{,}(x_{,}y_{,})=z_{,}$,

and $F(x_{,}y_{,}) \pm i F_{,}(x_{,}y_{,})=0$.

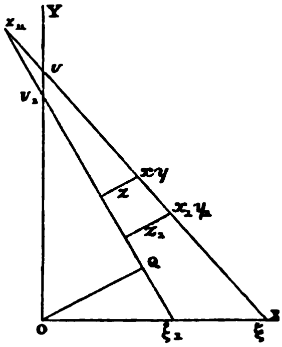
Seeing that it is the expression for the perpendicular from the point $x_{,}y_{,}$, which is *on* the line,

$$F(xy)+iF_{,}(xy)=0.$$

Hence $z+iz_{,}=0$,

or $\frac{z}{z_{,}}=-i$.

Fig. 6.



Tangential Coordinates.

Let $V(\xi, v)=0$, $V_{,}(\xi, v)=0$ be the *tangential* equations of *two points.*

Let a *third point* be assumed on the line passing through them whose *tangential* equation shall be

$$V(\xi, v)+jV_{,}(\xi, v)=0;$$

this point will be on the straight line passing through the given points whose equations are $V(\xi, v)=0$, $V_{,}(\xi, v)=0$; from these two points let perpendiculars be let fall to a line whose tangential coordinates are $\xi_{,}$ and $v_{,}$, then $P.V(\xi_{,}v_{,})$ and $P.V_{,}(\xi_{,}v_{,})$ will be the length of these perpendiculars. Let them be put z and $z_{,}$. Let the third point be assumed as not only on the line passing through the two given points, but also on the line whose tangential coordinates are $\xi_{,}$ and $v_{,}$. Hence the tangential equation of this point becomes $V(\xi v) \pm jV_{,}(\xi v)=0$; but as this point is on the line whose coordinates are $\xi_{,}$ and $v_{,}$, the perpendicular from it on this line must be 0. Hence

$$V(\xi_{,}v_{,}) \pm jV_{,}(\xi_{,}v)=0;$$

but it has been shown that

$$V(\xi_{,}v_{,})=\frac{z}{P},$$

$$V_{,}(\xi_{,}v_{,})=\frac{z_{,}}{P};$$

hence $z+jz_{,}=0$,

or $\frac{z}{z_{,}}=-j$.

As the principles involved in the preceding theory may appear somewhat obscure to beginners, especially how the *form* of an equation between two variables, when equated to 0, may by a substitution of like quantities become a line of given *length*, it would seem better to err rather in fulness of explanation than to assume as easy that which to some minds may at first sight be difficult of comprehension.

CHAPTER II.

ON THE TANGENTIAL EQUATIONS OF THE CONIC SECTIONS.

11.] We shall include the tangential equations of the circle in those of the central conic sections, as nothing is gained in facility of investigation by taking them separately; and we shall commence with the simplest forms of the equations of those curves, taking the central sections apart from the parabola, as their tangential equations are *essentially* distinct. We shall assume the projective equation of the ellipse, referred to its centre and axes, as the basis of investigation, and proceed thence to the more general forms of the equations of these curves. We shall commence with the equation of the ellipse, as we can always pass from it to that of the hyperbola, by changing b into $\sqrt{-1}\,b$.

12.] The projective equation of a tangent to an ellipse passing through the point $(x_{\prime}y_{\prime})$ on the ellipse, whose equation is $\frac{x_{\prime}^2}{a^2}+\frac{y_{\prime}^2}{b^2}=1$, may be written $\frac{x_{\prime}x}{a^2}+\frac{y_{\prime}y}{b^2}=1$.

In this equation x and y are the current coordinates, and the *limiting tangent* meets the axis of x at the point where $y=0$; at this point $x=\frac{a^2}{x_{\prime}}$; let this distance be $\frac{1}{\xi}$, hence $x_{\prime}=a^2\xi$. In like manner $y_{\prime}=b^2v$;

$$\text{hence } a^2\xi^2+b^2v^2=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

is the tangential equation of the ellipse referred to its centre and axes.

Let the axes of coordinates now be conceived to revolve, through the angle θ, round the origin, and be then translated, parallel to themselves, to a point whose coordinates are $-p$ and $-q$.

The formulæ of transformation are, see (4),

$$\xi=\frac{\cos\theta\,.\,\xi_{\prime}-\sin\theta\,.\,v_{\prime}}{1-p\xi_{\prime}-qv_{\prime}},\quad v=\frac{\sin\theta\,.\,\xi_{\prime}+\cos\theta\,.\,v_{\prime}}{1-p\xi_{\prime}-qv_{\prime}} \quad . \quad . \quad . \qquad (11)$$

If we substitute these values of ξ and v in equation (10), omitting the traits as no longer necessary, we shall find

$$\left.\begin{array}{l}[a^2\cos^2\theta+b^2\sin^2\theta-p^2]\xi^2+[b^2\cos^2\theta+a^2\sin^2\theta-q^2]v^2\\ \quad+2[(a^2-b^2)\sin\theta\cos\theta-pq]\xi v+2p\xi+2qv=1.\end{array}\right\} \quad . \quad . \quad (12)$$

Hence, the tangential equation of a conic section being in its most general form

$$a\xi^2+a_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{,}v=1, \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and equating the coefficients of this equation, term by term, with those of the preceding one, we shall have

$$\left.\begin{array}{l}a^2\cos^2\theta+b^2\sin^2\theta-p^2=a\,;\ a^2\sin^2\theta+b^2\cos^2\theta-q^2=a_{,}\,;\\ \quad(a^2-b^2)\sin\theta\cos\theta-pq=\beta\,;\ p=\gamma\,;\ q=\gamma_{,}.\end{array}\right\} \quad . \quad (14)$$

In the first place, we may observe that the halves of the linear coefficients of the general equation represent the projective coordinates of the centre; for p and q, the projective coordinates of the centre, are equal to γ and $\gamma_{,}$, which are the halves of the coefficients of the linear terms in ξ and v.

Comparing the three remaining coefficients, and introducing the values of p and q, we shall have

$$a^2\cos^2\theta+b^2\sin^2\theta=a+\gamma^2,\ a^2\sin^2\theta+b^2\cos^2\theta=a_{,}+\gamma_{,}^2,$$

and $$(a^2-b^2)\sin\theta\cos\theta=\beta+\gamma\gamma_{,}\,; \quad . \quad . \quad . \quad . \quad (15)$$

$$\left.\begin{array}{ll}\text{hence} & (a^2-b^2)\cos 2\theta=(a+\gamma^2)-(a_{,}+\gamma_{,}^2),\\ \text{and} & (a^2-b^2)\sin 2\theta=2(\beta+\gamma\gamma_{,})\,;\\ \text{and also} & \tan 2\theta=\dfrac{2(\beta+\gamma\gamma_{,})}{(a+\gamma^2)-(a_{,}+\gamma_{,}^2)}.\end{array}\right\} \quad . \quad . \quad (16)$$

Since

$$\left.\begin{array}{l}a^2+b^2=(a+\gamma^2)+(a_{,}+\gamma_{,}^2),\\ (a^2+b^2)^2=(a+\gamma^2)^2+2(a+\gamma^2)(a_{,}+\gamma_{,}^2)+(a_{,}+\gamma_{,}^2)^2,\\ \text{and}\\ (a^2-b^2)^2=(a+\gamma^2)^2-2(a+\gamma^2)(a_{,}+\gamma_{,}^2)+(a_{,}+\gamma_{,}^2)^2\,4(\beta+\gamma\gamma_{,})^2:\end{array}\right\}(17)$$

subtracting, we obtain the result,

$$a^2b^2=(a+\gamma^2)(a_{,}+\gamma_{,}^2)-(\beta+\gamma\gamma_{,})^2. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Again, since $a^2+b^2=(a+\gamma^2)+(a_{,}+\gamma_{,}^2)$,

and $a^2-b^2=\sqrt{[(a+\gamma^2)-(a_{,}+\gamma_{,}^2)]^2+4(\beta+\gamma\gamma_{,})^2}$,

adding these equations together, we obtain the result,

$$2a^2=(a+\gamma^2)+(a_{,}+\gamma_{,}^2)\pm\sqrt{[(a+\gamma^2)-(a_{,}+\gamma_{,}^2)]^2+4(\beta+\gamma\gamma)^2}, \ (19)$$

the upper sign being taken for the major axis, the lower when we require the value of the minor axis.

13.] When the section is an hyperbola, b^2 must be negative, or in (18) we must have

$$(\beta+\gamma\gamma_{,})^2>(\alpha+\gamma^2)\,(\alpha_{,}+\gamma_{,}^2). \quad . \quad . \quad . \quad . \quad . \quad (20)$$

When the conic section becomes a circle, the two semiaxes in (19) become equal; hence the quantity under the radical must vanish; and as this quantity is the sum of two squares, we must have each square separately equal to 0, or

$$\alpha+\gamma^2=\alpha_{,}+\gamma_{,}^2, \text{ and } \beta+\gamma\gamma_{,}=0. \quad . \quad . \quad . \quad . \quad (21)$$

When these two relations hold, the conic section becomes a circle.

The corresponding property in projective coordinates shows that the origin of projective coordinates must be at a focus.

Hence the relations between the coefficients of the general tangential equation of the conic section which indicate that the curve is a circle, namely

$$\alpha+\gamma^2=\alpha_{,}+\gamma_{,}^2 \text{ and } \beta+\gamma\gamma_{,}=0,$$

when translated into the projective equations of a conic, namely

$$A+C^2=A_{,}+C_{,}^2 \text{ and } B+CC_{,}=0,$$

show that the origin must be at a focus.

14.] Let $c^2=a^2-b^2$, then from (19)

$$c^4=(a^2-b^2)^2=\left[(\alpha+\gamma^2)-(\alpha_{,}+\gamma_{,}^2)\right]^2+4(\beta+\gamma\gamma_{,})^2.$$

Let D be the distance of the origin from the centre, then

$$D^2=\gamma^2+\gamma_{,}^2.$$

Now let $\alpha=\alpha_{,}$ and $\beta=0$, then we shall have $c^4=(\gamma^2+\gamma_{,}^2)^2$; hence $c=D$, or the origin is at a focus of the curve; and as

$$\tan 2\theta=\frac{2\gamma\gamma_{,}}{\gamma^2-\gamma_{,}^2}, \text{ or } \tan\theta=\frac{\gamma_{,}}{\gamma},$$

thus the axis of the curve passes also through the new origin.

When the two conditions $\alpha+\gamma^2=\alpha_{,}+\gamma_{,}^2$ and $\beta+\gamma\gamma_{,}=0$ are satisfied, the curve is a circle; and the origin is at a focus when

$$\alpha=\alpha_{,}, \text{ and } \beta=0.$$

15.] The origin of coordinates is on the curve when

$$\alpha\alpha_{,}-\beta^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

When the origin of coordinates is on the curve, through this point there can be drawn only one tangent to it, and at this point $\xi=\infty$, $v=\infty$. Let $v=n\xi$; then the general tangential equation of the curve, $\alpha\xi^2+\alpha_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{,}v=1$, may be changed into $(\alpha+n^2\alpha_{,})\,\xi^2+2\beta n\xi^2+2(\gamma+n\gamma_{,})\,\xi=1$, or, dividing by ξ^2,

$$(\alpha+n^2\alpha_{,}+2\beta n)+2\left(\frac{\gamma+n\gamma_{,}}{\xi}\right)=\frac{1}{\xi^2},$$

or $$\alpha+n^2\alpha_{,}+2\beta n=0, \text{ since } \xi=\infty.$$

Solving this equation for n, $n = \frac{-\beta \pm \sqrt{\beta^2 - aa_{\prime}}}{a_{\prime}}$.

Now, in order that n may have only one value, we must have

$$\beta^2 = aa_{\prime}, \text{ or } n = \sqrt{\frac{a}{a_{\prime}}};$$

when $\beta^2 < aa_{\prime}$ the origin must be within the curve; when the origin is outside the curve, n must have two values, and therefore $\beta^2 > aa_{\prime}$.

16.] To find an expression for the angle which the asymptote to an hyperbola makes with the axis of X.

Let
$$P\xi^2 + Q\upsilon^2 + 2R\xi\upsilon = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$
be the tangential equation of an hyperbola, referred to its centre and any rectangular axes passing through the centre. As the asymptote may be defined as a tangent to the curve at an infinite distance, and which passes through the centre, at which point $\frac{1}{\xi} = 0$, $\frac{1}{\upsilon} = 0$, let τ be the tangent of the angle which the asymptote makes with the axis of X, then $\tau = \frac{\xi}{\upsilon}$ or $\xi = \tau\upsilon$; substituting we obtain

$$P\tau^2 + 2R\tau + Q = \frac{1}{\upsilon^2} = 0,$$

or, solving for τ,
$$\tau = \frac{-R \pm \sqrt{R^2 - PQ}}{P}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Now, the direction of the coordinates continuing unchanged, let them be translated to a new origin, such that the projective coordinates of the centre of the curve on the new axes shall be $-\gamma$ and $-\gamma_{\prime}$; then

$$\xi = \frac{\xi_{\prime}}{1 - \gamma\xi_{\prime} - \gamma_{\prime}\upsilon_{\prime}}, \qquad \upsilon = \frac{\upsilon_{\prime}}{1 - \gamma\xi_{\prime} - \gamma_{\prime}\upsilon_{\prime}}. \quad . \quad . \quad . \quad \text{(c)}$$

Substituting these values in (a) and reducing, we find

$$(P - \gamma^2)\xi^2 + (Q - \gamma_{\prime}^2)\upsilon^2 + 2(R - \gamma\gamma_{\prime})\xi\upsilon + 2\gamma\xi + 2\gamma_{\prime}\upsilon = 1.$$

Comparing this expression with the general equation of the curve,

$$a\xi^2 + a_{\prime}\upsilon^2 + 2\beta\xi\upsilon + 2\gamma\xi + 2\gamma_{\prime}\upsilon = 1,$$

we shall have $P - \gamma^2 = a$, $Q - \gamma_{\prime}^2 = a_{\prime}$, $R - \gamma\gamma_{\prime} = \beta$;

hence $P = a + \gamma^2$, $Q = a_{\prime} + \gamma_{\prime}^2$, $R = \beta + \gamma\gamma_{\prime}$.

Substituting in the expression (b) for the angle which the asymptote makes with the axis of X, we obtain

$$\tau = \frac{-(\beta + \gamma\gamma_{\prime}) \pm \sqrt{(\beta + \gamma\gamma_{\prime})^2 - (a + \gamma^2)(a_{\prime} + \gamma_{\prime}^2)}}{a + \gamma^2}. \quad . \quad (23)$$

In order that this value of τ may be real, we must have

$$(\beta+\gamma\gamma_{,})^2 > (a+\gamma^2)(a_{,}+\gamma_{,}^2). \quad . \quad . \quad . \quad . \quad (24)$$

17.] It is not difficult to show that when we have the relation

$$(\beta+\gamma\gamma_{,})^2 = (a+\gamma^2)(a_{,}+\gamma_{,}^2),$$

the tangential equation breaks up into two linear equations, the tangential equations of two points. Let the assumed linear equations be

$$m\xi+n\upsilon-1=0, \text{ and } m_{,}\xi+n_{,}\upsilon+1=0. \quad . \quad . \quad . \quad (a)$$

Multiplying them together, the resulting equation becomes

$$mm_{,}\xi^2+m_{,}n\xi\upsilon-m_{,}\xi+mn_{,}\xi\upsilon+nn_{,}\upsilon^2-n_{,}\upsilon+m\xi+n\upsilon=1. \quad . \quad (b)$$

Comparing this equation with the normal equation of the conic section,

$$a\xi^2+a_{,}\upsilon^2+2\beta\xi\upsilon+2\gamma\xi+2\gamma_{,}\upsilon=1,$$

and equating like coefficients, we get

$$mm_{,}=a,\ nn_{,}=a_{,},\ m_{,}n+mn_{,}=2\beta,\ m-m_{,}=2\gamma,\ n-n_{,}=2\gamma_{,}; \quad . \quad (c)$$

hence $\quad m_{,}+m=2\sqrt{a+\gamma^2},\ n_{,}+n=2\sqrt{a_{,}+\gamma_{,}^2},$

and $\quad (m_{,}+m)^2(n_{,}+n)^2=16(a+\gamma^2)(a_{,}+\gamma_{,}^2).$

Now $\quad m-m_{,}=2\gamma,\ n-n_{,}=2\gamma_{,};$

hence $\quad (m-m_{,})(n-n_{,})=4\gamma\gamma_{,},$

and $\quad 4\beta=2(m_{,}n+mn_{,});$

hence $\quad 4(\beta+\gamma\gamma_{,})=(m+m_{,})(n+n_{,}).$

Consequently, equating these values, we find

$$(\beta+\gamma\gamma_{,})^2=(a+\gamma^2)(a_{,}+\gamma_{,}^2). \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Since $\quad m+m_{,}=2\sqrt{a+\gamma^2},$ and $m-m_{,}=2\gamma,$

we find $\quad m=\gamma+\sqrt{a+\gamma^2},\ m_{,}=-\gamma+\sqrt{a+\gamma^2},$

and $\quad n=\gamma_{,}+\sqrt{a_{,}+\gamma_{,}^2},\ n_{,}=-\gamma_{,}+\sqrt{a_{,}+\gamma_{,}^2}.$

Hence the equation of the curve is broken up into

$$\left.\begin{array}{l}\sqrt{a+\gamma^2}\,\xi+\sqrt{a_{,}+\gamma_{,}^2}\,\upsilon-\gamma\xi-\gamma_{,}\upsilon-1=0, \\ \text{and} \quad \sqrt{a+\gamma^2}\,\xi+\sqrt{a_{,}+\gamma_{,}^2}\,\upsilon+\gamma\xi+\gamma_{,}\upsilon+1=0.\end{array}\right\} \quad . \quad . \quad . \quad (e)$$

18.] When $a=0$, the axis of Y touches the curve, and when $a_{,}=0$ the axis of X touches the curve.

When $\frac{1}{\xi}$ is a tangent, the curve coincides with the axis of X; then $\frac{1}{\upsilon}=0$, since this tangent meets the axis of Y at the origin.

The general tangential equation of the curve may be written

$$a_{,}+\frac{1}{\upsilon^2}\{a\xi^2+2\gamma\xi-1\}+\frac{1}{\upsilon}\{2\beta\xi+2\gamma\}=0;$$

hence, when $\frac{1}{\upsilon}=0$, $a_{,}$ must be $=0$.

In like manner it may be shown that when the axis of Y touches the curve we must have $a=0$.

19.] To find the values of ξ and v when they are tangents to the curve.

When the limiting tangent coincides with the axis of X, then (as in the preceding article) $a_{\prime}=0$, and the tangential equation becomes, when divided by v (which in this case is infinite), $\frac{1}{v}\{a\xi^2+2\gamma\xi-1\}+2\beta\xi+2\gamma_{\prime}=0$. Hence, when the axes of co-ordinates are tangents, we find

$$\beta\xi+\gamma_{\prime}=0, \quad \beta v+\gamma=0. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

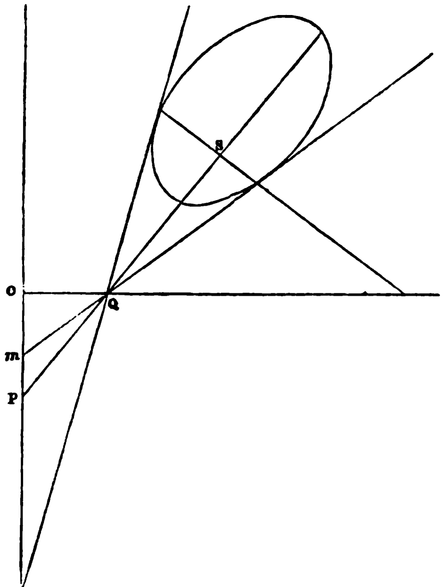
20.] Resuming the general tangential equation of the conic section (13), let it be solved for v.

$$v=-\left(\frac{\beta\xi+\gamma_{\prime}}{a_{\prime}}\right)\pm\frac{\sqrt{\mathrm{M}}}{a_{\prime}}.$$

Now let $\frac{1}{\mathrm{O}m}=-\left(\frac{\beta\xi+\gamma_{\prime}}{a_{\prime}}\right)+\frac{\sqrt{\mathrm{M}}}{a_{\prime}}$, $\frac{1}{\mathrm{OP}}=-\left(\frac{\beta\xi+\gamma_{\prime}}{a_{\prime}}\right)$,

and $\frac{1}{\mathrm{O}n}=-\left(\frac{\beta\xi+\gamma_{\prime}}{a_{\prime}}\right)-\frac{\sqrt{\mathrm{M}}}{a_{\prime}}$; . . . . (26)

Fig. 7.



hence $\frac{1}{\mathrm{O}m}$, $\frac{1}{\mathrm{OP}}$, $\frac{1}{\mathrm{O}n}$ are in arithmetical progression; therefore

O*m*, OP, O*n* are in harmonical progression; hence QO being $=\frac{1}{\xi}$, QO, Q*m*, QP, Q*n* constitute an harmonic pencil; and as Q*m*, Q*n* are tangents to the curve, QO, QP pass respectively through the poles, one of the other; hence QP passes through the pole of QO, which is the axis of X; hence

$$\frac{1}{\mathrm{OP}}=v=-\left(\frac{\beta\xi+\gamma_{,}}{a_{,}}\right), \text{ or } a_{,}v+\beta\xi+\gamma_{,}=0$$

is the equation of the pole of QO which is the axis of X. In the same way it may be shown that $a\xi+\beta v+\gamma=0$ is the equation of the pole of the axis of Y, and the two simultaneous equations

$$a\xi+\beta v+\gamma=0,\ a_{,}v+\beta\xi+\gamma_{,}=0\ \ .\ \ .\ \ .\ \ .\ \ (27)$$

determine the polar of the origin.

If we solve these equations, we find for ξ and v

$$\xi=\frac{\beta\gamma_{,}-a_{,}\gamma}{aa_{,}-\beta^2},\quad v=\frac{\beta\gamma-a\gamma_{,}}{aa_{,}-\beta^2}.\ \ .\ \ .\ \ .\ \ .\ \ (28)$$

These are the coordinates of the polar of the origin *.

21.] To find the species of the conic section when the reciprocal of the perpendicular on the tangent from the origin shall be a rational function of the tangential variables. Let the equation be

$$a\xi^2+a_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{,}v=1\,;\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (a)$$

then ϖ being the reciprocal of the perpendicular, and λ the angle it makes with the axis of x, we shall have $\varpi\cos\lambda=\xi$, $\varpi\sin\lambda=v$, hence

$$a\cos^2\lambda+a_{,}\sin^2\lambda+2\beta\sin\lambda\cos\lambda+2(\gamma\cos\lambda+\gamma_{,}\sin\lambda)\frac{1}{\varpi}=\frac{1}{\varpi^2},$$

or

$$\varpi=\frac{1-\gamma\xi-\gamma_{,}v}{\sqrt{(a+\gamma^2)\cos^2\lambda+(a_{,}+\gamma_{,}^2)\sin^2\lambda+2(\beta+\gamma\gamma_{,})\sin\lambda\cos\lambda}}.$$

Now, in order that this may be a rational function of ξ and v independently of λ, we must have

$$a+\gamma^2=a_{,}+\gamma^2 \text{ and } \beta+\gamma\gamma_{,}=0\,;$$

hence

$$\varpi=\frac{1-\gamma\xi-\gamma_{,}v}{\sqrt{a+\gamma^2}},\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ (29)$$

---

[*] Let $Ax^2+A_{,}y^2+2Bxy+2Cx+2C_{,}y=1$ be the projective equation of a central conic section, $\bar{x}$ and $\bar{y}$ being the coordinates of the centre; then, as is shown in all works on the subject,

$$\bar{x}=\frac{BC_{,}-A_{,}C}{AA_{,}-B^2},\quad \bar{y}=\frac{BC-AC_{,}}{AA_{,}-B^2},$$

which expressions for the ordinates of the centre are analogous to those for the tangential coordinates of the polar of the origin.

or the curve must be a circle, in order that ϖ may be a rational function of ξ and v[*].

22.] We shall show further on that if $F(x,y)=0$ and $V(\xi,v)=0$ be the projective and tangential equations of the *same* curve, we may pass from the one system to the other by the help of the following relations, taking the partial differentials as follows:—

$$\left.\begin{array}{ll} \xi=\dfrac{\dfrac{dF}{dx}}{\dfrac{dF}{dx}x+\dfrac{dF}{dy}y}, & v=\dfrac{\dfrac{dF}{dy}}{\dfrac{dF}{dx}x+\dfrac{dF}{dy}y}, \\[3ex] x=\dfrac{\dfrac{dV}{d\xi}}{\dfrac{dV}{d\xi}\xi+\dfrac{dV}{dv}v}, & y=\dfrac{\dfrac{dV}{dv}}{\dfrac{dV}{d\xi}\xi+\dfrac{dV}{dv}v}. \end{array}\right\} \quad . \ . \quad (30)$$

23.] Let $F(x,y)=0$ and $V(\xi,v)=0$ be the projective and tangential equations of the same curve, then

$$\frac{\dfrac{dV}{d\xi}v-\dfrac{dV}{dv}\xi}{\dfrac{dV}{d\xi}\xi+\dfrac{dV}{dv}v}+\frac{\dfrac{dF}{dx}y-\dfrac{dF}{dy}x}{\dfrac{dF}{dx}x+\dfrac{dF}{dy}y}=0. \quad . \ . \ . \ . \ . \ . \quad (31)$$

For brevity put

$$U\equiv\frac{dV}{d\xi}\xi+\frac{dV}{dv}v, \quad W\equiv\frac{dF}{dx}x+\frac{dF}{dy}y.$$

[*] Let $$Ax^2+A_{,}y^2+2Bxy+2Cx+2C_{,}y=1 \quad . \ . \ . \ . \ . \ . \quad (a)$$
be the projective equation of a conic section. Let $x=r\cos\omega$, $y=r\sin\omega$. Substituting these values and reducing, we find

$$A\cos^2\omega+A_{,}\sin^2\omega+2B\sin\omega\cos\omega+2(C\cos\omega+C_{,}\sin\omega)\frac{1}{r}=\frac{1}{r^2},$$

or, solving,

$$\left.\begin{array}{l} \dfrac{1}{r}=C\cos\omega+C_{,}\sin\omega \\ \quad \pm\sqrt{[A+C^2]+[(A_{,}+C_{,}^2)-(A+C^2)]\sin^2\omega+2(B+CC_{,})\sin\omega\cos\omega}. \end{array}\right\}. \quad (b)$$

Multiply this equation by r, putting x for $r\cos\omega$, and y for $r\sin\omega$, there finally results

$$r=\frac{1-Cx-C_{,}y}{\sqrt{(A+C^2)+[(A+C_{,}^2)-(A_{,}+C_{,}^2)]\sin^2\omega+2(B+CC_{,})\sin\omega\cos\omega}}. \ . \quad (c)$$

Now in order that r may be a rational function of x and y, the coefficients of the trigonometrical quantities under the radical sign must vanish, or we must have $A+C^2=A_{,}+C_{,}^2$, and $B+CC_{,}=0$, and

$$r=\frac{1-Cx-C_{,}y}{\sqrt{A+C^2}}. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (\ )$$

But when r is a rational function of x and y, it may easily be shown that the origin is at a focus.

Then, as shown in (30), $x=\dfrac{\dfrac{dV}{d\xi}}{U}$, $y=\dfrac{\dfrac{dV}{dv}}{U}$;

hence
$$xv-y\xi=\frac{\dfrac{dV}{d\xi}v-\dfrac{dV}{dv}\xi}{U}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

In like manner we may show that
$$xv-y\xi=\frac{\dfrac{dF}{dy}x-\dfrac{dF}{dx}y}{W}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

hence the truth of the proposition. In fact each expression is the value of tan θ, the angle between the perpendicular and the vector line.

Of this formula an elementary proof may be easily given. Since $\theta=\lambda-\alpha$,
$$\tan\theta=\frac{\tan\lambda-\tan\alpha}{1+\tan\lambda\tan\alpha}=\frac{\dfrac{v}{\xi}-\dfrac{y}{x}}{1+\dfrac{yv}{x\xi}}=\frac{xv-y\xi}{x\xi+yv};$$

but
$$x\xi+yv=1;$$

hence
$$\tan\theta=xv-y\xi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

24.] To apply these formulæ. Let us assume the general projective equation of a central conic section,
$$Ax^2+A_/y^2+2Bxy-1=F(x,y)=0. \quad . \quad . \quad . \quad . \qquad (a)$$

Now
$$\frac{dF}{dx}=2Ax+2By,\ \frac{dF}{dy}=2Ay+2Bx, \text{ and } \frac{dF}{dx}x+\frac{dF}{dy}y=2. \quad . \qquad (b)$$

Hence
$$\xi=Ax+By, \text{ and } v=Ay+Bx. \quad . \quad . \quad . \quad . \qquad (c)$$

From these equations, finding the values of x and y, and substituting them in the preceding equation, the result becomes
$$A_/\xi^2+Av^2-2B\xi v=AA_/-B^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (d)$$

Comparing this formula with the general tangential equation of a conic section, its centre at the origin, namely
$$a\xi^2+a_/v^2+2\beta\xi v=1,$$

we find
$$a=\frac{A_/}{AA_/-B^2},\ a_/=\frac{A}{AA_/-B^2},\ \beta=\frac{-B}{AA_/-B^2}, \quad . \quad . \qquad (35)$$

so that we may at once pass from the projective to the tangential equation of the curve.

The coefficients of the projective equation appear with some slight alterations: thus $A_/$, the coefficient of y^2, becomes the numerator of the coefficient of ξ^2; and $+2B$ is changed into $-2B$, while the absolute term becomes $AA'-B^2$.

On Asymptotes.

25.] An asymptote may be defined as a tangent to a curve, one of the projective coordinates x or y of the point of contact being at an infinite distance. Let $x=\infty$; then in the dual equation $x\xi+y\upsilon=1$, if x be infinite, we shall have $\frac{x}{y}=-\frac{\upsilon}{\xi}$.

$$\text{Now} \qquad \frac{x}{y}=\frac{\frac{dV}{d\xi}}{\frac{dV}{d\upsilon}}=-\frac{\upsilon}{\xi}; \text{ hence } \frac{dV}{d\xi}\xi+\frac{dV}{d\upsilon}\upsilon=0. \quad . \quad . \quad . \quad (36)$$

This is the general equation of an asymptote to a curve.

To apply this theory. Assume the general tangential equation of a curve of the second order,

$$a\xi^2+a_{,}\upsilon^2+2\beta\xi\upsilon+2\gamma\xi+2\gamma_{,}\upsilon=1;$$

$$\text{then} \qquad \frac{dV}{d\xi}\xi+\frac{dV}{d\upsilon}\upsilon=0=2a\xi^2+2a_{,}\upsilon^2+4\beta\xi\upsilon+2\gamma\xi+2\gamma_{,}\upsilon; \quad . \quad (a)$$

and the equation of the curve, multiplied by 2, gives

$$2a\xi^2+2a_{,}\upsilon^2+4\beta\xi\upsilon+4\gamma\xi+4\gamma_{,}\upsilon=2. \quad . \quad . \quad . \quad . \quad (b)$$

Subtracting the preceding expression from this equation, there results

$$\gamma\xi+\gamma_{,}\upsilon=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

the tangential equation of the centre of the curve, since γ and $\gamma_{,}$ are the projective coordinates of the centre. From this we may infer that whether the asymptotes be real or imaginary, they must pass through the centre of the curve. To determine the angle which the asymptote makes with the axis of X, let τ be the tangent of this angle, then $\tau=\frac{\xi}{\upsilon}$. Substituting this value of ξ, and dividing by υ^2, we get

$$a\tau^2+a_{,}+2\beta\tau+(\gamma\tau+\gamma_{,})\frac{1}{\upsilon}=0;$$

but (c) gives $\frac{1}{\upsilon}=\gamma\tau+\gamma_{,}$. Introducing this value of $\frac{1}{\upsilon}$ in the preceding expression, we find

$$(a+\gamma^2)\tau^2+2(\beta+\gamma\gamma_{,})\tau+a_{,}+\gamma_{,}^2=0.$$

Now, as this is a quadratic in τ, there must be *two* asymptotes, *each* of which passes through the centre; and if we solve this equation for τ, we get

$$\tau=\frac{-(\beta+\gamma\gamma_{,})\pm\sqrt{(\beta+\gamma\gamma_{,})^2-(a+\gamma^2)(a_{,}+\gamma_{,}^2)}}{a+\gamma^2},$$

which is real only when $(a+\gamma^2)(a_{,}+\gamma_{,}^2)$ is less than $(\beta+\gamma\gamma_{,})^2$.

This expression is identical with that found by a very different method in sec. 16.

26.] Let us resume the consideration of the central conic referred to rectangular axes passing through the centre, namely

$$A\xi^2+A_{,}\upsilon^2+2B\xi\upsilon=1.$$

Retaining the axis of X, let us assume a new axis of Y′ passing through the centre and making the angle ω with the axis of X. Hence, by (5), $\upsilon=\frac{\upsilon_{,}-\cos\omega.\xi}{\sin\omega}$; and substituting in the preceding equation the value here assigned to υ, we get

$$(A\sin^2\omega+A_{,}\cos^2\omega-2B\sin\omega\cos\omega)\xi^2+A_{,}\upsilon_{,}^2$$
$$+2(B\sin\omega-A_{,}\cos\omega)\xi\upsilon_{,}=\sin^2\omega.$$

Let us assume such a value of ω as will cause the coefficient of the rectangle $\xi\upsilon$ to vanish; then $\tan\omega=\frac{A_{,}}{B}$, and substituting this value of $\tan\omega$ in the preceding equation, the equation of the curve referred to oblique axes becomes

$$[AA_{,}-B^2]\xi^2+(A_{,}^2+B^2)\upsilon^2=A_{,}. \quad . \quad . \quad . \quad . \quad (37)$$

If we draw the *limiting tangent* parallel to the axis of X, ξ is $=0$, and $\frac{1}{\upsilon^2}=\frac{A_{,}^2+B^2}{A_{,}}=b_{,}^2$ the semidiameter conjugate to the axis of ξ. If we draw the limiting tangent parallel to the axis of Y, or make $\upsilon^2=0$,

$$\frac{AA_{,}-B^2}{A_{,}}=a_{,}^2;\text{ hence } a_{,}^2+b_{,}^2=\frac{AA_{,}-B^2}{A_{,}}+\frac{A_{,}^2+B^2}{A_{,}},$$

or $a_{,}^2+b_{,}^2=A+A_{,}$; but $A+A_{,}$ is the sum of the squares of the semiaxes a^2 and b^2, hence $a_{,}^2+b_{,}^2=a^2+b^2$, a well-known theorem.

The same formulæ for the parallel translation of axes will hold whether the systems of coordinates be right-angled or oblique, the coordinates of the centre of the first system being drawn parallel to those of the second.

27.] To determine a general expression in any plane curve for the distance between the point of contact of a tangent and the foot of the perpendicular let fall from the origin upon it.

As this line is the projection of the radius vector upon the tangent, it may with propriety be called the *protangent,* and may be written t.

Let $V(\xi,\upsilon)=0$ be the tangential equation of the curve, then $t^2=x^2+y^2-p^2$ (a), and write V instead of V (ξ,υ).

$$\text{Now}\qquad x=\frac{\frac{dV}{d\xi}}{\frac{dV}{d\xi}\xi+\frac{dV}{d\upsilon}\upsilon},\quad\text{and } y=\frac{\frac{dV}{d\upsilon}}{\frac{dV}{d\xi}\xi+\frac{dV}{d\upsilon}\upsilon}. \quad . \quad . \quad . \quad (b)$$

Squaring these values of x and y, and subtracting from them $\frac{1}{\xi^2+\upsilon^2}$, the square of the perpendicular, we obtain this remarkable and useful formula,

$$t=\left.\frac{\frac{dV}{d\xi}\upsilon-\frac{dV}{d\upsilon}\xi}{\left\{\frac{dV}{d\xi}\xi+\frac{dV}{d\upsilon}\upsilon\right\}(\xi^2+\upsilon^2)^{\frac{1}{2}}}\right\} \quad . \quad . \quad . \quad . \quad . \quad (38)$$

If we apply this formula to the tangential equation $A\xi^2+A_{\prime}\upsilon^2+2B\xi\upsilon=1$, of a conic section, where t is the distance measured from the point of contact along the tangent to the foot of the perpendicular, we shall find this expression,

$$t=\frac{(A-A_{\prime})\xi\upsilon+B(\upsilon^2-\xi^2)}{(\xi^2+\upsilon^2)^{\frac{1}{2}}}.$$

Now, if the *limiting tangent* be drawn parallel to the axis of X, $\xi=0$, and $t=B\upsilon$; but $\frac{1}{\upsilon}$ is the perpendicular on this tangent from the centre; therefore $Pt=B$. Hence, in the general tangential equation of the conic section, B denotes the area of the triangle between the axis of Y, the perpendicular tangent to it, and the diameter drawn through the point of contact.

When the tangent is parallel to the axis of X, the general equation becomes $A_{\prime}\upsilon^2=1$, or $P^2=A_{\prime}$; hence

$$\frac{P^2}{Pt}=\frac{A_{\prime}}{B}=\frac{P}{t}=\tan\omega;$$

or ω, the coordinate angle, is the angle between the conjugate diameters of the curve. Hence by the use of oblique tangential coordinates we may derive the properties of the conjugate diameters, as we may those of the axes, by the help of rectangular coordinates*.

28.] Let λ be the angle which a perpendicular P, let fall on a tangent to an ellipse which touches at a point whose radius vector is r, makes with the axis of X, we shall have

$$a^4\cos^2\lambda+b^4\sin^2\lambda=P^2r^2; \quad . \quad . \quad . \quad . \quad . \quad (39)$$

---

* We may obtain a similar expression (in projective coordinates) for the tangent of the angle between the perpendicular on the tangent and the radius vector of the point of contact.

For $\qquad \tan^2\theta=\frac{t^2}{P^2}=\frac{r^2-P^2}{P^2}$; hence $\frac{r^2-P^2}{r^2P^2}=\frac{1}{P^2}-\frac{1}{r^2}$.

Or $\qquad \frac{\tan^2\theta}{r^2}=\frac{1}{P^2}-\frac{1}{r^2}=\xi^2+\upsilon^2-\frac{1}{x^2+y^2}$.

Now $\qquad \xi^2=\frac{\left(\frac{dF}{dx}\right)^2}{\left[\left(\frac{dF}{dx}\right)x+\left(\frac{dF}{dy}\right)y\right]^2}, \quad \upsilon^2=\frac{\left(\frac{dF}{dy}\right)^2}{\left[\left(\frac{dF}{dx}\right)x+\left(\frac{dF}{dy}\right)y\right]^2};$

for $P\xi = \cos\lambda$, and $Pv = \sin\lambda$, we have also $a^2\xi = x$, $b^2v = y$. Hence, substituting, $a^4\cos^2\lambda + b^4\sin^2\lambda = P^2r^2$.

In like manner we may show that if ω be the angle which the diameter of an ellipse makes with the axis X, we shall have

$$\frac{\cos^2\omega}{a^4} + \frac{\sin^2\omega}{b^4} = \frac{1}{P^2r^2}. \quad \cdots\cdots \quad (40)$$

To express the value of a semidiameter drawn to the point of contact of the limiting tangent in terms of ξ and v:—

Since $\cos\lambda = P\xi$, $\sin\lambda = Pv$, substituting these values in (39),

$$r^2 = a^4\xi^2 + b^4v^2. \quad \cdots\cdots\cdots \quad (41)$$

CHAPTER III.

We may now illustrate this theory by its application to a few examples.

29.] *The product of pairs of perpendiculars let fall from two points on a straight line is constant; the line envelopes a conic section.*

In the expression given for the perpendicular in (6),

$$P = \frac{1 - p\xi - qv}{\sqrt{\xi^2 + v^2}},$$

where p and q are the projective coordinates of the point, while ξ and v are the tangential coordinates of the line, let $p = \pm c$ and $q = 0$; that is, let c be the distance between the points and the middle point of this line, taken as origin; then

$$P = \frac{1 - c\xi}{\sqrt{\xi^2 + v^2}}, \quad P_{\prime} = \frac{1 + c\xi}{\sqrt{\xi^2 + v^2}};$$

and let $PP_{\prime} = b^2$, the resulting equation becomes

$$(b^2 + c^2)\xi^2 + b^2v^2 = 1,$$

the tangential equation of a conic section, of which $(b^2 + c^2)$ and b^2 are the squares of the semiaxes.

30.] *The vertex of a right angle moves along the circumference of*

---

adding these expressions, and subtracting $\frac{1}{x^2 + y^2}$, and taking the square root, we find

$$\tan\theta = \frac{\frac{dF}{dx}y - \frac{dF}{dy}x}{\frac{dF}{dx}x + \frac{dF}{dy}y}.$$

If we apply this expression to the equation $Ax^2 + Ay^2 + 2Bxy = 1$, we shall find $\tan\theta = (A - A_{\prime})xy + B(x^2 - y^2)$.

a circle; one side passes through a fixed point; the other envelopes a conic section.

Let the line joining the fixed point with the centre of the circle be taken as the axis of X, let this distance be c, the equation of the circle being $x^2+y^2=a^2$; then the tangent of the angle which the line that passes through the fixed point makes with the axis of X is $\frac{y}{c+x}$, and the tangent of the angle which the *limiting tangent* makes with the same axis is $-\frac{\xi}{v}$; and these angles are complements one of the other; hence $\frac{y\xi}{v(c+x)}=1$. The dual equation (1) gives

$$x\xi+yv=1;$$

finding the values of y and x, substituting them in the equation of the circle $x^2+y^2=a^2$, we find

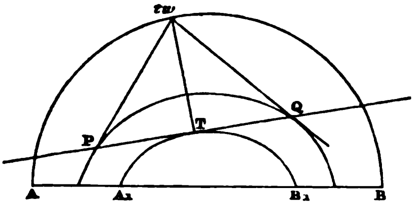
$$a^2\xi^2+(a^2-c^2)v^2=1,$$

when $c=a$, $a\xi=1$, or $\xi=\frac{1}{a}$, the tangential equation of a point in the axis of x, at the distance a from the origin; when $c>a$ the curve becomes an hyperbola.

31.] *Tangents are drawn to an ellipse from any point of a concentric circle whose radius is* $\sqrt{a^2+b^2}$; *the line joining the points of contact envelopes a confocal conic.*

Let t and u be the projective coordinates of the given point on the circumference of the circle, then $t^2+u^2=a^2+b^2$ is the equation of the circle; and the polar of this point with reference to the ellipse is $\frac{tx}{a^2}+\frac{uy}{b^2}=1$; and this gives $t=a^2\xi, u=b^2v$. Substituting these values of t and u in the equation of the circle, we find

Fig. 8.



$$a^4\xi^2+b^4v^2=a^2+b^2,$$

the tangential equation of an ellipse whose semiaxes $\bar{a}$ and $\bar{b}$ are given by the equations

$$\bar{a}^2=\frac{a^4}{a^2+b^2},\quad \bar{b}^2=\frac{b^4}{a^2+b^2};$$

and these sections are confocal for

$$\bar{a}^2-\bar{b}^2=\frac{a^4-b^4}{a^2+b^2}=a^2-b^2.$$

The line drawn from the point $(t\,u)$ to the point of contact of the polar of (t, u) with the interior confocal curve is a normal to the latter.

The tangent of the angle which the polar of $(t\,u)$, the tangent to the interior confocal curve, makes with the axis of X is $-\frac{\xi}{v}$; but the tangent of the angle which the line joining the point $(t\,u)$ with $(x_{,}\,y_{,})$, the point of contact of the polar of $(t\,u)$ with the confocal curve, is

$$\frac{u-y_{,}}{t-x_{,}}=\frac{b^2v-\bar{b}^2v}{a^2\xi-\bar{a}^2\xi}=\left(\frac{b^2-\dfrac{b^4}{a^2+b^2}}{a^2-\dfrac{a^4}{a^2+b^2}}\right)\frac{v}{\xi}=\frac{v}{\xi}.$$

Hence these lines are at right angles, the one to the other.

32.] *The sum of the perpendiculars let fall from n given points in a plane to a straight line in the same plane is constant. The straight line envelopes a circle.*

Let the projective coordinates of the n given points on the axes of coordinates, their origin and direction being arbitrary, be pq, $p_{,}q_{,}$, $p_{,,}q_{,,}$, &c. Then the length of one of the perpendiculars on the given line is $\frac{1-p\xi-qv}{\sqrt{\xi^2+v^2}}$; for the next point it will be $\frac{1-p\xi-q_{,}v}{\sqrt{\xi^2+v^2}}$ &c.

Let the sum of the perpendiculars be nc, then

$$n-(p+p_{,}+p_{,,}\ \&c.)\xi-(q+q_{,}+q_{,,}\ \&c.)v=nc\sqrt{\xi^2+v^2}. \quad . \qquad (a)$$

Let P and Q be the coordinates of the centre of gravity of all the points; then

$$p+p_{,}+p_{,,}\ \&c.=n\mathrm{P}, \quad q+q_{,}+q_{,,}=n\mathrm{Q}.$$

Substituting these values in (a), and dividing by n, we get

$$1-\mathrm{P}\xi-\mathrm{Q}v=c\sqrt{\xi^2+v^2}.$$

Reducing,

$$(c^2-\mathrm{P}^2)\xi^2+(c^2-\mathrm{Q}^2)v^2-2\mathrm{PQ}\xi v+2\mathrm{P}\xi+2\mathrm{Q}v=1, \quad . \qquad (b)$$

comparing this expression with the normal form,

$$a\xi^2+a_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{,}v=1,$$

which becomes the equation of a circle when

$$a+\gamma^2=a_{,}+\gamma_{,}^2, \quad \text{and } \beta+\gamma\gamma_{,}=0, \text{ see (21)},$$

relations which hold between the coefficients of the preceding equation.

When the sum of the perpendiculars is 0, $c=0$, and the tangential equation of the locus becomes $\mathrm{P}\xi+\mathrm{Q}v=1$, the tangential equation of a point of which the projective coordinates are P and Q, the projective coordinates of the centre of gravity of the system of n points.

33.] *Perpendiculars are drawn to the extremities of the diameters of an ellipse. They envelope a curve; to determine its tangential equation.*

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(a)}$$

and as the diameter of the ellipse r is perpendicular to the line whose coordinates are ξ and v, we get $r^2=x^2+y^2=\frac{1}{\xi^2+v^2}$; and as r is a mean proportional between x and $\frac{1}{\xi}$, $x=r^2\xi=\frac{\xi}{\xi^2+v^2}$; in the same way $y=\frac{v}{\xi^2+v^2}$. Hence, substituting,

$$(\xi^2+v^2)^2=\frac{\xi^2}{a^2}+\frac{v^2}{b^2}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(b)}$$

The rectification of this curve gives one of the best illustrations of the geometrical interpretation of the first elliptic integral.

34.] *Two semidiameters of a conic section and the chord joining their extremities contain a given area. The curve enveloped by this chord is a similar conic section.*

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a) be the equation of the conic section; $x_{\prime}\, y_{\prime}$ and $x_{\prime\prime}\, y_{\prime\prime}$ the coordinates of the extremities of the semidiameters; then

$$y_{\prime}=\frac{1-x_{\prime}\xi}{v}, \quad y_{\prime\prime}=\frac{1-x_{\prime\prime}\xi}{v}. \quad . \; . \; . \; . \quad \text{(b)}$$

Substituting these values of $y_{\prime}$ and $y_{\prime\prime}$ successively in the equation (a), we shall have

$$[a^2\xi^2+b^2v^2]x_{\prime}^2-2a^2\xi x_{\prime}+a^2(1-b^2v^2)=0; \quad . \; . \; . \quad \text{(c)}$$

the substitution of $y_{\prime\prime}$ would give an equation of precisely the same form. Hence

$$x_{\prime}+x_{\prime\prime}=\frac{2a^2\xi}{a^2\xi^2+b^2v^2}, \quad x_{\prime}x_{\prime\prime}=\frac{a^2(1-b^2v^2)}{a^2\xi^2+b^2v^2}; \quad . \; . \quad \text{(d)}$$

combining these expressions, we get

$$(x_{\prime}-x_{\prime\prime})^2=\frac{4a^2b^2v^2[a^2\xi^2+b^2v^2-1]}{(a^2\xi^2+b^2v^2)^2}. \quad . \; . \; . \quad \text{(e)}$$

Let the area of the given triangle be $\frac{nab}{2}$; it may be shown that it is also equal to $\frac{x_{\prime}-x_{\prime\prime}}{2v}$.

Hence $\frac{nab}{2}=\frac{x_{\prime}-x_{\prime\prime}}{2v}$, or $n^2=\frac{(x_{\prime}-x_{\prime\prime})^2}{a^2b^2v^2}$.

Substituting the preceding value of $(x_{,}-x_{,,})$, we obtain

$$n^2=\frac{4(a^2\xi^2+b^2v^2-1)}{(a^2\xi^2+b^2v^2)^2}.$$

Let $a^2\xi^2+b^2v^2=\mathrm{M}$. Then $n^2\mathrm{M}^2=4(\mathrm{M}-1)$, or

$$a^2\xi^2+b^2v^2=\frac{2}{n^2}\{1\pm\sqrt{1-n^2}\}. \quad . \quad . \quad . \quad . \quad \text{(f)}$$

Since the area of a triangle generally is $\frac{ab\sin\phi}{2}$, ϕ being the contained angle, and since it has been assumed equal to $\frac{nab}{2}$, $n=\sin\phi$; hence

$$\frac{2}{n^2}[1\pm\sqrt{1-n^2}]=\frac{2(1\pm\cos\phi)}{\sin^2\phi}=\frac{1}{\sin^2\frac{\phi}{2}} \text{ or } \frac{1}{\cos^2\frac{\phi}{2}};$$

consequently the equation of the sought curve becomes

$$\sin^2\frac{\phi}{2}\{a^2\xi^2+b^2v^2\}=1, \text{ or } \cos^2\frac{\phi}{2}[a^2\xi^2+b^2v^2]=1, \quad . \quad \text{(g)}$$

accordingly as we take the upper or lower sign. Thus there are two concentric ellipses enveloped by the revolving chord, such that the sum of the squares of the coincident axes will be equal to the squares of the axes of the original ellipse; for

$$a^2\sin^2\frac{\phi}{2}+a^2\cos^2\frac{\phi}{2}=a^2, \text{ and } b^2\sin^2\frac{\phi}{2}+b^2\cos^2\frac{\phi}{2}=b^2.$$

Hence if a polygon of n sides be inscribed in a conic section, the sides being inversely as the perpendiculars let fall upon them from the centre, this polygon will circumscribe a conic section similar to the given one.

35.] The straight line which joins the points of intersection of two focal vectors, containing a given angle θ, with a conic section, envelopes two conic sections having their foci coincident with the focus of the given section; and if ϵ and $\epsilon_{,}$ be the eccentricities of the loci, e that of the given section, p and $p_{,}$ the parameters of the loci, P that of the given section, we shall have the following relations between the eccentricities and parameters of the three conic sections,

$$\epsilon^2+\epsilon_{,}^2=e^2, \quad p^2+p_{,}^2=\mathrm{P}^2.$$

Let the equation of the given section be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2ex}{a}=\frac{b^2}{a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

the origin being placed at a focus, and the axes drawn parallel to the principal axes of the section.

Let $(y_{\prime}\,x_{\prime})$, $(y_{\prime\prime}\,x_{\prime\prime})$ be the coordinates of the points in which the sides of the given angle θ intersect the curve: the equation of the line passing through those points is

$$y-y_{\prime}=\frac{y_{\prime}-y_{\prime\prime}}{x_{\prime}-x_{\prime\prime}}(x-x_{\prime})\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

or if

$$y_{\prime}=mx_{\prime}, \quad . \quad . \quad \text{(c)} \qquad y_{\prime\prime}=m_{\prime}x_{\prime\prime} \quad . \quad . \quad . \quad . \quad \text{(c}_{\prime}\text{)}$$

be the equations of the sides of the angle, we shall find, eliminating $y_{\prime}$, $y_{\prime\prime}$ between (b), (c), and (c$_{\prime}$),

$$y-mx_{\prime}=\frac{mx_{\prime}-m_{\prime}x_{\prime\prime}}{x_{\prime}-x_{\prime\prime}}(x-x_{\prime}). \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Let ξ and v denote the reciprocals of the intercepts of the axes of X and Y by the limiting tangent (d); then

$$\frac{1}{x_{\prime}}=mv+\xi, \quad . \quad . \quad \text{(e)} \qquad \frac{1}{x_{\prime\prime}}=m_{\prime}v+\xi. \quad . \quad . \quad . \quad \text{(f)}$$

Now, eliminating $(x_{\prime}, y_{\prime})$ from the three equations (a), (c), (e), we shall find the quadratic equation

$$(a^2-b^4v^2)m^2-2(b^2aev+b^4\xi v)m+b^2-2aeb^2\xi-b^4\xi^2=0. \quad . \quad \text{(g)}$$

But this is precisely the equation we should have found for $m_{\prime}$; hence m and $m_{\prime}$ are the roots of (g), or

$$m+m_{\prime}=\frac{b^2aev+b^4\xi v}{a^2-b^4v^2}, \quad mm_{\prime}=\frac{b^2-2ab^2\iota\xi-b^4\xi^2}{a^2-b^4v^2}\,;$$

hence $$m-m_{\prime}=\frac{2ab\{b^2(\xi^2+v^2)+2ae\xi-1\}^{\frac{1}{2}}}{a^2-b^4v^2}.$$

Let the quantity under the radical sign be written M; then

$$\tan\theta=\frac{m-m_{\prime}}{1+mm_{\prime}}=\frac{\pm 2ab\sqrt{(\mathrm{M})}}{a^2-b^2\mathrm{M}},$$

or, solving this quadratic equation, we shall find

$$\mathrm{M}=\frac{a^2(1\pm\cos\theta)^2}{b^2\sin^2\theta},$$

or, replacing for M its value, reducing and taking the lower sign, we find

$$\frac{b^4(\xi^2+v^2)}{\left(b^2+a^2\tan^2\frac{\theta}{2}\right)}+\frac{2b^2ae.\xi}{b^2+a^2\tan^2\frac{\theta}{2}}=1. \quad . \quad . \quad . \quad \text{(h)}$$

Had we taken the upper sign, we should have found for the tangential equation of the locus

$$\frac{b^4(\xi^2+v^2)}{b^2+a^2\cot^2\frac{\theta}{2}}+\frac{2b^2ae.\xi}{b^2+a^2\cot^2\frac{\theta}{2}}=1. \quad . \quad . \quad . \quad . \quad \text{(k)}$$

Now in these equations, as the coefficients of ξ and v are equal, the foci of these sections are at the origin, or coincide with the focus of the given section.

To determine the axes &c. of these loci. The tangential equation of a conic section whose semiaxes and eccentricity are A, B, and ϵ, the origin of coordinates being at a focus and parallel to the axes of the section, is

$$B^2(\xi^2+v^2)+2A\epsilon . \xi=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (m)$$

Comparing this equation (m) with (h), we get

$$B^2=\frac{b^4}{b^2+a^2\tan^2\frac{\theta}{2}}, \quad A\epsilon=\frac{b^2ae}{b^2+a^2\tan^2\frac{\theta}{2}};$$

hence $\quad \epsilon=e\cos\frac{\theta}{2}$, and $\frac{B^2}{A}=\frac{b^2}{a}\cos\frac{\theta}{2}$, or $p=P\cos\frac{\theta}{2}$.

Had we taken the upper sign, we should have found

$$\epsilon_{,}=e\sin\frac{\theta}{2}, \quad p_{,}=P\sin\frac{\theta}{2};$$

$$\text{hence} \quad \epsilon^2+\epsilon_{,}^2=e^2, \quad p^2+p_{,}^2=P^2:$$

when θ is a right angle, the two loci coincide.

Had any other point except one of the foci been chosen, we should have found for the locus a curve whose tangential equation would be of the fourth degree—the curve in this particular case separating into two distinct curves, each of which is a conic section.

Had the given section been an equilateral hyperbola, and θ a right angle, a parabola would have been the locus.

When the given angle θ revolves round the centre instead of the focus, the tangential equation of the locus is

$$\{a^2b^2(\xi^2+v^2)-(a^2+b^2)\}^2=4a^2b^2\cot^2\frac{\theta}{2}\{a^2\xi^2+b^2v^2-1\}.$$

36.] *A straight line revolves in a conic section, having always a constant ratio to the parallel diameter; it will envelope a similar conic section.*

Let c be the chord, $2r$ the parallel diameter, n the ratio. Let the tangential equation of the conic section be

$$a^2\xi_{,}^2+b^2v_{,}^2=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

Let c, the limiting tangent, cut the axes of coordinates at the distances $\frac{1}{\xi}$ and $\frac{1}{v}$ from the centre. Let $2a_{,}$ be the diameter conjugate to $2r_{,}$, and x the distance between c and r measured along $a_{,}$. Then

$$c^2:r^2::a_{,}^2-x^2:a_{,}^2, \text{ or } \frac{x_{,}^2}{a_{,}^2}=\frac{r^2-c^2}{r^2}.$$

Let the tangent to the curve be parallel to the chord c, then

$$\frac{1}{\xi_{,}} : \frac{1}{\xi} :: a_{,} : x_{,}, \text{ or } \xi_{,}^2 : \xi^2 :: x_{,}^2 : a_{,}^2,$$

or

$$\xi_{,}^2 = \frac{\xi^2 x_{,}^2}{a_{,}^2} = \xi^2 \left(\frac{r^2 - c^2}{r^2}\right).$$

In the same way $v_{,}^2 = v^2 \left(\frac{r^2-c^2}{r^2}\right)$. Hence, substituting in (a),

$$a^2\xi^2 + b^2v^2 = \frac{r^2}{r^2-c^2}; \text{ but } \frac{c^2}{r^2} = n^2; \text{ hence}$$

$$a^2(1-n^2)\xi^2 + b^2(1-n^2)v^2 = 1; \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

when the line is indefinitely small $n=0$, and we get the original equation of the curve. When the revolving chord is equal to the parallel diameter, $n=1$, and the equation becomes $0 . \xi + 0 . v = 1$. In order that this relation may hold, we must have $\xi = \infty$, $v = \infty$, or $\frac{1}{\xi} = 0$, $\frac{1}{v} = 0$, or the chord c must pass through the centre.

37.] *The product of the sides of a right-angled triangle diminished by fixed quantities, is constant; the hypotenuse will envelope a conic section.*

Let the sides of the triangle be taken as the axes of coordinates, and let the subtracted lines be a and b. Then by the terms of the question $\left(\frac{1}{\xi} - a\right)\left(\frac{1}{v} - b\right) = c^2$. Since the sides of the triangle are $\frac{1}{\xi}$ and $\frac{1}{v}$, reducing,

$$(c^2 - ab)\xi v + a\xi + bv = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

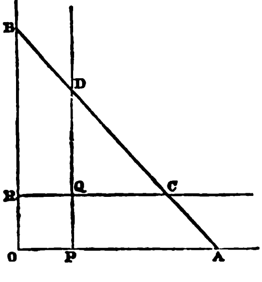
the tangential equation of a conic section.

Hence $\frac{1}{2}a$ and $\frac{1}{2}b$ are the coordinates of the centre.

Since a and $a_{,}$, the coefficients of the squares of the variables, do not appear in this equation [sec. 18.], the sides of the triangle are tangents to the curve.

When $c^2 = ab$, the equation becomes $a\xi + bv = 1$, the tangential equation of a point.

Fig. 9.



38.] *Let the sides of the rectangle* OPQR *be produced, and cut by the transversal* ABCD; *to find the tangential equation of the curve to which this line is always a tangent, under certain conditions.*

Let

$$\text{OP} = a, \ \text{PQ} = b, \ \text{OA} = \frac{1}{\xi}, \ \text{OB} = \frac{1}{v};$$

then we shall have

$$\left.\begin{array}{l}
\mathrm{RC}=\dfrac{1-bv}{\xi},\quad \mathrm{QC}=\dfrac{1-a\xi-bv}{\xi},\quad \mathrm{QD}=\dfrac{1-a\xi-bv}{v},\\
\mathrm{RB}=\dfrac{1-bv}{v},\quad \mathrm{PA}=\dfrac{1-a\xi}{\xi},\quad \mathrm{PD}=\dfrac{1-a\xi}{v},\\
\overline{\mathrm{AB}}^2=\dfrac{1}{\xi^2}+\dfrac{1}{v^2},\quad \overline{\mathrm{CD}}^2=\left(\dfrac{1}{\xi^2}+\dfrac{1}{v^2}\right)(1-a\xi-bv)^2,\\
\overline{\mathrm{AD}}^2=\left(\dfrac{1}{\xi^2}+\dfrac{1}{v^2}\right)(1-a\xi)^2,\quad \overline{\mathrm{CB}}^2=\left(\dfrac{1}{\xi^2}+\dfrac{1}{v^2}\right)(1-bv)^2,\\
\overline{\mathrm{AC}}^2=b^2v^2\left(\dfrac{1}{\xi^2}+\dfrac{1}{v^2}\right),\quad \overline{\mathrm{BD}}^2=a^2\xi^2\left(\dfrac{1}{\xi^2}+\dfrac{1}{v^2}\right).
\end{array}\right]\ .\quad \text{(a)}$$

(α) Assume $\dfrac{\mathrm{AD}\times\mathrm{BC}}{\mathrm{AB}\times\mathrm{CD}}=n$.

Let $\xi^2+v^2=\varpi^2$. Now, if we substitute the values of these lines as given above in ξ and v, we shall find

$$\frac{(1-bv)(1-a\xi)}{(1-a\xi-bv)}=n,$$

or

$$\frac{ab}{n-1}\xi v+a\xi+bv=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

If we submit this equation to the test in (20), we shall find that it is an hyperbola, since $n+1>n-1$, and the curve touches the axes of coordinates, since the coefficients of the squares of the variables are wanting.

(β) Let $\dfrac{\mathrm{AC}\times\mathrm{BD}}{\mathrm{AB}\times\mathrm{CD}}=n$. Substituting the values of these lines above given, we find $\dfrac{ab}{n}\xi v+a\xi+bv=1$. It may easily be shown that this is the tangential equation of the hyperbola. For, assuming the form of the general equation

$$a\xi^2+a_{\prime}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{\prime}v=1,$$

we find

$$a=0,\ a_{\prime}=0,\ \frac{ab}{2n}=\beta,\ \frac{a}{2}=\gamma,\ \frac{b}{2}=\gamma_{\prime};$$

and as the curve will be an hyperbola when

$$(\beta+\gamma\gamma_{\prime})^2>(a+\gamma^2)(a_{\prime}+\gamma_{\prime}^2),$$

we shall find on substituting, $2+n>n\cdot$

(γ) To find the curve when $\dfrac{\overline{\mathrm{AD}}^2+\overline{\mathrm{CB}}^2}{\overline{\mathrm{AB}}^2}=n^2$. Substituting the values of these lines, the resulting equation becomes

$$a^2\xi^2+b^2v^2-2a\xi-2bv=n^2-2,$$

the tangential equation of an ellipse, parabola or hyperbola, accordingly as $n^2>2$, $n^2=2$, or $n^2<2$.

(δ) Let $\frac{\overline{CB}^2+\overline{AD}^2}{AB^2+CD^2}=n^2$. Substituting the values of these straight lines, we get by reduction

$$-\frac{a^2\xi^2}{2}-\frac{b^2v^2}{2}-\frac{n^2}{n^2-1}ab\xi v+a\xi+bv=1,$$

the equation of an hyperbola, since $\frac{n^2+1}{n^2-1}$ is greater than 1.

39.] *The vertex of an angle of constant magnitude moves along the circumference of a circle; one side passes through a fixed point; to determine the curve that will be enveloped by the other.*

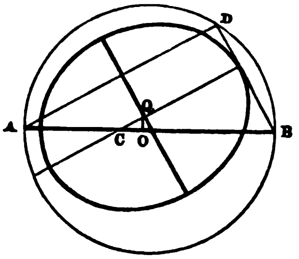
Let C B D $=\theta$, then while one side of the given angle θ passes through C, the other side B D touches the locus.

Let the given point C, whose distance from the centre O of the circle is c, be taken as the origin; then the equation of the circle, whose radius is a, will be

$$(x-c)^2+y^2=a^2. \quad . \quad . \quad \text{(a)}$$

Let the constant angle be θ, whose tangent is m, and let ϕ, $\phi_{,}$ be the angles which the moving lines make with the axis of X. Then $\theta=\phi+\phi_{,}$.

Fig. 10.



Now $\tan\theta=m$, $\tan\phi=\frac{y}{x}$, $\tan\phi_{,}=\frac{\xi}{v}$.

$$\text{Hence}\qquad m=\frac{\frac{y}{x}+\frac{\xi}{v}}{1-\frac{y\xi}{xv}}, \quad \text{or } m=\frac{\xi x+vy}{vx-\xi y}. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

But $x\xi+yv=1$, hence $m=\frac{1}{vx-\xi y}$. Eliminating y and x successively, we get

$$x=\frac{m\xi+v}{m(\xi^2+v^2)}, \quad y=\frac{mv-\xi}{m(\xi^2+v^2)}. \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Substituting these values in the equation of the circle, putting for m its value $\tan\theta$, we shall obtain

$$(a^2-c^2)\sin^2\theta(\xi^2+v^2)+2c\sin\theta(\sin\theta.\xi+\cos\theta.v)=1. \quad . \quad \text{(d)}$$

This is the equation of a conic section whose focus is at the origin.

If we compare this equation term by term with the general tangential equation of a conic section,

$$a\xi^2+a_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma v=1, \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

we shall have

$$\alpha=\alpha_{,}=(a^2-c^2)\sin^2\theta,\ \beta=0,\ \gamma=c\sin^2\theta,\ \gamma_{,}=c\sin\theta\cos\theta. \quad \text{(f)}$$

To determine the semiaxes and eccentricity of this curve. In (19), the general equation of the axes of a conic section, we find

$$2\mathrm{A}^2=\alpha+\gamma^2+\alpha_{,}+\gamma_{,}\pm\{[(\alpha+\gamma^2)-(\alpha_{,}+\gamma^2)]^2+4(\beta+\gamma\gamma_{,})^2\}^{\frac{1}{2}}.$$

If we substitute the preceding values, we shall have

$$\left.\begin{array}{l} \mathrm{A}^2=a^2\sin^2\theta,\ \mathrm{B}^2=a^2\sin^2\theta-c^2\sin^2\theta, \\ \text{and therefore } e^2=\dfrac{\mathrm{A}^2-\mathrm{B}^2}{\mathrm{A}^2},\ \text{ or } e=\dfrac{c}{a}. \end{array}\right\} \quad . \quad . \quad . \quad \text{(g)}$$

Hence, as e is independent of θ, all the enveloped curves will be similar and unifocal.

The coordinates of the centre are manifestly

$$\gamma=c\sin^2\theta \ \text{ and } \ \gamma_{,}=c\sin\theta\cos\theta.$$

Hence the distance of the centre of the curve from the origin is $\mathrm{D}=c\sin\theta$.

When $c=a$, or the origin is on the circle, equation (d) becomes

$$2a\sin^2\theta\,.\,\xi+2a\sin\theta\cos\theta\,.\,v=1,$$

the tangential equation of a point on the circumference of the circle. It is manifest that the line joining this point with the origin is the chord of the segment of the circle which contains the angle θ.

40.] *An angle of given magnitude revolves round a fixed point, intersecting by its sides two given straight lines; the line which joins the point of intersection envelopes a conic section.*

Let the fixed point be taken as origin of coordinates, the axes of coordinates being rectangular. Let

$$\lambda x+\mu y=1, \quad . \quad . \quad . \quad \text{(a)} \quad \text{and} \quad \lambda_{,}x+\mu_{,}y=1 \quad . \quad . \quad . \quad (\mathrm{a}_{,})$$

be the projective equations of the two fixed straight lines. Let

$$y=mx, \quad . \quad . \quad . \quad . \quad \text{(b)} \quad \text{and} \quad y=m_{,}x \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{b}_{,})$$

be the equations of the sides of the moving angle, and let

$$x\xi+yv=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

be the dual equation. Eliminating x and y between (a), (b), and (c), we get

$$m=\frac{\xi-\lambda}{\mu-v}, \quad \text{and also } m_{,}=\frac{\xi-\lambda_{,}}{\mu_{,}-v}. \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Let the revolving angle be θ; then $\tan\theta=\frac{m-m_{,}}{1+mm_{,}}$; or, substituting the preceding values of m and $m_{,}$, we obtain

$$\tan\theta=\frac{(\mu_{,}-\mu)\xi+(\lambda-\lambda_{,})v+\lambda_{,}\mu-\lambda\mu_{,}}{\xi^2+v^2-(\lambda+\lambda_{,})\xi-(\mu+\mu_{,})v+\lambda\lambda_{,}+\mu\mu_{,}}. \quad . \quad . \quad \text{(e)}$$

Should the two revolving lines coincide, $\theta=0$, and the numerator becomes 0; or it becomes the tangential equation of a point, and the point is the intersection of the two given lines. Or it becomes 0 when the denominator of (e) is infinite, or $\xi=\infty$, $v=\infty$, and the limiting tangent passes through the origin. When the angle θ is a right angle, the denominator becomes 0; but this expression is the tangential equation of a conic section whose focus is at the origin; and the projective coordinates of the centre, γ and $\gamma_{\prime}$, are given by the equations,

$$2\gamma=\frac{\lambda+\lambda_{\prime}}{\lambda\lambda_{\prime}+\mu\mu_{\prime}},\quad 2\gamma_{\prime}=\frac{\mu+\mu_{\prime}}{\lambda\lambda_{\prime}+\mu\mu_{\prime}}.$$

When the two fixed lines are parallel, and equidistant from the origin, $\lambda_{\prime}=-\lambda$, $\mu_{\prime}=-\mu$, and the denominator of (e) becomes $\xi^2+v^2=\lambda^2+\mu^2$, or the locus becomes a circle.

When the lines are at right angles, the constant term in the denominator vanishes and the curve becomes a parabola, as we shall show further on.

If in the equation (e) we substitute λ or $\lambda_{\prime}$ for ξ, and μ or $\mu_{\prime}$ for v, we shall find the equation satisfied independently of θ; hence it follows that the fixed lines themselves are tangents to the locus.

Taking the polar of the above, we get the elementary proposition, that if two lines, each passing through a fixed point, contain a constant angle the locus will be a circle, since the primitive has its focus at the origin.

This theorem gives, perhaps, the simplest method of describing a conic section by means of a ruler. Let any point be assumed in a plane, in which let two straight lines be drawn. If a right angle with sides of indefinite length be made to revolve round this point, cutting the fixed lines always in two points, the line which always joins these points will envelope a conic section, of which a focus is at the origin.

41.] *An angle of given magnitude θ revolves round a point in the plane of a conic section, cutting the curve in two points; the line joining these points will envelope a curve whose tangential equation is of the fourth order.*

Let the projective equation of the conic section be

$$Ax^2+A_{\prime}y^2+2Bxy+2Cx+2C_{\prime}y=1, \quad . \quad . \quad . \quad . \quad (a)$$

the vertex of the angle being taken as origin.

Let $$y_{\prime}=mx_{\prime} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

be the equation of one of the sides of the angle; substituting and dividing by x^2,

$$A+A_{\prime}m^2+2Bm+2(C+C_{\prime}m)\frac{1}{x_{\prime}}=\frac{1}{x_{\prime}^2};$$

but as $x\xi+y\upsilon=1$, $\frac{1}{x_{\prime}}=\xi+m\upsilon$; eliminating x, we shall obtain the resulting equation, arranging according to powers of m,

$$[A_{\prime}+2C_{\prime}\upsilon-\upsilon^2]m^2+2[B+C\upsilon+C_{\prime}\xi-\xi\upsilon]m+A+2C\xi-\xi^2=0. \quad \text{(c)}$$

Now the tangents of the angles which the revolving lines make with the axes of X are the roots of this quadratic equation; hence

$$m\ +m_{\prime\prime}=\frac{-2[B+C_{\prime}\xi+C\upsilon-\xi\upsilon]}{A\ +2C\ \upsilon-\upsilon^2}, \quad . \quad . \quad . \quad . \quad \text{(d)}$$

and

$$m_{\prime}m_{\prime\prime}=\frac{A+2C\xi-\xi^2}{A_{\prime}+2C_{\prime}\upsilon-\upsilon^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

It is obvious that

$$\tan\theta=\frac{m_{\prime}-m_{\prime\prime}}{1+m_{\prime}m_{\prime\prime}}.$$

To obtain the value of $m_{\prime}-m_{\prime\prime}$, we must square (d) and subtract 4 (e) from it.

Hence, after some reductions, we shall find

$$\begin{aligned} m_{\prime}-m_{\prime\prime}= [(C_{\prime}^2+A_{\prime})\xi^2+(C^2+A)\upsilon^2-2(B+CC_{\prime})\xi\upsilon \\ +2(BC_{\prime}-CA_{\prime})\xi+2(BC-C_{\prime}A)\upsilon+B^2-AA_{\prime}]^{\frac{1}{2}} \end{aligned}$$

divided by the coefficient of m^2, and

$$1+m_{\prime}m_{\prime\prime}=-[\xi^2+\upsilon^2-2C\xi-2C_{\prime}\upsilon-A-A_{\prime}]$$

divided by the same coefficient of m^2. Hence

$$\left.\begin{aligned} \tan\theta[\xi^2+\upsilon^2-2C\xi-2C_{\prime}\upsilon-A-A_{\prime}]=[(C_{\prime}^2+A_{\prime})\xi^2+(C^2+A)\upsilon^2 \\ -2(B+CC_{\prime})\xi\upsilon+2(BC_{\prime}-CA_{\prime})\xi+2(BC-C_{\prime}A)\upsilon+B^2-AA_{\prime}]^{\frac{1}{2}}. \end{aligned}\right\} \text{(f)}$$

If we now square both sides of this equation, the resulting formula will be of the fourth degree of the tangential coordinates ξ and υ.

Without proceeding to this expansion, we may make two suppositions, which will lead to remarkable results. If we suppose the two sides of the revolving angle to approximate and finally to coalesce, the line which joins their extremities will ultimately become a tangent to the curve itself, and therefore exhibit its tangential equation; but if $\tan\theta=0$, the second member of the preceding equation (f) becomes 0.

So that if $Ax^2+A_{\prime}y^2+2Bxy+2Cx+2C_{\prime}y=1$, be the projective equation of a conic section, the tangential equation of the same section, referred to the same axes, will be

$$\left.\begin{aligned} (A_{\prime}+C_{\prime}^2)\xi^2+(A+C^2)\upsilon^2-2(B+CC_{\prime})\xi\upsilon+2(BC_{\prime}-A_{\prime}C)\xi \\ +2(BC-AC_{\prime})\upsilon+B^2-AA_{\prime}=0. \end{aligned}\right\}. \quad \text{(g)}$$

Hence also, as in the general tangential equation of a conic section, the halves of the coefficients of the linear terms are the projective coordinates of the centre; therefore $\frac{BC_{,}-A_{,}C}{AA_{,}-B^2}$ and $\frac{BC-AC}{AA_{,}-B^2}$ are the projective coordinates of the centre of the conic section, whose projective equation is (a)—a result obtained by a very different method in the theory of projective coordinates.

When θ is a right angle, $\tan\theta$ becomes infinite, the second side of the equation vanishes by division, and the first member becomes

$$\frac{\xi^2}{A+A_{,}}+\frac{v^2}{A+A_{,}}-\frac{2C\xi}{A+A_{,}}-\frac{2C_{,}v}{A+A_{,}}=1, \quad . \quad . \quad \text{(h)}$$

the tangential equation of a conic section whose focus is at the origin, the coordinates of whose centre are $\frac{-C}{A+A_{,}}$ and $\frac{-C}{A+A_{,}}$.

When the angle θ is a right angle, the two branches of the curve whose tangential equation is of the fourth degree coalesce, so to speak, into one conic section.

42.] *A right-angled triangle has its right angle at a focus of a conic section, while the hypotenuse envelopes the curve; one acute angle of the triangle moves along a given straight line, the other will describe a conic section.*

Let the origin be at the focus. Then the tangential equation of the given conic section will be

$$a(\xi^2+v^2)+2\gamma\xi+2\gamma_{,}v=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let

$$px+qy=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

be the projective equation of the given straight line, and let

$$y=mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

be the equation of one of the sides of the right angle; and as the hypotenuse meets the straight line in the point (xy), we shall have from the *dual equation*

$$x\xi+yv=1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

eliminating x and y between (c), (b), and (d), we shall have

$$m=\frac{-(\xi-p)}{(v-q)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

Now as the other side of the right-angled triangle is separated by a right angle from the former, we shall have

$$y_{,}=\frac{-x_{,}}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

and

$$x_{,}\xi+y_{,}v=1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

eliminating m between (e) and (f), we obtain the resulting equation,

$$y_{,}\xi-x_{,}v=py_{,}-qx_{,};$$

combining this with the dual equation, $x_{,}\xi+y_{,}v=1$, we obtain

$$\xi=\frac{x_{,}+y_{,}(py_{,}-qx_{,})}{x_{,}^2+y_{,}^2}, \quad v=\frac{y_{,}-x_{,}(py_{,}-qx_{,})}{x_{,}^2+y_{,}^2}. \quad . \quad . \quad . \quad \text{(h)}$$

Now, if the hypotenuse were a fixed line, ξ and v would be constant quantities, and from the last equation we might determine the corresponding values of $x_{,}$ and $y_{,}$.

Let us assume that ξ and v are connected by a linear equation such as

$$P\xi+Qv=1.$$

Substituting in this equation the preceding values of ξ and v, the resulting projective equation becomes

$$Px+Qy+(Py-Qx)(py-qx)=x^2+y^2, \quad . \quad . \quad . \quad \text{(i)}$$

the equation of a conic section passing through the origin; hence, if a right-angled triangle revolve round a given point, and one angle move along a given straight line while the hypotenuse passes through a fixed point, the other angle will describe a conic section passing through the origin.

Again, let us assume that ξ and v are the tangential coordinates of the limiting tangent to the curve. Substituting in (a) the values of ξ and v given in (h), we shall find, after some reductions,

$$\begin{gathered}[1+2\gamma_{,}p-aq^2]x^2+[1+2\gamma q-ap^2]y^2+2[apq-\gamma p-\gamma_{,}q]xy \\ -2\gamma x-2\gamma_{,}y=a,\end{gathered}$$

the projective equation of a conic section.

43.] Assume the tangential equation of a conic section referred to its axes as axes of coordinates, namely

$$a^2\xi^2+b^2v^2=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let the axes of coordinates be conceived to be *first* turned round through an angle θ, and then translated, in parallel directions, to a point whose projective coordinates are $-p$ and $-q$. The formulæ by which this double translation is made are given in (4), and are

$$\xi=\frac{\cos\theta.\xi_{,}-\sin\theta.v_{,}}{1-p\xi_{,}-qv_{,}}, \quad v=\frac{\sin\theta.\xi_{,}+\cos\theta.v_{,}}{1-p\xi_{,}-qv_{,}}. \quad . \quad . \quad . \quad \text{(b)}$$

Substituting these values in the original equation, we get

$$\left.\begin{aligned}&[a^2\cos^2\theta+b^2\sin^2\theta-p^2]\xi^2+[a^2\sin^2\theta+b^2\cos^2\theta-q^2]v^2\\ &\quad+2[(a^2-b^2)\sin\theta\cos\theta-pq]\xi v+2p\xi+2qv=1.\end{aligned}\right\} \quad \text{(c)}$$

This is the tangential equation of the conic section referred to a new origin and other rectangular axes. Hitherto no relation has been assumed as existing between p, q, and θ; but if we assume

$$\tan\theta=\frac{q}{p}, \text{ and } p^2+q^2=a^2-b^2, \quad . \quad . \quad . \quad . \quad \text{(d)}$$

and substitute these expressions in the preceding equation, we shall obtain

$$\left.\begin{array}{c} a^2\cos^2\theta+b^2\sin^2\theta-p^2=b^2,\ a^2\sin^2\theta+b^2\cos^2\theta-q^2=b^2, \\ \text{and } 2[(a^2-b^2)\sin\theta\cos\theta-pq]=0. \end{array}\right\} \quad \text{(e)}$$

Hence the preceding equation is reduced to

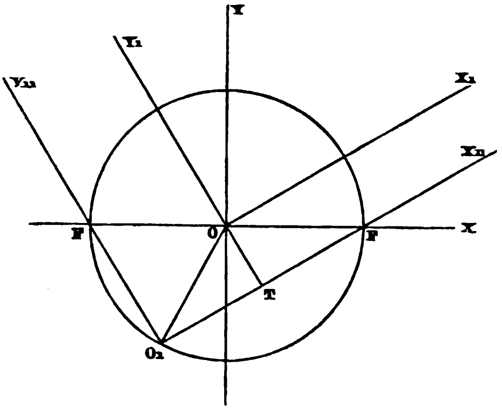
$$b^2(\xi^2+v^2)+2p\xi+2qv=1. \quad \ldots\ldots \quad \text{(f)}$$

Now this is exactly the form of the tangential equation of the conic section when the focus is the origin of coordinates. We may hence infer that if a concentric circle be described passing through the foci of the ellipse, and any point on this circle be taken as that round which the right angle revolves, we shall have the same results as if the focus had been selected. Let θ be the angle through which the first system of rectangular coordinates is turned, the radius of the circle being

$$\sqrt{a^2-b^2}, \text{ then } p=\sqrt{a^2-b^2}\cos\theta, \text{ and } q=\sqrt{a^2-b^2}\,.\sin\theta.$$

There is no difficulty in making this construction. Construct the focal circle. Let the coordinates be turned through the angle θ, make the angle $O_{\prime}O\,Y_{\prime}=X\,O\,Y_{\prime\prime}$, and draw $O_{\prime}X_{\prime\prime}$ parallel with $O\,X_{\prime}$; $O_{\prime}X_{\prime\prime}$, $O_{\prime}Y_{\prime\prime}$ will be the new system of coordinates, and $O_{\prime}T=p$, $O\,T=q$. A point taken anywhere in this circle will enjoy the tangential properties of the focus.

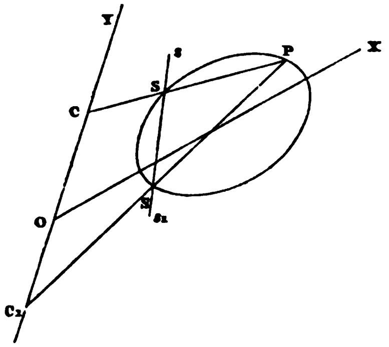
Fig. 11.



44.] *A triangle is inscribed in a conic section; two of its sides always pass through two fixed points, the third side envelopes a conic section.*

Let the line joining the fixed points be taken as the axis of Y,

Fig. 12.



and the diameter conjugate to it as the axis of X. Let the distances of the fixed points to the origin be h and $h_{,}$.

Let the projective equation of the conic section be

$$\mathrm{A}x^2+\mathrm{A}_{,}y^2+2\mathrm{C}x=1\,; \quad . \; . \; . \; . \; . \; . \quad \text{(a)}$$

and as the line (xy) $(x_{,}y_{,})$ passes through the point (x, y), we shall have the dual equation

$$x\xi+y\upsilon=1. \quad . \; . \; . \; . \; . \; . \; . \quad \text{(b)}$$

Eliminating x and y successively between these two equations, and putting

$$\mathrm{M}=(\mathrm{A}+\mathrm{C}^2)\upsilon^2+\mathrm{A}_{,}\xi^2-2\mathrm{CA}_{,}\xi-\mathrm{AA}_{,}, \quad . \; . \; . \quad \text{(c)}$$

we shall have

$$\left.\begin{aligned} x&=\frac{\mathrm{A}_{,}\xi-\mathrm{C}\upsilon^2-\upsilon\sqrt{\mathrm{M}}}{\mathrm{A}_{,}\xi^2+\mathrm{A}\upsilon^2}, \quad & y&=\frac{\mathrm{A}\upsilon+\mathrm{C}\upsilon\xi+\xi\sqrt{\mathrm{M}}}{\mathrm{A}_{,}\xi^2+\mathrm{A}\upsilon^2}, \\ x_{,}&=\frac{\mathrm{A}_{,}\xi-\mathrm{C}\upsilon^2+\upsilon\sqrt{\mathrm{M}}}{\mathrm{A}_{,}\xi^2+\mathrm{A}\upsilon^2}, \quad & y_{,}&=\frac{\mathrm{A}\upsilon+\mathrm{C}\upsilon\xi-\xi\sqrt{\mathrm{M}}}{\mathrm{A}_{,}\xi^2+\mathrm{A}\upsilon^2}. \end{aligned}\right\} \; . \; . \quad \text{(d)}$$

The signs of the radical $\sqrt{\mathrm{M}}$ must be so assumed as to fulfil the conditions

$$(x_{,}-x)\xi+(y_{,}-y)\upsilon=0, \text{ and } x\xi+y\upsilon=1. \; . \quad . \; . \quad \text{(e)}$$

We have also

$$\left.\begin{aligned} x+x_{,}-2\mathrm{C}xx_{,}&=\frac{2\mathrm{A}_{,}(\xi-\mathrm{C})}{\mathrm{A}_{,}\xi^2+\mathrm{A}\upsilon^2}; \quad & x-x_{,}&=\frac{-2\upsilon\sqrt{\mathrm{M}}}{\mathrm{A}_{,}\xi^2+\mathrm{A}\upsilon^2}, \\ \text{and} \quad yx_{,}&=\frac{\upsilon(\xi-\mathrm{C})+\sqrt{\mathrm{M}}}{\mathrm{A}_{,}\xi^2+\mathrm{A}\upsilon^2}; \quad & y_{,}x&=\frac{\upsilon(\xi-\mathrm{C})-\sqrt{\mathrm{M}}}{\mathrm{A}_{,}\xi^2+\mathrm{A}\upsilon^2}. \end{aligned}\right\} \; . \quad \text{(f)}$$

Now, as the point (u, t) is on the conic section, we shall have

$$At^2+A_{\prime}u^2+2Ct=1, \text{ and also } Ax^2+A_{\prime}y^2+2Cx=1. \quad . \quad (g)$$

Subtracting the latter from the former, and dividing by $(t-x)$, we shall have

$$A(t+x)+A_{\prime}(u+y)\left(\frac{u-y}{t-x}\right)+2C=0;$$

and as this line passes through the fixed points of which the coordinates are $y=h$, $x=0$, we shall have

$$\left(\frac{y-h}{x}\right)=\frac{u-y}{t-x}.$$

Eliminating u from the preceding equations, we shall have for the value of t,

$$t=\frac{x(A_{\prime}h^2-1)}{1-2Cx-2A_{\prime}hy+A_{\prime}h^2}. \quad . \quad . \quad . \quad . \quad . \quad (h)$$

In the same way we shall obtain for the other side of the triangle, passing through the points (u, t) and $(y_{\prime}\, x_{\prime})$,

$$t=\frac{x_{\prime}(A_{\prime}h_{\prime}^2-1)}{1-2Cx_{\prime}-2A_{\prime}h_{\prime}y_{\prime}+A_{\prime}h_{\prime}^2}. \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Comparing these two values of t, we shall have, all necessary reductions made, the following equation:

$$\left.\begin{aligned}&A_{\prime}(h^2-h_{\prime}^2)\,[x+x_{\prime}-2Cxx_{\prime}]+(A_{\prime}^2h^2h_{\prime}^2-1)(x-x_{\prime})\\&\quad=2A_{\prime}h_{\prime}(A_{\prime}h^2-1)y_{\prime}x-2A_{\prime}h(A_{\prime}h_{\prime}^2-1)yx_{\prime}.\end{aligned}\right\}. \quad . \quad (j)$$

Substituting the values of x, y, $x_{\prime}$, and $y_{\prime}$ given in the preceding formulæ (d), we shall obtain the resulting condition

$$\left.\begin{aligned}&A_{\prime}(h^2-h_{\prime}^2)(\xi-C)-(A_{\prime}^2h^2h_{\prime}^2-1)v\sqrt{M}+A_{\prime}h(A_{\prime}h^2-1)v(\xi-C)\\&\quad+A_{\prime}h(A_{\prime}h_{\prime}^2-1)\sqrt{M}+A_{\prime}h_{\prime}(A_{\prime}h^2-1)\sqrt{M}\\&\quad-A_{\prime}h_{\prime}(A_{\prime}h^2-1)v(\xi-C)=0;\end{aligned}\right\}(k)$$

or reducing,

$$\begin{aligned}&A_{\prime}(h-h_{\prime})\,[A_{\prime}(h+h_{\prime})-(A_{\prime}hh_{\prime}+1)v]\,(\xi-C)\\&\quad=(A_{\prime}hh_{\prime}-1)\,[A_{\prime}(h+h_{\prime})-(A_{\prime}hh_{\prime}+1)v]\,\sqrt{M},\end{aligned}$$

or, eliminating the common factor between the brackets,

$$A_{\prime}(h-h_{\prime})(\xi-C)=(A_{\prime}hh_{\prime}-1)\sqrt{M}.$$

Now, substituting for M its value given in (c), namely

$$M=(A+C^2)v^2+A_{\prime}\xi^2-2CA_{\prime}\xi-AA_{\prime},$$

we shall have

$$\frac{v^2}{A_{\prime}}+\frac{(A_{\prime}h^2-1)(A_{\prime}h_{\prime}^2-1)}{(A_{\prime}hh_{\prime}-1)^2(A+C^2)}(\xi-C)^2=1, \quad . \quad . \quad . \quad (l)$$

the tangential equation of a conic section.

Now, if we invert the preceding demonstration step by step, substituting tangential coordinates for projective coordinates, and reciprocally, we shall establish the reciprocal theorem, that if a triangle be circumscribed to a conic section, two of its angles always resting on fixed straight lines, the third angle will describe a conic section.

45.] *A series of central conic sections having the same centre, and their axes in the same direction, but such that the differences of the reciprocals of the squares of their axes is constant, the tangents drawn to a point on each, their intersection with a common diameter, envelope a concentric hyperbola, if the intersected curves be a series of ellipses, and an ellipse if the intersected curves be hyperbolas.*

Let $$a^2\xi^2+b^2v^2=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$
be the tangential equation of one of the curves; let $y=nx$. (b)
be the equation of one of the diameters; and let $\frac{1}{b^2}=\frac{1}{a^2}+\frac{1}{h^2}$ (c)
be the relation of the semiaxes. Then
$$a^2\xi=x, \quad . \quad . \quad . \quad \text{(d)} \qquad b^2v=y. \quad . \quad . \quad . \quad . \quad \text{(d}_{\prime}\text{)}$$
Between the five equations (a), (b), (c), (d), and (d$_{\prime}$) eliminating a, b, x, and y, we get for the equation of the envelope
$$h^2(v^2-\xi^2)+h^2\left(\frac{1-n^2}{n}\right)\xi v=1. \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$
When the angle which the common diameter makes with the axis of X is half a right angle, $n=1$, and the equation becomes $h^2(v^2-\xi^2)=1$, the equation of an equilateral hyperbola referred to its axes.

In any series of concentric conic sections, in which $\frac{1}{b^2}=\frac{1}{a^2}+\frac{1}{h^2}$, we shall have $h^2=\frac{a^2b^2}{a^2-b^2}=\frac{b^2}{e^2}$, or all the conic sections will have the same *Minor Directrices.*

Curves and curved surfaces having the same minor directrices, or minor directrix planes possess properties reciprocal to those of confocal curves and confocal surfaces, as shall be shown further on*.

---

* It may be instructive to compare, with the brevity and simplicity of the preceding solution, the ordinary method by which questions of this kind are solved.

The equation of the tangent through the point $x_{\prime}$, $y_{\prime}$ is
$$\frac{x_{\prime}}{a^2}\cdot x+\frac{y_{\prime}}{b^2}\,y=1, \text{ or as } \frac{1}{b^2}=\frac{1}{a^2}+\frac{1}{h^2};$$
it becomes, since $y_{\prime}=nx_{\prime}$, $\frac{x_{\prime}}{a^2}x+\frac{nx_{\prime}}{a^2}\cdot y+\frac{nx_{\prime}}{h^2}\,y=1$.

Eliminating $x_{\prime}$ between this equation and that of the curve, we find
$$\left[\frac{x+ny}{a^2}+\frac{ny}{h^2}\right]^2-\frac{n^2}{h}-\frac{(1+n^2)}{a^4}=0=\text{V},$$
for the equation of the tangent.

Eliminating a between the two equations $\text{V}=0$, and $\frac{d\text{V}}{da}=0$, we obtain, after

On Polygons inscribed *and* circumscribed *to Conic Sections.*

46.] There is a large class of questions having reference to polygons *in*scribed and *circum*scribed to curves and also to curved surfaces, which in many cases may be very simply treated. Questions relating to polygons *inscribed* in conic sections must be solved by the ordinary procedure of projective coordinates; while those which have reference to *circumscribed* polygons must be investigated by the help of the methods and formulæ of tangential coordinates, as explained in the preceding pages. As the properties of space are dual, so must the methods of investigation be dual also. It would be bootless to apply tangential coördinates to the investigation of the properties of *in*scribed polygons, and equally futile to use projective coordinates in the discussion of *circum*scribed polygons. It is further to be observed that in this class of questions the variables, such as x and y or ξ and v, become given constants, while the usual coefficients A, $A_{,}$, &c. or α, $\alpha_{,}$, &c. become unknown but determinate quantities of the first order. Thus if it were required to show that a conic section may be determined by five given points through which it is to pass, or by five given straight lines which it is required to touch, the variables x and y or ξ and v in the formal equations of the curve $Ax^2+A_{,}y^2+2Bxy+2Cx+2C_{,}y=1$, or $\alpha\xi^2+\alpha_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{,}v=1$, become given constants, while the coefficients become the unknown quantities; and as there are five of them, there must be five equations to determine them, and therefore there must be five sets of values of x, y or ξ, v. And as all the unknown coefficients, such as A, B, C or α, β, γ, are linear, the equations by which their values will be ultimately determined are linear also. Hence they are always real, though their values may sometimes be 0 or ∞, as the given values of x and y or ξ and v may turn out.

We shall apply the method to one or two examples.

Conic sections are inscribed in the same quadrilateral, the polar of any point in their plane envelopes a conic section.

The fixed point being assumed as origin of coordinates, let the tangential equation of one of the sections be

$$\alpha\xi^2+\alpha_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{,}v=1. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The equations of the polar of the origin are, see (27),

$$\alpha\xi+\beta v+\gamma=0 \quad . \quad . \quad (b) \text{ and } \alpha_{,}v+\beta\xi+\gamma_{,}=0. \quad . \quad . \quad (b_{,})$$

---

some considerable reductions,

$$y^2-x^2+\left(\frac{1-n^2}{n}\right)xy=\frac{h^2(1+n^2)^2}{4n^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

the projective equation of the locus.

The elimination of α between $V=0$, and $\frac{dV}{d\alpha}=0$, is frequently a matter of great complexity, and is often quite impracticable.

Now there are four linear equations under the form (a) to determine the five unknown quantities a, $a_{,}$, β, γ, $\gamma_{,}$; we may then eliminate any three, and connect the fourth and fifth by a linear equation. Eliminating then a, $a_{,}$, γ, $\gamma_{,}$, three by three successively, we shall have

$$\left.\begin{array}{ll} a=\mathrm{K}\beta+\mathrm{L}, & a_{,}=\mathrm{K}_{,}\beta+\mathrm{L}_{,}, \\ \gamma=\mathrm{M}\beta+\mathrm{N}, & \gamma_{,}=\mathrm{M}_{,}\beta+\mathrm{N}_{,}, \end{array}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{c})$$

where K, L, M, N, $\mathrm{K}_{,}$, $\mathrm{L}_{,}$, $\mathrm{M}_{,}$, $\mathrm{N}_{,}$ are known functions of the constant tangential coordinates of the four given straight lines. Substituting these values in (b), $(\mathrm{b}_{,})$, we shall have

$$\left.\begin{array}{l} (\mathrm{K}\xi+v+\mathrm{M})\beta+(\mathrm{L}\xi+\mathrm{N})=0, \\ (\mathrm{K}_{,}v+\xi+\mathrm{M}_{,})\beta+(\mathrm{L}_{,}v+\mathrm{N}_{,})=0; \end{array}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{d})$$

eliminating β from those equations, we obtain the tangential equation of a conic section.

When the point chosen is at one of the angles of the quadrilateral, the section becomes a point.

The origin being placed at this point, as two of the sides of the quadrilateral which are tangents to the curve pass through it, we shall have at the origin for one of the lines $\frac{1}{\xi}=0$, $\frac{1}{v}=0$. Let $\xi=nv$, n being the tangent which one of these lines makes with the axis of X. In the general equation, substituting nv for ξ, we find

$$an^2+a_{,}+2\beta n+2(\gamma n+\gamma_{,})\frac{1}{v}=\frac{1}{v^2}, \text{ or as } \frac{1}{v}=0,\ an^2+a_{,}+2\beta n=0.$$

In like manner for the other tangent, we shall find

$$an_{,}^2+a_{,}+2\beta n_{,}=0.$$

Eliminating successively $a_{,}$ and a between these equations, we shall find

$$a=\mathrm{K}\beta,\quad a_{,}=\mathrm{K}_{,}\beta.$$

Substituting these values of a and $a_{,}$ in (a), the result becomes

$$\begin{array}{ll} a=\mathrm{K}\beta, & a_{,}=\mathrm{K}_{,}\beta, \\ \gamma=\mathrm{M}\beta+\mathrm{N}, & \gamma_{,}=\mathrm{M}_{,}\beta+\mathrm{N}_{,}; \end{array}$$

hence eliminating β, having substituted the preceding values of a and $a_{,}$, γ and $\gamma_{,}$ in the equations of the polar of the origin, (b) and $(\mathrm{b}_{,})$,

$$\mathrm{N}_{,}\mathrm{K}\xi+\mathrm{N}_{,}v+\mathrm{M}\mathrm{N}_{,}=\mathrm{N}\mathrm{K}_{,}\xi+\mathrm{N}v+\mathrm{M}_{,}\mathrm{N},$$

the tangential equation of a point.

47.] *A series of conic sections are inscribed in the same quadrilateral, their centres range on the same straight line.*

From the four tangential equations of the sides of the quadrilateral we may eliminate three of the five unknown constants a, $a_{,}$,

β, γ, and $\gamma_{\prime}$; eliminating the three former, we shall obtain a linear resulting equation in γ and $\gamma_{\prime}$, namely

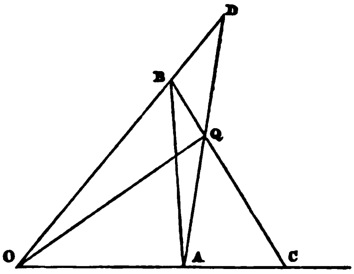
$$L\gamma+M\gamma_{\prime}+N=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

the projective equation of a straight line, the coefficients γ and $\gamma_{\prime}$ being, as shown in (14), the projective coordinates of the centre; therefore the centres of all these inscribed curves will be found on the same straight line.

48.] The method of tangential coordinates supplies a short and simple demonstration of a theorem of some difficulty and much celebrity, due to Newton, that, *The centres of conic sections inscribed in the same quadrilateral all range on the straight line which joins the points of bisection of the two diagonals of the quadrilateral.*

Let O A Q B be the quadrilateral in which a conic section is inscribed. Let A B and O Q be the diagonals, the line which joins their middle points will pass through the centre of the curve.

Fig. 13.



Let the tangential equation of the conic section be

$$a\xi^2+a_{\prime}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{\prime}v=1. \quad . \quad . \quad . \quad . \qquad (a)$$

Now if O be taken as origin, and O A, O B as oblique axes, then as the curve touches the axis of X, $a_{\prime}=0$, and as the curve touches the axis of y, $a=0$, and thus the tangential equation of the curve is reduced to

$$2\beta\xi v+2\gamma\xi+2\gamma_{\prime}v=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

Let O A $=a$, O B $=b$, O C $=c$, O D $=d$; then, as B C is a tangent to the curve, O B $=b=\frac{1}{v}$, O C $=c=\frac{1}{\xi}$; hence, substituting in the equation (b),

$$\frac{2\beta}{cb}+\frac{2\gamma}{c}+\frac{2\gamma_{\prime}}{b}=1, \text{ or } 2\beta+2b\gamma+2c\gamma_{\prime}=cb, \quad . \quad . \quad . \qquad (c)$$

in the same way, as A D is a tangent to the curve, $d=\frac{1}{v}$, $a=\frac{1}{\xi}$; hence substituting,

$$2\beta+2d\gamma+2a\gamma_{\prime}=ad. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (d)$$

Subtracting (d) from (c),

$$2(b-d)\gamma+2(c-a)\gamma_{\prime}=cb-ad. \quad . \quad . \quad . \qquad (e)$$

Now γ and $\gamma_{\prime}$ in the general tangential equation always denote the projective coordinates of the centre.

Again, the coordinates of the middle point of O Q are $y_{,}=\frac{b}{2}$, $x_{,}=\frac{a}{2}$.

$$\left.\begin{array}{l}\text{The equation of the line C B is } \frac{x}{c}+\frac{y}{b}=1,\\ \text{and the equation of the line A D is } \frac{x}{a}+\frac{y}{d}=1,\end{array}\right\} \quad . \ . \ . \ . \ . \ . \quad \text{(f)}$$

and therefore the coordinates of the point Q, their intersection, are

$$\bar{x}=\frac{ac(b-d)}{ab-cd}, \quad \bar{y}=\frac{bd(a-c)}{ab-cd},$$

while the coordinates of the middle point of O Q are $\frac{\bar{x}}{2}, \frac{\bar{y}}{2}$.

Now the equation of a straight line passing through two given points being $\frac{y-y_{,}}{(x-x_{,})}=\frac{y_{,}-y_{,,}}{x_{,}-x_{,,}}$, we shall have

$$y-\frac{b}{2}=\frac{\frac{bd(a-c)}{2(ab-cd)}-\frac{b}{2}}{\frac{ac(b-d)}{2(ab-cd)}-\frac{a}{2}}\left(x-\frac{a}{2}\right),$$

or

$$y-\frac{b}{2}=\frac{d-b}{c-a}\left(x-\frac{a}{2}\right),$$

or

$$2(b-d)x+2(c-a)y=cb-ad,$$

which is identically the same as (e).

The above proof, it will be seen, rests on the simplest elementary principles.

CHAPTER IV.

ON THE TANGENTIAL EQUATIONS OF THE PARABOLA.

49.] Let the focus and axis of the parabola be taken as the origin and axis of X. The projective equation of the parabola, referred to its focus, is

$$x^2+y^2=(2k+x)^2. \ . \ . \ . \ . \ . \ . \ . \quad \text{(a)}$$

Now the tangent to the curve drawn through the point $x_{,}y_{,}$ on the curve being $\frac{y-y_{,}}{x-x_{,}}=\frac{2k}{y_{,}}$, we shall have $x=\frac{1}{\xi}$ when $y=0$, and $y=\frac{1}{v}$ when $x=0$; hence $x_{,}=-\frac{(1+k\xi)}{\xi}$, $y_{,}=-\frac{2kv}{\xi}$.

Making these substitutions in the projective equation (a), we shall have for the tangential equation of the parabola, one fourth of the parameter being equal to k, and the origin at the focus,

$$k\xi^2+kv^2+\xi=0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(b)}$$

in this equation there is no absolute term.

The tangential equation is satisfied by $\xi=0$, $v=0$; for the parabola admits a tangent at infinity.

Let the axes of coordinates now be conceived to turn round the origin through the angle θ, and then translated, in parallel directions, to a point of which the coordinates are $-p$ and $-q$; then using the formulæ given in (4) for the transformation of coordinates, namely

$$\xi=\frac{\cos\theta.\xi_{,}-\sin\theta.v_{,}}{1-p\xi_{,}-qv_{,}}, \qquad v=\frac{\sin\theta.\xi_{,}+\cos\theta.v_{,}}{1-p\xi_{,}-qv_{,}}.$$

Substituting these values in (b), the general tangential equation of the parabola becomes

$$\left.\begin{array}{l}(k-p\cos\theta)\xi^2+(k-q\sin\theta)v^2 \\ \quad-(p\sin\theta+q\cos\theta)\xi v+\cos\theta.\xi+\sin\theta.v=0.\end{array}\right\} \quad . \quad . \qquad \text{(c)}$$

Assuming the most general form of the tangential equation of the parabola, let us suppose

$$f\xi^2+f_{,}v^2+g\xi v+h\xi+h_{,}v=0. \quad . \quad . \quad . \quad . \quad . \qquad \text{(d)}$$

Now, as the absolute term no longer affords a guide in comparing the general form of the tangential equation of the parabola with that derived by the transformation of coordinates, and as in this latter form the coefficients of ξ and v are $\cos\theta$ and $\sin\theta$, we must reduce the coefficients of ξ and v to the same form as in (c); hence, dividing by $\sqrt{h^2+h_{,}^2}$, the general form becomes

$$\frac{f\xi^2}{\sqrt{h^2+h_{,}^2}}+\frac{f_{,}v^2}{\sqrt{h^2+h_{,}^2}}+\frac{g\xi v}{\sqrt{h^2+h_{,}^2}}+\frac{h\xi}{\sqrt{h^2+h_{,}^2}}+\frac{h_{,}v}{\sqrt{h^2+h_{,}^2}}=0\,; \qquad \text{(e)}$$

hence

$$\left.\begin{array}{l}k-p\cos\theta=\dfrac{f}{\sqrt{h^2+h_{,}^2}}, \quad k-q\sin\theta=\dfrac{f_{,}}{\sqrt{h^2+h_{,}^2}}, \\ \qquad p\sin\theta+q\cos\theta=\dfrac{-g}{\sqrt{h^2+h_{,}^2}}, \\ \qquad \cos\theta=\dfrac{h}{\sqrt{h^2+h_{,}^2}}, \quad \sin\theta=\dfrac{h_{,}}{\sqrt{h^2+h_{,}^2}},\end{array}\right\} \quad . \qquad \text{(f)}$$

or, reducing,

$$\left.\begin{array}{l}k=\dfrac{fh_{,}^2+f_{,}h^2-ghh_{,}}{(h^2+h_{,}^2)^{\frac{3}{2}}}, \\ p=\dfrac{(f_{,}-f)h-gh_{,}}{h^2+h_{,}^2}, \quad q=\dfrac{(f-f_{,})h_{,}-gh}{h^2+h_{,}^2}.\end{array}\right\} \quad . \quad . \quad . \quad . \qquad \text{(g)}$$

Let D be the distance of the focus from the origin,

$$D^2=\frac{(f-f_{\prime})^2+g^2}{h^2+h_{\prime}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(h)}$$

Hence, when $f=f_{\prime}$ and $g=0$, $D=0$, or the origin is at the focus.

When the origin is at the vertex of the parabola, and its axis coincides with the axis of X, the tangential equation becomes

$$kv^2+\xi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

We may also express h and $h_{\prime}$ in terms of p and q. Solving (g),

$$h=\frac{p(f_{\prime}-f)-gq}{p^2+q^2}, \quad h_{\prime}=\frac{q(f-f_{\prime})-gp}{p^2+q^2}. \quad . \quad . \quad . \quad \text{(j)}$$

In the projective equation of a conic section, when the absolute term is 0, the origin of coordinates is on the curve; while in tangential coordinates, when the absolute term is 0, the curve is the parabola. Again, in projective coordinates the condition $AA_{\prime}-B^2=0$ indicates that the curve is a parabola; so in tangential coordinates the condition $aa_{\prime}-\beta^2=0$ indicates that the origin is on the curve, as shown in (22).

50.] In the tangential equation of the parabola $f_{\prime}$ is $=0$ when the curve touches the axis of X, and f is $=0$ when the curve touches the axis of Y.

To determine the distance of the point of contact on the axis of X from the origin. Let the equation of the curve divided by v be

$$\frac{f\xi^2}{v}+g\xi+\frac{h\xi}{v}+h_{\prime}=0 \text{ or } (f\xi^2+h\xi)\frac{1}{v}+g\xi+h_{\prime}=0. \quad . \quad \text{(k)}$$

Now, as $\frac{1}{v}=0$, the preceding equation is reduced to $g\xi+h_{\prime}=0$; in the same way, $gv+h=0$, gives the distance when the axis Y touches the curve.

Hence the distances from the origin of the points at which the axes of X and Y touch the parabola are given by the equations

$$g\xi+h_{\prime}=0, \text{ and } gv+h=0. \quad . \quad . \quad . \quad . \quad . \quad \text{(l)}$$

51.] We shall apply this theory to a few examples.

The sum of the sides of a right angle is constant. The hypotenuse envelopes a parabola.

By the terms of the proposition $\frac{1}{\xi}+\frac{1}{v}=c$, or $-c\,.\,\xi v+\xi+v=0$, the equation of a parabola, since the absolute term is $=0$.

Comparing this equation with the general tangential equation of the parabola, we get

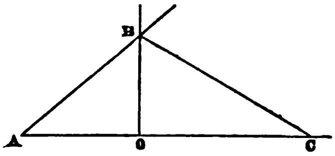
$$f=0,\ f_{\prime}=0,\ g=-c,\ h=1,\ h_{\prime}=1, \text{ and } k=\frac{c}{2^{\frac{3}{2}}}=\frac{\sqrt{2}c}{4}.$$

The locus is a parabola whose parameter is $\sqrt{2}c$, which touches both sides of the right angle, and whose axis bisects the right angle.

52.] *An angle of given magnitude moves along a fixed straight line, one side always passes through a fixed point, the other side will envelope a parabola.*

Let the fixed line O B and the perpendicular a from the fixed point C be taken as axes of coordinates, let the tangent of the constant angle at B be m; then, by the conditions of the question,

Fig. 14.



$$\frac{\xi}{v}=\frac{m+\dfrac{1}{av}}{1-\dfrac{m}{av}},$$

or, reducing, $$mav^2-a\xi v+m\xi+v=0,$$

the equation of a parabola, as the absolute term is 0.

Comparing the terms of this equation with those of the general form (c) in [49], we shall have $f=0$, $f_{\prime}=ma$, $g=-a$, $h=m$, $h_{\prime}=1$; hence

$$\tan\theta=\frac{1}{m},\quad k=\frac{ma}{\sqrt{1+m^2}}=a\cos\theta,\quad p=a,\quad q=0.$$

53.] *Parabolas are inscribed in a triangle; the locus of their foci is the circumscribing circle.*

Let the base of the triangle be taken as the axis of X, the origin being placed at an angle of the triangle, n being the tangent of the angle which the second side, passing through the origin, makes with the axis of X.

Let the base of the triangle be a, and let the two other sides make, with the base, angles whose tangents are n and m, the former line passing through the origin.

As the base a measured along the axis of X is a tangent to the curve, $f_{\prime}=0$; and as $\xi=nv$, the general equation (d) becomes

$$\{fn^2+gn\}v^2+hnv+h_{\prime}v=0;$$

and as at the origin, $\frac{1}{v}=0$, $fn+g=0$, or $g=-fn$.

Hence the general equation now becomes

$$f\xi^2-fn\xi v+h\xi+h_{\prime}v=0. \quad \ldots\ldots \quad \text{(a)}$$

Again, the tangential coordinates of the third side of the triangle are $\xi=\frac{1}{a}$, $v=\frac{1}{am}$. Substituting these values in the last equation, we shall have

$$fm-fn+ham+h_{\prime}a=0. \quad \ldots\ldots \quad \text{(b)}$$

Substituting in this latter equation the values of h and $h_{,}$ given in (j), first making $f_{,}=0$, $g=-fn$,

$$h=\frac{fnq-pf}{p^2+q^2},\quad h_{,}=\frac{qf+fnp}{p^2+q^2}. \quad . \quad . \quad . \quad . \quad . \qquad \text{(c)}$$

Substituting these values in the preceding equation, we get

$$fm-fn+\frac{amfnq-ampf+aqf+afnp}{p^2+q^2}=0\,; \quad . \quad . \quad . \qquad \text{(d)}$$

or dividing by f and $(m-n)$, we obtain the final result,

$$p^2+q^2-ap+aq\frac{(1+mn)}{m-n}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(e)}$$

Now, as p and q are the projective coordinates of the focus in the general equation of the parabola, and as they are in this equation the variables of the projective equation of a circle, it follows that the locus of the foci of the inscribed parabolas is the circle circumscribing the triangle.

54.] *An angle of given magnitude revolves round the focus of a parabola, to determine the curve enveloped by the cord which joins the points in which the parabola is intersected by the sides of the angle.*

Let $y^2+x^2=(2a+x)^2$ be the projective equation of the parabola referred to its focus as origin, and having the axes of coordinates parallel with those of the curve.

Let $y=nx$ be the equation of one of the lines, then

$$\sqrt{1+n^2}=1+\frac{2a}{x}, \text{ and } \frac{1}{x}=\xi+nv\,;$$

eliminating x,

$$(4a^2v^2-1)n^2+4av(1+2a\xi)n+4a\xi(1+a\xi)=0. \quad . \quad . \qquad \text{(a)}$$

Now the roots of this equation are

$$n_{,}+n_{,,}=-\frac{4av(1+2a\xi)}{4a^2v^2-1},\quad n_{,}n_{,,}=\frac{4a\xi(1+a\xi)}{4a^2v^2-1}\,; \quad . \quad . \qquad \text{(b)}$$

hence

$$(n_{,}+n_{,,})^2=\frac{16a^2v^2(1+2a\xi)^2}{(4a^2v^2-1)^2},$$

and

$$4n_{,}n_{,,}=\frac{64a^3\xi v^2+64a^4\xi^2v^2-16a\xi-16a^2\xi^2}{(4a^2v^2-1)^2}.$$

Hence

$$n_{,}-n_{,,}=\frac{4\{a^2v^2+a^2\xi^2+a\xi)^{\frac{1}{2}}}{4a^2v^2-1},$$

and

$$1+n_{,}n_{,,}=\frac{4a^2v^2+4a^2\xi^2+4a\xi-1}{4a^2v^2-1}\,;$$

consequently

$$\frac{n_{,}-n_{,,}}{1+n_{,}n_{,,}}=\frac{4\{a^2\xi^2+a^2v^2+a\xi\}^{\frac{1}{2}}}{4(a^2\xi^2+a^2v^2+a\xi)-1}. \quad . \quad . \quad . \quad . \qquad \text{(c)}$$

Let the expression under the radical be put M, and as $\frac{n_{,}-n_{,,}}{1+n_{,}n_{,,}}$ is equal to the tangent of the revolving angle θ, we shall have

$$\tan\theta=\frac{4\sqrt{\mathrm{M}}}{4\mathrm{M}-1};$$

reducing and solving for M, we shall find, taking the upper sign,

$$\mathrm{M}=\cot^2\frac{\theta}{2}; \text{ taking the lower sign, } \mathrm{M}=\tan^2\frac{\theta}{2},$$

or, substituting for M its value,

$$a^2\xi^2+a^2v^2+a\xi=\cot^2\frac{\theta}{2}, \text{ or } a^2\xi^2+a^2v^2+a\xi=\tan^2\frac{\theta}{2}.$$

When θ is a right angle, $\tan\frac{\theta}{2}=\cot\frac{\theta}{2}=1$, and the two curves coincide. Its equation then becomes $a^2\xi^2+a^2v^2+\xi=0$.

Since the coefficients of ξ^2 and v^2 are equal, and the rectangle disappears, the origin must be at the focus of the curve. Hence the loci and the parabola have the same foci.

We may determine the axis and eccentricity of the two loci as follows. While the tangential equation of one section is

$$a^2\xi^2+a^2v^2+a\xi=\cot^2\frac{\theta}{2}, \text{ that of the other is}$$

$$a^2\xi^2+a^2v^2+a\xi=\tan^2\frac{\theta}{2}.$$

Let $\tan^2\frac{\theta}{2}=t^2$. Multiplying the first equation by t^2, we get

$$a^2t^2\xi^2+a^2t^2v^2+at^2\xi=1.$$

Now let A and B be the semiaxes of this curve, and, comparing the coefficients of this equation with the general equation

$$a\xi^2+a_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{,}v=1,$$

we shall have

$$a=a^2t^2,\ a_{,}=a^2t^2,\ \gamma=\frac{at^2}{2},\ \beta=0,\ \gamma_{,}=0.$$

Substituting these values in (19), the formula for finding the axes, we shall have

$$2\mathrm{A}^2=2a^2t^2+\frac{a^2t^4}{4}\pm\sqrt{\frac{a^4t^8}{16}} \text{ or } 2\mathrm{A}^2=2a^2t^2+\frac{a^2t^4}{2},$$

or

$$4\mathrm{A}^2=4a^2t^2+a^2t^4 \text{ and } 4\mathrm{B}^2=4a^2t^2;$$

hence $\frac{A^2-B^2}{A^2}=\frac{t^2}{4+t^2}=e_{\prime}^2$; in the same way we may find, for the eccentricity of the other ellipse,

$e_{\prime\prime}^2=\frac{1}{1+4t^2}$, as $\frac{4+t^2}{t^2}=\frac{1}{e_{\prime}^2}$ and $1+4t^2=\frac{1}{e_{\prime\prime}^2}$, $5(1+t^2)=\frac{1}{e_{\prime\prime}^2}+\frac{t^2}{e_{\prime}^2}$, or

$$5=\frac{\cos^2\frac{\theta}{2}}{e_{\prime\prime}^2}+\frac{\sin^2\frac{\theta}{2}}{e_{\prime}^2}.$$

When the two curves coincide, or when θ is a right angle, the eccentricity of the two coinciding loci will be found to be $\frac{1}{\sqrt{5}}$; hence, naming the eccentricity e, we shall have

$$\frac{1}{e^2}=\frac{\cos^2\frac{\theta}{2}}{e_{\prime\prime}^2}+\frac{\sin^2\frac{\theta}{2}}{e_{\prime}^2}.$$

55.] It is almost needless to observe that, as in the case of projective coordinates, analogous formulæ may be established when the axes of coordinates are oblique, by reasonings precisely similar. A single application of such formulæ may suffice.

A series of parabolas are inscribed in the same triangle, the line joining the points of contact of each parabola with the opposite vertices of the triangle meet in a point. The locus of this point is the minimum ellipse circumscribing the triangle.

Let the sides a, b of the triangle be taken as axes of coordinates, then, in the general equation of the parabola

$$f\xi^2+f_{\prime}v^2+g\xi v+h\xi+h_{\prime}v=0, \quad \ldots\ldots \quad \text{(a)}$$

as the curve touches the axis of X, $f_{\prime}=0$; and as it touches the axis of Y, $f=0$. Hence this equation becomes

$$g\xi v+h\xi+h_{\prime}v=0; \quad \ldots\ldots\ldots \quad \text{(b)}$$

and as the third side of the triangle is a tangent also to the parabola, its coordinates being $\xi=\frac{1}{a}$, $v=\frac{1}{b}$, the preceding equation becomes

$$g+hb+h_{\prime}a=0. \quad \ldots\ldots\ldots \quad \text{(c)}$$

The parabola touches the axis of X, see (l) in [50], at a distance from the origin $=\frac{-g}{h_{\prime}}$, and the axis of Y at the distance $\frac{-g}{h}$. The projective equations of the straight lines joining these points with the opposite vertices, namely $y=0$, and $x=\frac{-g}{h_{,}}$, and $x=0$, $y=b$, are

$$\frac{y}{b}-\frac{h_{,}}{g}x=1, \text{ and } \frac{x}{a}-\frac{h}{g}y=1. \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Taking these as simultaneous equations, the values of x and y determined on this supposition are the coordinates of the common point. Substituting the values of h and $h_{,}$ derived from these equations in the formula $g+hb+h_{,}a=0$, we get

$$a^2y^2+b^2x^2+abxy-a^2by-ab^2x=0, \quad . \quad . \quad . \quad . \quad \text{(e)}$$

the projective equation of an ellipse circumscribing the given triangle; for this equation is satisfied by the three sets of values

$$\begin{array}{lll} y=0, & y=b, & y=0, \\ x=0, & x=0, & x=a, \end{array}$$

the coordinates of the three vertices of the triangle.

Let the origin be translated to the centre of gravity of the triangle, then $x=x_{,}+\frac{a}{3}$, $y=y_{,}+\frac{b}{3}$, and the resulting equation becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{xy}{ab}=\frac{1}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

Now, that the ellipse circumscribing the triangle is a minimum when its centre coincides with the centre of gravity of the triangle may be thus simply shown. Let a circle be circumscribed to an equilateral triangle, it will have its centre in the centre of gravity of this triangle. On the circle and inscribed equilateral triangle let a right cylinder and right prism be erected, and let them be cut by any inclined plane; this plane will cut the cylinder in an ellipse and the inscribed prism in a triangle, which will be inscribed in the ellipse, and every line that is bisected or divided in any given ratio in the equilateral triangle or circle will have its projection similarly divided in the triangle and its circumscribing ellipse, and as the circle is the orthogonal projection of the ellipse, while the equilateral triangle is the projection of the triangle inscribed in the ellipse, and which has its centre of gravity in the axis of the cylinder and therefore in the centre of the ellipse, it follows that the minimum ellipse circumscribing a triangle has its centre in the centre of gravity of the triangle.

CHAPTER V.

ON THE TANGENTIAL EQUATIONS OF THE POINT, THE PLANE, AND THE STRAIGHT LINE IN SPACE.

THROUGHOUT the preceding Chapters, the point and the straight line have been considered as pole and polar. This is but a partial and inadequate conception, because, in the complete duality of the properties of space, the point, the straight line, and the plane are the polars of the plane, the straight line, and the point. When a curve or a rectilinear figure is given in the same plane with the centre of the *polarizing sphere*, the reciprocal polar is neither a curve nor a rectilinear figure, but a cylinder or prism standing at right angles to the plane; and when the centre of the polarizing sphere is not in the plane of the given curve or other figure, the reciprocal polar is a cone or pyramid whose vertex is the pole of the given plane. When discussing the properties of figures *in plano* or in the geometry of two dimensions as it is called, we leave out of consideration the cylindrical surfaces, the polars of the curves described on the given plane, and deal only with their bases in that plane.

As a point and a plane may be pole and polar one to the other, so may a straight line be a reciprocal polar to a straight line. Such a line may easily be found; for, assume any two points in the given straight line, the polar planes of these two points will intersect in a straight line which is the reciprocal polar of the former, and the planes drawn from the centre of the polarizing sphere through these straight lines will be at right angles, one to the other, and the polar plane of any point assumed on one of the straight lines will pass through the other, as we shall show further on.

On the Tangential Equations of a Point and Plane in Space.

56.] Let p, q, r be the projective coordinates of a point on three rectangular planes. The tangential equation of the point is

$$p\xi+qv+r\zeta=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

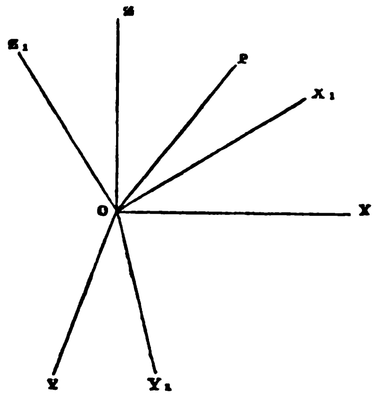
The tangential equations of a fixed plane are

$$\xi=\text{constant}, \quad v=\text{constant}, \quad \zeta=\text{constant}. \quad . \quad . \quad \text{(b)}$$

On the Transformations of Tangential Coordinates in Space.

Let three rectangular axes, O X, O Y, O Z, be drawn through a fixed point O meeting a given plane in three points; let the reciprocals of the distances of these points from the origin be denoted by ξ, v, ζ; let the reciprocals of the distances of the corresponding points for three other rectangular axes passing through the same origin and meeting the *same plane* be denoted by $\xi_{\prime}$, $v_{\prime}$, $\zeta_{\prime}$. Let the axis of $X_{\prime}$ make with the original axes the angles λ, μ, ν; and let the axes of $Y_{\prime}$ and $Z_{\prime}$ make with the same axes the angles $\lambda_{\prime}$, $\mu_{\prime}$, $\nu_{\prime}$; $\lambda_{\prime\prime}$, $\mu_{\prime\prime}$, $\nu_{\prime\prime}$ respectively. We are required to express ξ, v, ζ in terms of $\xi_{\prime}$, $v_{\prime}$, $\zeta_{\prime}$.

Fig. 15.



We shall previously give an expression for the cosine of the angle contained between two lines drawn from the origin. Let r and $r_{\prime}$ be the lines, ϕ the angle between them, D the distance between their extremities, λ, μ, ν, $\lambda_{\prime}$, $\mu_{\prime}$, $\nu_{\prime}$ the angles they make with the axes of coordinates. Then

$$D^2 = r^2 + r_{\prime}^2 - 2rr_{\prime}\cos\phi,$$

and $$D^2 = (x - x_{\prime})^2 + (y - y_{\prime})^2 + (z - z_{\prime})^2,$$

or $$D^2 = x^2 + y^2 + z^2 + x_{\prime}^2 + y_{\prime}^2 + z_{\prime}^2 - 2(xx_{\prime} + yy_{\prime} + zz_{\prime});$$

but $$r^2 = x^2 + y^2 + z^2, \quad r_{\prime}^2 = x_{\prime}^2 + y_{\prime}^2 + z_{\prime}^2.$$

Equating the values of D,

$$\cos\phi = \frac{xx_{\prime}}{rr_{\prime}} + \frac{yy_{\prime}}{rr_{\prime}} + \frac{zz_{\prime}}{rr_{\prime}} = \cos\lambda\cos\lambda_{\prime} + \cos\mu\cos\mu_{\prime} + \cos\nu\cos\nu_{\prime}. \quad \text{(c)}$$

When ϕ is a right angle,

$$\cos\lambda\cos\lambda_{\prime} + \cos\mu\cos\mu_{\prime} + \cos\nu\cos\nu_{\prime} = 0;$$

and when $\phi = 0$,

$$\cos^2\lambda + \cos^2\mu + \cos^2\nu = 1.$$

Let P be the perpendicular let fall from the origin on the given limiting plane, then the angles which the axis of X makes with the three new axes are λ, $\lambda_{\prime}$, $\lambda_{\prime\prime}$, and the cosines of the angles which the perpendicular P makes with the same three axes are $P\xi_{\prime}$, $Pv_{\prime}$, $P\zeta_{\prime}$, while $P\xi$ is the cosine of the angle between P and X; hence from (c)

$$P\xi = P\xi_{\prime}\cos\lambda + Pv_{\prime}\cos\lambda_{\prime} + P\zeta_{\prime}\cos\lambda_{\prime\prime},$$

or, dividing by the common factor P,

$$\xi = \xi_{\prime}\cos\lambda + v_{\prime}\cos\lambda_{\prime} + \zeta_{\prime}\cos\lambda_{\prime\prime}.$$

By the help of the following formulæ we may express the values of the original coordinates in terms of those derived from them; thus

$$\left.\begin{aligned}\xi &= \cos\lambda.\xi_{,} + \cos\lambda_{,}.\upsilon_{,} + \cos\lambda_{,,}.\zeta_{,},\\ \upsilon &= \cos\mu.\xi_{,} + \cos\mu_{,}.\upsilon_{,} + \cos\mu_{,,}.\zeta_{,},\\ \zeta &= \cos\nu.\xi_{,} + \cos\nu_{,}.\upsilon_{,} + \cos\nu_{,,}.\zeta_{,}.\end{aligned}\right\} \quad \dots\dots \quad (d)$$

Square these equations and add them, bearing in mind that $\xi^2+\upsilon^2+\zeta^2=\xi_{,}^2+\upsilon_{,}^2+\zeta_{,}^2$, since each is the value of the square of the reciprocal of the same perpendicular P, let fall from the origin on the plane whose tangential coordinates are ξ, υ, ζ, and also $\xi_{,}$, $\upsilon_{,}$, $\zeta_{,}$.

Hence $\cos^2\lambda+\cos^2\mu+\cos^2\nu=1$, $\cos^2\lambda_{,}+\cos^2\mu_{,}+\cos^2\nu_{,}=1$, $\cos^2\lambda_{,,}+\cos^2\mu_{,,}+\cos^2\nu_{,,}=1$, and

$$\left.\begin{aligned}\cos\lambda\cos\lambda_{,} + \cos\mu\cos\mu_{,} + \cos\nu\cos\nu_{,} &= 0,\\ \cos\lambda\cos\lambda_{,,} + \cos\mu\cos\mu_{,,} + \cos\nu\cos\nu_{,,} &= 0,\\ \cos\lambda_{,}\cos\lambda_{,,} + \cos\mu_{,}\cos\mu_{,,} + \cos\nu_{,}\cos\nu_{,,} &= 0.\end{aligned}\right\} \quad \dots \quad (e)$$

Again, since λ, μ, ν are the angles which the axis of $X_{,}$ makes with the original axes, and since P makes with the same axes angles whose cosines are $P\xi$, $P\upsilon$, $P\zeta$, we shall have for the cosine of the angle between P and $X_{,}$ the expression $P\xi_{,}$; hence

$$P\xi_{,} = P\xi\cos\lambda + P\upsilon\cos\mu + P\zeta\cos\nu\,;$$

or, dividing by P,

$$\left.\begin{aligned}\xi_{,} &= \cos\lambda.\xi + \cos\mu.\upsilon + \cos\nu.\zeta,\\ \upsilon_{,} &= \cos\lambda_{,}.\xi + \cos\mu_{,}.\upsilon + \cos\nu_{,}.\zeta,\\ \zeta_{,} &= \cos\lambda_{,,}.\xi + \cos\mu_{,,}.\upsilon + \cos\nu_{,,}.\zeta.\end{aligned}\right\} \quad \dots\dots \quad (f)$$

Here we must also necessarily have $\cos^2\lambda+\cos^2\lambda_{,}+\cos^2\lambda_{,,}=1$, $\cos^2\mu+\cos^2\mu_{,}+\cos^2\mu_{,,}=1$, $\cos^2\nu+\cos^2\nu_{,}+\cos^2\nu_{,,}=1$, and

$$\left.\begin{aligned}\cos\lambda\cos\mu + \cos\lambda_{,}\cos\mu_{,} + \cos\lambda_{,,}\cos\mu_{,,} &= 0,\\ \cos\lambda\cos\nu + \cos\lambda_{,}\cos\nu_{,} + \cos\lambda_{,,}\cos\nu_{,,} &= 0,\\ \cos\mu\cos\nu + \cos\mu_{,}\cos\nu_{,} + \cos\mu_{,,}\cos\nu_{,,} &= 0.\end{aligned}\right\} \quad \dots \quad (g)$$

Several demonstrations have been given of the preceding relations between the nine direction cosines; but nothing can well be simpler or more elementary than the above.

57.] To find an expression for the perpendicular from the origin on the plane of which the tangential coordinates are ξ, υ, ζ. Let P make the angles λ, μ, ν with the axes, then

$$\cos^2\lambda+\cos^2\mu+\cos^2\nu=1\,;$$

but $\qquad \cos\lambda=P\xi, \quad \cos\mu=P\upsilon, \quad \cos\nu=P\zeta;$

hence $\qquad P^2(\xi^2+\upsilon^2+\zeta^2)=1, \text{ or } \xi^2+\upsilon^2+\zeta^2=\frac{1}{P^2}. \quad \dots \quad (a)$

To determine the area of the triangle whose vertices are the three points in which a plane is pierced by the axes O X, O Y, O Z.

Let S be twice the area of this triangle, P the perpendicular on it from the origin. Then the solid contents of the pyramid of which the three rectangular edges are $\frac{1}{\xi}, \frac{1}{\upsilon}, \frac{1}{\zeta}$ is $\frac{1}{6\xi\upsilon\zeta}$; but it is also $\frac{\text{SP}}{6}$; hence

$$\text{S}=\frac{\{\xi^2+\upsilon^2+\zeta^2\}^{\frac{1}{2}}}{\xi\upsilon\zeta} \text{ or } \text{S}=\left\{\frac{1}{\zeta^2\upsilon^2}+\frac{1}{\xi^2\zeta^2}+\frac{1}{\xi^2\upsilon^2}\right\}^{\frac{1}{2}} . \quad . \quad . \quad \text{(b)}$$

To determine an expression for the perpendicular let fall on a plane from a point of which the projective coordinates are p, q, r.

Let ξ, υ, ζ be the tangential coordinates of the given plane; let a point be assumed of which the projective coordinates are p, q, r; then the volume of the pyramid of which the faces are the three coordinate planes and the limiting tangential plane is manifestly $=\frac{1}{6\xi\upsilon\zeta}$. Let P be the perpendicular on the tangential plane from the point O; then the volume of the pyramid is also equal to the sum of the four pyramids of which the altitudes are p, q, r, and P, while the bases are the triangles made by the axes of coordinates and the limiting plane; and it has been shown that the area of the triangle of which the vertices are X, Y, Z is $\left\{\frac{1}{\zeta^2\upsilon^2}+\frac{1}{\xi^2\zeta^2}+\frac{1}{\xi^2\upsilon^2}\right\}^{\frac{1}{2}}$. Equating the volume of the whole pyramid with the sum of the volumes of the four component pyramids,

$$\frac{1}{6\xi\upsilon\zeta}=\frac{1}{6}\left(\frac{r}{\zeta\upsilon}+\frac{q}{\xi\zeta}+\frac{p}{\xi\upsilon}\right)+\frac{1}{6}\,\text{P}\left\{\frac{1}{\zeta^2\upsilon^2}+\frac{1}{\xi^2\zeta^2}+\frac{1}{\xi^2\upsilon^2}\right\}^{\frac{1}{2}}$$

Multiply by $6\xi\upsilon\zeta$, and the resulting equation becomes

$$1=p\xi+q\upsilon+r\zeta+\text{P}\{\xi^2+\upsilon^2+\zeta^2\}^{\frac{1}{2}}.$$

Hence, finally,

$$\text{P}=\frac{1-p\xi-q\upsilon-r\zeta}{\{\xi^2+\upsilon^2+\zeta^2\}^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

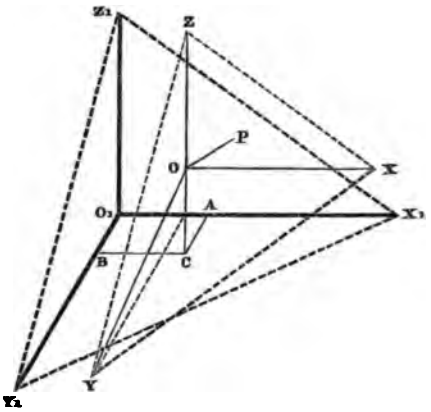
When the perpendicular is let fall from the origin $\text{O}_{\prime}$, $p=0$, $q=0$, $r=0$, and

$$\text{P}=\frac{1}{\{\xi^2+\upsilon^2+\zeta^2\}^{\frac{1}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

58.] We may use the preceding formulæ obtained for the perpendiculars let fall from the points O and $\text{O}_{\prime}$ to determine the translation of the coordinate planes in parallel directions.

Let the coordinates, through O, of the given plane be ξ, υ, ζ; and let the coordinates of the same plane passing through the point $\text{O}_{\prime}$ be $\xi_{\prime}$, $\upsilon_{\prime}$, $\zeta_{\prime}$; and let the projective coordinates of the point O on these planes be p, q, r.

Fig. 16.



Now the perpendicular P from the point O on the given limiting plane is

$$\frac{1-p\xi_{,}-qv_{,}-r\zeta_{,}}{\{\xi_{,}^2+v_{,}^2+\zeta_{,}^2\}^{\frac{1}{2}}},$$

and the perpendicular $P_{,}$ from the new origin $O_{,}$ is

$$\frac{1}{\{\xi_{,}^2+v_{,}^2+\zeta_{,}^2\}^{\frac{1}{2}}},$$

hence

$$\frac{P}{P_{,}}=1-p\xi_{,}-q_{,}v-r\zeta;$$

but it is manifest that these perpendiculars on the limiting plane from the points O and $O_{,}$ are proportional to the segments of the parallel axes of coordinates, through the same points, or $\frac{1}{\xi}\div\frac{1}{\xi_{,}}=\frac{P}{P_{,}}$ or $\xi=\frac{P_{,}}{P}\xi_{,}$. Hence

$$\xi=\frac{\xi_{,}}{1-p\xi_{,}-qv_{,}-r\zeta_{,}},\quad v=\frac{v_{,}}{1-p\xi_{,}-qv_{,}-r\zeta_{,}},\quad \zeta=\frac{\zeta_{,}}{1-p\xi_{,}-qv_{,}-r\zeta_{,}}. \qquad \text{(a)}$$

In like manner, by the help of (d) and (f) in sec. [56] we can always turn the axes of coordinates through certain given angles, and then translate them in parallel directions to another origin.

59.] *On the tangential equations of a plane passing through the origin of coordinates.*

When the plane, whose position is to be determined, passes through the origin of coordinates, its tangential coordinates become $\xi=\infty$, $v=\infty$, $\zeta=\infty$, values which are illusory. Yet the plane must have a determinate position in space; how is this position to be ascertained?

Let P, a perpendicular to the plane, drawn through the origin, make the angles λ, μ, ν with the axes of coordinates; and let the intersections or *traces* of this plane with two of the coordinate planes (ZX and ZY) suppose make the angles χ and ψ with the axes of X and Y. Then the angles which this trace on the plane of ZX makes with the axes of coordinates OX, OY, OZ are χ, $\frac{\pi}{2}$, and $\left(\frac{\pi}{2}-\chi\right)$.

But as this trace is also in the plane whose position is to be determined, P and this trace must be at right angles. Hence

$$\cos\lambda\,\cos\chi+\cos\mu\,\cos\frac{\pi}{2}+\cos\nu\,\cos\left(\frac{\pi}{2}-\chi\right)=0,$$

or
$$\cos\lambda\,\cos\chi+\cos\nu\,\sin\chi=0.$$

But $\frac{\cos\lambda}{\cos\nu}=\frac{\xi}{\zeta}$; hence $\frac{\xi}{\zeta}+\tan\chi=0$. In like manner, $\frac{\upsilon}{\zeta}+\tan\psi=0$.

Consequently, when the plane whose position is to be determined passes through the origin, the ratios of the tangential coordinates $\xi\div\zeta$ and $\upsilon\div\zeta$ denote the tangents of the angles which the traces of the plane make with the axes of X and Y. Hence the position of the plane passing through the origin may be determined by the equations

$$\xi+\tan\chi\,.\,\zeta=0, \quad \upsilon+\tan\psi\,.\,\zeta=0. \;.\;.\;.\;.\; \text{(a)}$$

CHAPTER VI.

ON THE TANGENTIAL EQUATIONS OF THE STRAIGHT LINE IN SPACE.

60.] In defining the position of a straight line given in space, there are two methods we may follow. We may conceive of any two planes out of three passing through the given line and perpendicular to two of the coordinate planes; and the traces on these coordinate planes made by the perpendicular planes let fall through the given line enable us, by conceiving planes to be erected on these traces, to determine the position of the straight line in space.

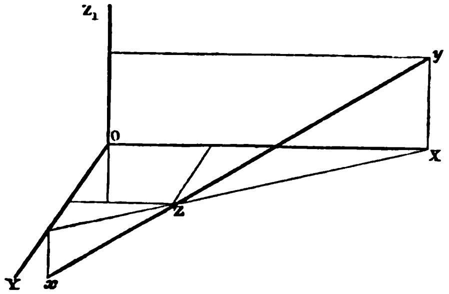
But there is another method we may follow. Instead of passing *planes* through the straight line, perpendicular to the coordinate planes, we may determine the *points* in which the straight line *pierces* the coordinate planes; and if we can ascertain any *two* of these *three* points on the coordinate planes, we can fix the position of the straight line. The former is the projective, the latter is the tangential method. Thus the *three points* which are on the straight line, and also on the coordinate planes, are analogous to the *three* *planes* all passing through the straight line and perpendicular to the three coordinate planes.

61.] Let $p, q, p_{\prime}, r, q_{\prime}, r_{\prime}$ denote the projective coordinates of the three points z, y, x in the three coordinate planes XY, XZ, and YZ through which the straight line passes. Then the tangential equations of these points in the straight line will be

$$p\xi+qv=1,\ p_{\prime}\xi+r\zeta=1,\ \text{and}\ q_{\prime}v+r_{\prime}\zeta=1. \quad . \quad . \quad . \quad \text{(a)}$$

We may deduce any one of these from the other two. Thus, eliminating ξ from the first two, we get for the third $\frac{p_{\prime}qv}{p_{\prime}-p}-\frac{pr\zeta}{p_{\prime}-p}=1$. This formula may be shown very simply by a diagram.

Fig. 17.



It will simplify the notation if, instead of writing the tangential equations in the normal form, we put them under the forms

$$\xi=\mu\zeta+a, \quad v=\nu\zeta+\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

If we compare these forms with those given above,

$$\xi=-\frac{r}{p_{\prime}}\zeta+\frac{1}{p_{\prime}}\ \text{and}\ v=-\frac{r_{\prime}}{q_{\prime}}\zeta+\frac{1}{q_{\prime}}, \quad . \quad . \quad . \quad . \quad \text{(c)}$$

we shall see that $\mu=-\frac{r}{p_{\prime}}=$tangent of the angle which the line drawn to the origin from the point in which the straight line pierces the plane of XZ makes with the axis of X, while a is the reciprocal of the projective coordinate of the same point on the axis of X; and like values may be found for the constants in the planes of the tangential equations of the points in YZ and XY.

It is obvious that to determine the position of the straight line two equations are required, namely the tangential equations of two of the three points in which the coordinate planes are pierced by the given straight line. While one tangential equation determines the point, three are necessary to fix the tangential plane.

62.] *To express the cosines of the angles which a straight line makes with the axes of coordinates in terms of the constants μ, ν, a, β of the tangential equations of the given straight line,*

$$\xi=\mu\zeta+a, \quad \upsilon=\nu\zeta+\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Assume the general tangential equations of the points in which the straight line pierces the planes of XY, XZ, and ZY, namely

$$p\xi+q\upsilon=1, \; p_i\xi+r\zeta=1, \text{ and } \upsilon=\frac{pr}{p_iq}\zeta+\frac{p_i-p}{p_iq}, \quad . \quad . \quad \text{(b)}$$

derived from the two preceding.

Comparing the coefficients of these equations with those of (a), we shall have

$$\mu=-\frac{r}{p_i}, \quad ap_i=1, \quad \nu=\frac{pr}{p_iq}, \quad \beta=\frac{p_i-p}{p_iq},$$

or

$$\mu p_i+r=0, \quad ap_i=1, \quad p\mu+q\nu=0, \quad ap+\beta q=1. \quad . \quad \text{(c)}$$

Hence

$$\left.\begin{aligned} &q=\frac{\mu}{\beta\mu-a\nu}, \quad p=\frac{-\nu}{\beta\mu-a\nu}, \quad p_i=\frac{1}{a}, \\ &p_i-p=\frac{\beta\mu}{a(\beta\mu-a\nu)}, \quad \mu+ar=0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(d)}$$

If we now turn to the diagram in page 55, a short inspection will show (denoting the angles which a parallel line through the origin makes with the axes of coordinates by the letters lOZ, lOY, lOX) that

$$\left.\begin{aligned} &\cos l\text{OZ}=\frac{r}{\{r^2+q^2+(p_i-p)^2\}^{\frac{1}{2}}}, \quad \cos l\text{OY}=\frac{q}{\{r^2+q^2+(p_i-p)^2\}^{\frac{1}{2}}}, \\ &\cos l\text{OX}=\frac{p_i-p}{\{r^2+q^2+(p_i-p)^2\}^{\frac{1}{2}}}. \end{aligned}\right\}\text{(e)}$$

Hence, substituting the values of p, q, r, p_i, as given in the preceding equations (d), we shall have, putting

$$\{a^2+\beta^2+(\beta\mu-a\nu)^2\}^{\frac{1}{2}}=\Delta,$$

$$\cos l\text{OZ}=\frac{-(\beta\mu-a\nu)}{\Delta}, \quad \cos l\text{OY}=\frac{-a}{\Delta}, \quad \cos l\text{OX}=\frac{\beta}{\Delta}. \quad . \quad \text{(f)}$$

63.] *To determine the conditions that two straight lines may meet in space.*

When two straight lines meet, a plane can always be drawn through them. Let

$$\xi=\mu\zeta+a, \; \upsilon=\nu\zeta+\beta; \; \xi=\mu_i\zeta+a_i \text{ and } \upsilon=\nu_i\zeta+\beta_i \quad . \quad . \quad \text{(a)}$$

be the tangential equations of the two given straight lines, and let ξ, υ, ζ be the tangential coordinates of the plane which passes through both straight lines. Then, from the four preceding equations, eliminating the three quantities ξ, υ, ζ, we shall have the following equation of condition between the eight constants of the two given lines,

$$\frac{\alpha-\alpha_{\prime}}{\beta-\beta_{\prime}}=\frac{\mu-\mu_{\prime}}{\nu-\nu_{\prime}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

64.] We may also define a straight line as the locus of the points in which two given planes intersect.

As the straight line is wholly in one of the planes, this plane will pass through the three points in which the straight line pierces the coordinate planes, and the tangential equations of these points will be satisfied by the tangential coordinates of the two given planes.

Hence, the equations of two of these points being

$$\xi=\mu\zeta+\alpha \text{ and } \upsilon=\nu\zeta+\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

let the tangential coordinates of the two given planes be $\xi_{\prime}$, $\upsilon_{\prime}$, $\zeta_{\prime}$ and $\xi_{\prime\prime}$, $\upsilon_{\prime\prime}$, $\zeta_{\prime\prime}$.

Now, as the former plane intersects the plane of XZ in the point where it is pierced by the common line of intersection of the two planes, the equation (a) will be satisfied by the coordinates of the given planes. Consequently $\xi_{\prime}=\mu\zeta_{\prime}+\alpha$, and also $\xi_{\prime\prime}=\mu\zeta_{\prime\prime}+\alpha$; hence $\mu=\frac{\xi_{\prime}-\xi_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}}$; but we have also $\xi-\xi_{\prime}=\mu(\zeta-\zeta_{\prime})$.

Hence the tangential equation (a) becomes

$$\frac{\xi-\xi_{\prime}}{\zeta-\zeta_{\prime}}=\frac{\xi_{\prime}-\xi_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}}. \quad \text{In like manner } \frac{\upsilon-\upsilon_{\prime}}{\zeta-\zeta_{\prime}}=\frac{\upsilon_{\prime}-\upsilon_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}}.^{*} \quad . \quad \text{(b)}$$

65.] *To investigate the conditions that a given line may be found in a given plane.*

Let the tangential equations of the straight line be

$$\xi=\mu\zeta+\alpha, \quad \upsilon=\nu\zeta+\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let $\xi_{\prime}$, $\upsilon_{\prime}$, $\zeta_{\prime}$ be the required coordinates to determine the position of the plane. Then, substituting the coordinates of this plane in the equations of the straight line,

$$\xi_{\prime}=\mu\zeta_{\prime}+\alpha, \quad \upsilon_{\prime}=\nu\zeta_{\prime}+\beta.$$

But there are only two equations to determine the three unknown quantities $\xi_{\prime}$, $\upsilon_{\prime}$, $\zeta_{\prime}$. Hence the problem is indeterminate, as is antecedently manifest.

66.] *To find the tangential coordinates of a plane which shall pass through a given point and a given straight line.*

Let p, q, r be the projective coordinates of the given point;

---

[*] It is obvious that these equations of a *line*, the intersection of *two given planes*, are analogous to the projective equations of a *line* joining *two given points*, namely

$$\frac{x-x_{\prime}}{z-z_{\prime}}=\frac{x_{\prime}-x_{\prime\prime}}{z_{\prime}-z_{\prime\prime}} \text{ and } \frac{y-y'}{z-z_{\prime}}=\frac{y_{\prime}-y_{\prime\prime}}{z_{\prime}-z_{\prime\prime}}.$$

then the tangential equation of the given point may be written $p\xi+q\upsilon+r\zeta=1$.

Let $\xi=\mu\zeta+\alpha$, $\upsilon=\nu\zeta+\beta$ be the tangential equations of the points in the planes of XZ and YZ where they are pierced by the straight line; then the plane must of necessity pass through these points, if the given line is to be in the plane; hence, substituting the values of ξ and υ, we get

$$(p\mu+q\nu+r)\zeta=1-p\alpha-q\beta: \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

this equation will determine the value of ζ; hence those of ξ and υ may be obtained.

When the point is in the straight line, ζ must be indeterminate, or

$$p\mu+q\nu+r=0, \quad 1-p\alpha-q\beta=0. \quad . \quad . \quad . \quad . \quad \text{(b)}$$

67.] *To determine the angle between two given planes.*

Let $\xi_{,}$, $\upsilon_{,}$, $\zeta_{,}$ and $\xi_{,,}$, $\upsilon_{,,}$, and $\zeta_{,,}$ be the tangential coordinates of the two given planes. Let fall two perpendiculars P and $P_{,}$ from the origin, making the angles λ, μ, ν and $\lambda_{,}$, $\mu_{,}$, $\nu_{,}$ with the axes.

Then, ϕ being the angle between the planes or between the perpendiculars to them,

$$\cos\phi=\cos\lambda\cos\lambda_{,}+\cos\mu\cos\mu_{,}+\cos\nu\cos\nu_{,}. \quad . \quad . \quad \text{(a)}$$

Now, as $P\xi_{,}=\cos\lambda$, $P_{,}\xi_{,,}=\cos\lambda_{,}$, and making like substitutions for the other angles, we find

$$PP_{,}\cos\phi=\xi_{,}\xi_{,,}+\upsilon_{,}\upsilon_{,,}+\zeta_{,}\zeta_{,,},$$

$$\cos\phi=\frac{\xi_{,}\xi_{,,}+\upsilon_{,}\upsilon_{,,}+\zeta_{,}\zeta_{,,}}{\sqrt{\xi_{,}^{2}+\upsilon_{,}^{2}+\zeta_{,}^{2}}\sqrt{\xi_{,,}^{2}+\upsilon_{,,}^{2}+\zeta_{,,}^{2}}}. \quad . \quad . \quad \text{(b)}$$

68.] *A straight line is perpendicular to a given plane, to determine the relations between the coefficients of the given straight line and the given plane.*

Let $\xi=\mu\zeta+\alpha$, and $\upsilon=\nu\zeta+\beta$, be the equations of the straight line, and let $\xi_{,}$, $\upsilon_{,}$ $\zeta_{,}$ be the tangential coordinates of the given plane. Then, as a perpendicular to this plane through the origin is parallel to the straight line, the angle between them is $=0$.

Now the cosines of the angles which P makes with the axes of coordinates are $\xi_{,}$, $\upsilon_{,}$, $\zeta_{,}$ divided severally by $\{\xi_{,}^{2}+\upsilon_{,}^{2}+\zeta_{,}^{2}\}^{\frac{1}{2}}$; and the cosines of the angles which the straight line makes with the same axes are β, $-\alpha$, $-\gamma$, divided severally by $\{\alpha^{2}+\beta^{2}+\gamma^{2}\}^{\frac{1}{2}}=\Delta$, putting $-\gamma$ for $\beta\mu-\alpha\nu$. Hence the direction cosines may be written

$$\frac{\beta}{\Delta}, \ \frac{-\alpha}{\Delta}, \text{ and } \frac{-\gamma}{\Delta}, \text{ or } 1=\frac{\beta\xi_{,}-\alpha\upsilon_{,}-\gamma\zeta_{,}}{\sqrt{\alpha^{2}+\beta^{2}+\gamma^{2}}\sqrt{\xi_{,}^{2}+\upsilon_{,}^{2}+\zeta_{,}^{2}}}; \quad \text{(a)}$$

or, reducing this expression, we shall have

$$(\alpha\xi_{,}+\beta\upsilon_{,})^{2}+(\alpha\zeta_{,}-\gamma\upsilon_{,})^{2}+(\gamma\xi_{,}+\beta\zeta_{,})^{2}=0. \quad . \quad . \quad \text{(b)}$$

Hence, as this expression is the sum of three squares, we must have each term separately equal to 0, or

$$a\xi_{\prime}+\beta\upsilon_{\prime}=0,\quad a\zeta_{\prime}-\gamma\upsilon_{\prime}=0,\quad \gamma\xi_{\prime}+\beta\zeta_{\prime}=0.\quad .\ .\ .\ \text{(c)}$$

These are the relations that exist between the coefficients in order that a given line may be perpendicular to a given plane.

We may obtain a much shorter solution of this problem, but one neither so simple nor so elegant.

Let $\xi_{\prime}$, $\upsilon_{\prime}$, $\zeta_{\prime}$ be the coordinates of the given plane, and $\xi=\mu\zeta+a$, $\upsilon=\nu\zeta+\beta$ the tangential equations of the given straight line.

Then as any plane which passes through this straight line will be at right angles to the given plane, we must have $\xi\xi_{\prime}+\upsilon\upsilon_{\prime}+\zeta\zeta_{\prime}=0$, but $\xi=\mu\zeta+a$, $\upsilon=\nu\zeta+\beta$; substituting,

$$(\xi_{\prime}\mu+\upsilon_{\prime}\nu+\zeta_{\prime})\zeta+a\xi_{\prime}+\beta\upsilon_{\prime}=0;$$

but as the plane through the given straight line is manifestly indeterminate, we must have

$$\xi_{\prime}\mu+\upsilon_{\prime}\nu+\zeta_{\prime}=0,\quad a\xi_{\prime}+\beta\upsilon_{\prime}=0,\ .\ .\ .\ .\ .\ \text{(d)}$$

or two equations between the four constants of the equations of the given straight line.

69.] *To determine the angles which the straight line, in which two given planes intersect, makes with the axes of coordinates.*

Let the tangential coordinates of the two given planes be $\xi_{\prime}$, $\upsilon_{\prime}$, $\zeta_{\prime}$ and $\xi_{\prime\prime}$, $\upsilon_{\prime\prime}$, $\zeta_{\prime\prime}$; then the tangential equations of the straight line in which they intersect are, as shown in (b), sec. [64],

$$\xi-\xi_{\prime}=\frac{\xi_{\prime}-\xi_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}}(\zeta-\zeta_{\prime})\ \text{ and }\ \upsilon-\upsilon_{\prime}=\frac{\upsilon_{\prime}-\upsilon_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}}(\zeta-\zeta_{\prime}).$$

Now, if we compare these equations with the general tangential equations of a straight line,

$$\xi=\mu\zeta+a,\quad \upsilon=\nu\zeta+\beta,$$

we shall have

$$\left.\begin{aligned}&\mu=\frac{\xi_{\prime}-\xi_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}},\ \nu=\frac{\upsilon_{\prime}-\upsilon_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}};\ a=\frac{\xi_{\prime\prime}\zeta_{\prime}-\xi_{\prime}\zeta_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}},\ \beta=\frac{\upsilon_{\prime\prime}\zeta_{\prime}-\upsilon_{\prime}\zeta_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}},\\ \text{and}\quad &-\gamma=(\beta\mu-a\nu)=\frac{\xi_{\prime\prime}\upsilon_{\prime}-\xi_{\prime}\upsilon_{\prime\prime}}{\zeta_{\prime}-\zeta_{\prime\prime}}.\end{aligned}\right\}\quad \text{(a)}$$

It is easy to show that perpendiculars to the two intersecting planes are perpendiculars to their intersection; for as

$$\cos l\text{OX}=\frac{\beta}{\Delta},\ \cos l\text{OY}=\frac{-a}{\Delta},\ \cos l\text{OZ}=\frac{-\gamma}{\Delta}.$$

Substituting the values of β, a, γ given in (a), we shall have

$$\cos l\text{OX}=\frac{\upsilon_{\prime\prime}\zeta_{\prime}-\upsilon_{\prime}\zeta_{\prime\prime}}{\Delta},\ \cos l\text{OY}=\frac{\xi_{\prime}\zeta_{\prime\prime}-\xi_{\prime\prime}\zeta_{\prime}}{\Delta},\ \cos l\text{OZ}=\frac{\xi_{\prime\prime}\upsilon_{\prime}-\xi_{\prime}\upsilon_{\prime\prime}}{\Delta}.$$

Now the perpendicular to one of the planes makes angles with the axes of coordinates, whose cosines are $\frac{\xi_{\prime}}{\Pi}$, $\frac{\upsilon_{\prime}}{\Pi}$, $\frac{\zeta_{\prime}}{\Pi}$, where Π is the reciprocal of P.

Hence ω being the angle between this perpendicular and the straight line in the two planes, we shall have

$$\cos\omega = \frac{\xi_{\prime}\upsilon_{\prime\prime}\zeta_{\prime} - \xi_{\prime}\upsilon_{\prime}\zeta_{\prime\prime} + \upsilon_{\prime}\xi_{\prime}\zeta_{\prime\prime} - \upsilon_{\prime}\xi_{\prime\prime}\zeta_{\prime} + \zeta_{\prime}\xi_{\prime\prime}\upsilon_{\prime} - \zeta_{\prime}\xi_{\prime}\upsilon_{\prime\prime}}{\Delta\Pi},$$

the numerator of which is identically $=0$; hence ω is a right angle.

We should have found a like result had we multiplied by $\xi_{\prime\prime}$, $\upsilon_{\prime\prime}$, $\zeta_{\prime\prime}$ the tangential coordinates of the second intersecting plane.

70.] *To determine the conditions in order that a given straight line and a given plane may be parallel.*

Let $\xi_{\prime}$, $\upsilon_{\prime}$, $\zeta_{\prime}$ be the tangential coordinates of the given plane, $\xi=\mu\zeta+a$, $\upsilon=\nu\zeta+\beta$ the equations of the given straight line. Now a plane which is perpendicular to this straight line will also be perpendicular to the given plane.

Let the coordinates of the second plane which is to be perpendicular to the given straight line be $\xi_{\prime\prime}$, $\upsilon_{\prime\prime}$, $\zeta_{\prime\prime}$. This plane and the given straight line will be perpendicular, see (d), sec. [68], where

$$\mu\xi_{\prime\prime}+\nu\upsilon_{\prime\prime}+\zeta_{\prime\prime}=0 \text{ and } a\xi_{\prime\prime}+\beta\upsilon_{\prime\prime}=0;$$

hence

$$\xi_{\prime\prime}=-\frac{\beta}{a}\upsilon_{\prime\prime} \text{ and } \zeta_{\prime\prime}=\left(\frac{\beta\mu-a\nu}{a}\right)\upsilon_{\prime\prime};$$

but as these two planes must be at right angles one to the other, we must have

$$\xi_{\prime}\xi_{\prime\prime}+\upsilon_{\prime}\upsilon_{\prime\prime}+\zeta_{\prime}\zeta_{\prime\prime}=0.$$

Substituting the values of $\xi_{\prime\prime}$ and $\zeta_{\prime\prime}$, we get

$$\beta\xi_{\prime}-a\upsilon_{\prime}-(\beta\mu-a\nu)\zeta_{\prime}=0. \quad . \quad . \quad . \quad . \quad . \quad (a)$$

71.] *A straight line is parallel to a given straight line, to determine the relation between the constants.*

Let $\xi=\mu\zeta+a$, $\upsilon=\nu\zeta+\beta$ be the equations of one of these lines, and $\xi=\mu_{\prime}\zeta+a_{\prime}$, $\upsilon=\nu_{\prime}\zeta+\beta_{\prime}$ those of the other; then, as these straight lines are perpendicular to the same plane, we must have

$$\mu\xi_{\prime}+\nu\upsilon_{\prime}+\zeta_{\prime}=0, \qquad a\xi_{\prime}+\beta\upsilon_{\prime}=0,$$
$$\mu_{\prime}\xi_{\prime}+\nu_{\prime}\upsilon_{\prime}+\zeta_{\prime}=0, \qquad +a_{\prime}\xi_{\prime}+\beta_{\prime}\upsilon_{\prime}=0,$$

or

$$\frac{a}{\beta}=\frac{a_{\prime}}{\beta_{\prime}}=\frac{\mu-\mu_{\prime}}{\nu_{\prime}-\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

72.] *To investigate an expression for the angle between two given straight lines whose equations are*

$$\xi=\mu\zeta+a,\ \upsilon=\nu\zeta+\beta;\ \xi=\mu_{\prime}\zeta+a_{\prime},\ \upsilon=\nu_{\prime}\zeta+\beta_{\prime}.$$

Now the angle between two straight lines in space is equal to that between two planes at right angles to them; and as this angle is equal to that between the perpendiculars let fall from the origin upon these planes, we shall have

$$\cos\omega = \frac{\xi\xi_{\prime\prime} + \upsilon_{\prime}\upsilon_{\prime\prime} + \zeta_{\prime}\zeta_{\prime\prime}}{\sqrt{\xi_{\prime}^2 + \upsilon_{\prime}^2 + \zeta_{\prime}^2}\,\sqrt{\xi_{\prime\prime}^2 + \upsilon_{\prime\prime}^2 + \zeta_{\prime\prime}^2}} \quad . \quad . \quad . \quad \text{(a)}$$

We have now to determine the tangential coordinates of these two planes; and as these planes are given only in direction but not in position, we can only obtain the ratios of the tangential coordinates, but not the coordinates themselves. Let $\xi_{\prime} = \phi\zeta_{\prime}$, $\xi_{\prime\prime} = \phi_{\prime}\zeta_{\prime\prime}$, $\upsilon_{\prime} = \psi\zeta_{\prime}$, $\upsilon_{\prime\prime} = \psi_{\prime}\zeta_{\prime\prime}$, and the preceding equation becomes

$$\cos\omega = \frac{1 + \phi\phi_{\prime} + \psi\psi_{\prime}}{\sqrt{1 + \phi^2 + \psi^2}\,\sqrt{1 + \phi_{\prime}^2 + \psi_{\prime}^2}} \quad . \quad . \quad . \quad . \quad \text{(b)}$$

But it was shown in (d), sec. [68], that when a straight line is perpendicular to a plane, we must have

$$\left.\begin{array}{ll} \xi_{\prime}\mu + \upsilon_{\prime}\nu + \zeta = 0, & \alpha\xi_{\prime} + \beta\upsilon_{\prime} = 0, \\ \text{or} & \\ \phi\mu + \psi\nu + 1 = 0, & \phi\alpha + \psi\beta = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Hence

$$\phi = \frac{-\beta}{\beta\mu - \alpha\nu}, \quad \psi = \frac{+\alpha}{\beta\mu - \alpha\nu}; \quad \text{also } \phi_{\prime} = \frac{-\beta_{\prime}}{\beta_{\prime}\mu_{\prime} - \alpha_{\prime}\nu_{\prime}}, \quad \psi_{\prime} = \frac{+\alpha_{\prime}}{\beta_{\prime}\mu_{\prime} - \alpha_{\prime}\nu_{\prime}}.$$

Substituting,

$$\cos\omega = \frac{\alpha\alpha_{\prime} + \beta\beta_{\prime} + (\beta\mu - \alpha\nu)(\beta_{\prime}\mu_{\prime} - \alpha_{\prime}\nu_{\prime})}{\sqrt{\alpha^2 + \beta^2 + (\beta\mu - \alpha\nu)^2}\,\sqrt{\alpha_{\prime}^2 + \beta_{\prime}^2 + (\beta_{\prime}\mu_{\prime} - \alpha_{\prime}\nu_{\prime})^2}} \quad . \quad . \quad \text{(d)}$$

73.] *To find an expression for the angle between a given plane and a given straight line.*

Let $\xi = \mu\zeta + \alpha$, $\upsilon = \nu\zeta + \beta$ be the tangential equations of the given straight line, and $\xi_{\prime}$, $\upsilon_{\prime}$, $\zeta_{\prime}$ the coordinates of the given plane. Let a plane be drawn perpendicular to the given line, then the angle between these planes will be the complement of the angle between the given plane and the given line. But since this last drawn plane is perpendicular to the given line, we must have

$$\mu\xi + \nu\upsilon + \zeta = 0, \quad \alpha\xi + \beta\upsilon = 0, \quad \text{or } \xi = \frac{-\beta}{\alpha}\upsilon, \quad \zeta = \frac{(\beta\mu - \alpha\nu)}{\alpha}\upsilon.$$

Substituting these expressions in the general formula for the angle between two planes, we get

$$\sin\omega = \frac{\alpha\upsilon - \beta\xi + (\beta\mu - \alpha\nu)\zeta}{\sqrt{\alpha^2 + \beta^2 + (\beta\mu - \alpha\nu)^2}\,\sqrt{\xi_{\prime}^2 + \upsilon_{\prime}^2 + \zeta_{\prime}^2}}.$$

CHAPTER VII.

ON THE TANGENTIAL EQUATIONS OF SURFACES OF THE SECOND ORDER.

74.] When the surface is referred to its centre and axes, as origin and axes of coordinates, the transformation of projective into tangential coordinates exhibits no difficulty. Thus, let the projective equation of an ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

the equation of the tangent plane passing through the point $(x\,y\,z)$ is

$$\frac{xx_,}{a^2}+\frac{yy_,}{b^2}+\frac{zz_,}{c^2}=1,$$

the current coordinates being $x_,$, $y_,$, and $z_,$.

Let $y_,=0$, $z_,=0$, then $x_,=\frac{1}{\xi}$, and $\frac{xx_,}{a^2}=1$. Hence $\frac{x}{a}=\frac{a}{x_,}=a\xi$. In like manner $\frac{y}{b}=b\upsilon$, and $\frac{z}{c}=c\zeta$, and by substitution

$$a^2\xi^2+b^2\upsilon^2+c^2\zeta^2=1, \quad \ldots\ldots \quad \text{(a)}$$

which is the tangential equation of an ellipsoid referred to its axes.

Let us now refer the surface of the second order to any rectangular axes passing through the centre.

Let its equation, in projective coordinates, be

$$Ax^2+A_,y^2+A_{,,}z^2+2Bzy+2B_,xz+2B_{,,}xy=1. \quad \ldots \quad \text{(b)}$$

Then the equation of the tangent plane passing through the point $(x\,y\,z)$ on the surface, is

$$[Ax+B_,z+B_{,,}y]x_,+[A_,y+B_{,,}x+Bz]y_,+[A_{,,}z+By+B_,z]z_,=1, \text{(c)}$$

in which equation $(x\,y\,z)$ is the point of contact on the surface, and $x_,\,y_,\,z_,$ are the current coordinates.

Assume

$$Ax+B_,z+B_{,,}y=\xi,\ A_,y+B_{,,}x+Bz=\upsilon,\ A_{,,}z+By+B_,x=\zeta; \quad \text{(d)}$$

solving these linear equations for x, y, and z, putting

$$D=AB^2+A_,B_,^2+A_{,,}B_{,,}^2-AA_,A_{,,}-2BB_,B_{,,},$$

we shall obtain the resulting equations,

$$\left.\begin{aligned} Dx&=(B^2-A_,A_{,,})\xi+(A_{,,}B_{,,}-BB_,)\upsilon+(A_,B_,-BB_{,,})\zeta,\\ Dy&=(B_,^2-A_{,,}A)\upsilon+(AB-B_,B_{,,})\zeta+(A_{,,}B_{,,}-B_,B)\xi,\\ Dz&=(B_{,,}^2-A_,A)\zeta+(A_,B_,-B_{,,}B)\xi+(AB-B_,B_{,,})\upsilon;\end{aligned}\right\} \quad . \quad \text{(e)}$$

or dividing by D, and substituting single symbols for the foregoing expressions, we shall have

$$\left.\begin{aligned} x&=a\xi+\beta_{ll}\upsilon+\beta_{l}\zeta,\\ y&=a_{l}\upsilon+\beta\zeta+\beta_{ll}\xi,\\ z&=a_{ll}\zeta+\beta_{l}\xi+\beta\upsilon. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad \text{(f)}$$

Multiply these equations respectively by the three following expressions, namely

$$Ax+B_{l}y+B_{ll}z=\xi, \quad A_{l}y+B_{ll}z+Bx=\upsilon, \quad A_{ll}z+Bx+B_{l}z=\zeta,$$

and adding them, we thus obtain

$$\left.\begin{aligned} &Ax^2+A_{l}y^2+A_{ll}z^2+2Bxy+2B_{l}xz+2B_{l}xy=\\ &\quad a\xi^2+a_{l}\upsilon^2+a_{ll}\zeta^2+2\beta\zeta\upsilon+2\beta_{l}\xi\zeta+2\beta_{ll}\xi\upsilon=1. \end{aligned}\right\} \quad . \; . \quad \text{(g)}$$

These are the relations which exist between the projective and tangential equations of the same surface of the second order.

75.] Let the surface of the second order be now referred to an origin of tangential coordinates other than the centre O. Let the point Q be so assumed, and through this point Q let three rectangular planes be drawn, *parallel* to the original rectangular planes, through the centre O. Let p, q, r be the projective coordinates of the centre O let fall on the new coordinate planes. Let the tangential equation of the surface, the origin being at the centre, be as in (g) in the preceding section,

$$a\xi^2+a_{l}\upsilon^2+a_{ll}\zeta^2+2\beta\zeta\upsilon+2\beta_{l}\xi\zeta+2\beta_{ll}\xi\upsilon=1\,;$$

and, assuming the values of the old tangential coordinates in terms of the new as given in (a), sec. [58], namely

$$\xi=\frac{\xi_{l}}{1-p\xi_{l}-q\upsilon_{l}-r\zeta_{l}}, \quad \upsilon=\frac{\upsilon_{l}}{1-p\xi_{l}-q\upsilon_{l}-r\zeta_{l}}, \quad \zeta=\frac{\zeta_{l}}{1-p\xi_{l}-q\upsilon_{l}-r\zeta_{l}},$$

making these substitutions in the preceding equation, we obtain for the tangential equation of a surface of the second order referred to Q, any point in space,

$$\left.\begin{aligned} &(a-p^2)\xi^2+(a_{l}-q^2)\upsilon^2+(a_{ll}-r^2)\zeta^2+2(\beta-qr)\zeta\upsilon+2(\beta_{l}-pr)\xi\zeta\\ &\quad +2(\beta_{ll}-pq)\xi\upsilon+2p\xi+2q\upsilon+2r\zeta=1. \end{aligned}\right\} \quad \text{(a)}$$

It is hence obvious that in the general tangential equation of a central surface of the second order, referred to any coordinate planes in space, the coefficients of the linear terms ξ, υ, ζ will be twice the projective coordinates of the centre on the new coordinate planes.

This is a matter of much importance, as without any calculation at all we can discern the position of the centre. Hence, if the tangential equation of a central surface of the second order referred to any system of rectangular coordinates in space be

$$L\xi^2+L_{,}\upsilon^2+L_{,,}\zeta^2+2M\zeta\upsilon+2M_{,}\xi\zeta+2M_{,,}\xi\upsilon+2p\xi+2q\upsilon+2r\zeta=1, \quad \text{(b)}$$

then, comparing this equation with the preceding (a) and equating like terms, we shall have $L=a-p^2$, $L_{,}=a_{,}-q^2$, $L_{,,}=a_{,,}-r^2$, $M=\beta-qr$, $M_{,}=\beta_{,}-pr$, $M_{,,}=\beta_{,,}-pq$.

Hence we may *return* to the system of coordinates passing through the centre of the surface, if we substitute for the coefficients of that equation, namely a, $a_{,}$, $a_{,,}$, β, $\beta_{,}$, $\beta_{,,}$, their values just given—that is to say,

$$\left.\begin{aligned} &a=L+p^2,\ a_{,}=L_{,}+q^2,\ a_{,,}=L_{,,}+r^2, \\ &\beta=M+qr,\ \beta_{,}=M_{,}+pr,\ B_{,,}=M_{,,}+pq. \end{aligned}\right\} \quad . \quad . \quad . \quad \text{(c)}$$

So that, passing from the equation of the surface referred to *any* system of axes to another parallel system passing through the centre, the general equation (b) will become

$$\left.\begin{aligned} &(L+p^2)\xi^2+(L_{,}+q^2)\upsilon^2+(L_{,,}+r^2)\zeta^2+2(M+qr)\zeta\upsilon \\ &\quad +2(M_{,}+pr)\xi\zeta+2(M_{,,}+pq)\xi\upsilon=1. \end{aligned}\right\} \quad . \quad . \quad \text{(d)}$$

76.] Let $\frac{dF}{dx}(x-x_{,})+\frac{dF}{dy}(y-y_{,})+\frac{dF}{dz}(z-z_{,})=0$, . . . (a)

be the equation of a tangent plane to a surface of the second order, $f(x, y, z)=0$, in which $x_{,}\,y_{,}\,z_{,}$ are the current coordinates and F is written for brevity instead of $f(x, y, z)$.

Since $x_{,}\,y_{,}\,z_{,}$ are the current coordinates, let $y_{,}=0$, $z_{,}=0$, therefore $x_{,}$ is the distance from the origin of the point where the axis of X is cut by the tangent plane. Hence $x_{,}=\frac{1}{\xi}$, and we thus get

$$\xi=\frac{\frac{dF}{dx}}{\frac{dF}{dx}x+\frac{dF}{dy}y+\frac{dF}{dz}z}, \text{ and also } \upsilon=\frac{\frac{dF}{dy}}{\frac{dF}{dx}x+\frac{dF}{dy}y+\frac{dF}{dz}z}\,;\, . \quad \text{(b)}$$

and a like expression for ζ may be found.

Combining these equations with the dual equation

$$x\xi+y\upsilon+z\zeta=1,$$

we may obtain the tangential equation of the same surface.

Thus, let the projective equation of the surface of the second degree $f(x, y, z)=0$, referred to three rectangular coordinate planes, be

$$\left.\begin{aligned} &Ax^2+A_{,}y^2+A_{,,}z^2+2Byz+2B_{,}xz+2B_{,,}xy \\ &\quad +2Cx+2C_{,}y+2C_{,,}z=1\,; \end{aligned}\right\} . \quad . \quad . \quad \text{(c)}$$

taking the partial differential coefficients, we get

$$\left.\begin{aligned}&\frac{dF}{dx}=2[Ax+B_{\prime}z+B_{\prime\prime}y+C],\ \frac{dF}{dy}=2[A_{\prime}y+B_{\prime\prime}x+Bz+C_{\prime}],\\&\frac{dF}{dz}=2[A_{\prime\prime}z+By+B_{\prime}x+C_{\prime\prime}],\text{ and therefore}\\&\frac{dF}{dx}x+\frac{dF}{dy}y+\frac{dF}{dz}z=2[1-Cx-C_{\prime}y-C_{\prime\prime}z];\end{aligned}\right\}\ (d)$$

hence

$$\left.\begin{aligned}\xi&=\frac{Ax+B_{\prime}z+B_{\prime\prime}y+C}{1-Cx-C_{\prime}y-C_{\prime\prime}z},\\ \upsilon&=\frac{A_{\prime}y+B_{\prime\prime}x+Bz+C_{\prime}}{1-Cx-C_{\prime}y-C_{\prime\prime}z},\\ \zeta&=\frac{A_{\prime\prime}z+By+B_{\prime}x+C_{\prime\prime}}{1-Cx-C_{\prime}y-C_{\prime\prime}z}.\end{aligned}\right\}\ .\ .\ .\ .\ .\ .\ (e)$$

If now from these equations we derive the values of x, y, z in terms of ξ, υ, and ζ, and substitute their values in the dual equation $x\xi+y\upsilon+z\zeta=1$, we shall obtain as the result the tangential equation of the same surface—that is to say, $\phi_{\prime}(\xi,\upsilon,\zeta)=0$.

If we divide, one by the other, the equations (b), we shall obtain

$$\frac{\upsilon}{\xi}=\frac{\frac{dF}{dy}}{\frac{dF}{dx}},\ \frac{\zeta}{\xi}=\frac{\frac{dF}{dz}}{\frac{dF}{dx}}.$$

Square these equations, and add,

$$\frac{\xi^2+\upsilon^2+\zeta^2}{\xi^2}=\frac{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2}{\left(\frac{dF}{dx}\right)^2}.$$

Now $P^2(\xi^2+\upsilon^2+\zeta^2)=1$, and $P^2\xi^2=\cos^2\lambda$; hence

$$\cos^2\lambda=\frac{\left(\frac{dF}{dx}\right)^2}{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2};\ .\ .\ .\ .\ .\ (f)$$

and like expressions may be found for $\cos\mu$, $\cos\nu$.

Hence the angles which the perpendicular on the tangent plane makes with the axes of coordinates derive their expressions from the partial differentiation of the *projective* equation, and not from that of the *tangential* equation of the surface as we might have anticipated. We shall find that like expressions for the angles which the diameter passing through the point of contact with the tangent plane makes with the axes of coordinates are derived from the tangential equation of the surface.

77]. Resuming the general tangential equation of a surface of the second order,

$$a\xi^2+a_{,}v^2+a_{,,}\zeta^2+2\beta v\zeta+2\beta_{,}\xi\zeta+2\beta_{,,}\xi v+2\gamma\xi+2\gamma_{,}v+2\gamma_{,,}\zeta=1, \quad \text{(a)}$$

let us solve this equation for one of the variables, ζ suppose, and the resulting equation becomes

$$\zeta=-\frac{(\beta v+\beta_{,}\xi+\gamma_{,,})}{a_{,,}}\pm\sqrt{\mathrm{M}}, \quad \text{. . . . . (b)}$$

writing M for the sum of the terms under the radical sign. ζ has evidently two values, because for the same values of ξ and v there must be two tangent planes to the surface. Now, using the figure in sec. [20], and the reasoning of that article, let

$$\left.\begin{aligned}\frac{1}{\mathrm{O}m}&=-\frac{(\beta v+\beta_{,}\xi+\gamma_{,,})}{a_{,,}}+\sqrt{\mathrm{M}},\\ \frac{1}{\mathrm{OP}}&=-\frac{(\beta v+\beta_{,}\xi+\gamma_{,,})}{a_{,,}},\\ \frac{1}{\mathrm{O}n}&=-\frac{(\beta v+\beta_{,}\xi+\gamma_{,,})}{a_{,,}}-\sqrt{\mathrm{M}}.\end{aligned}\right\} \quad \text{. . . . (c)}$$

Hence $\frac{1}{\mathrm{O}m}$, $\frac{1}{\mathrm{OP}}$, and $\frac{1}{\mathrm{O}n}$ are in arithmetical progression, and therefore Om, OP, and On are in harmonical progression. Consequently the plane of XY, the two tangent planes to the surface intersecting in the plane of XY, and the plane which passes through the point P in the axis of Z, and the common intersection of the two tangent planes, in the plane of XY, are four harmonic planes all intersecting in the same straight line in the plane of XY; therefore the plane of XY and the secant plane through the point P pass respectively through the pole, one of the other; hence the plane whose equation is

$$a_{,,}\zeta+\beta v+\beta_{,}\xi+\gamma_{,,}=0$$

passes through the pole of the plane of XY.

Consequently the following equations,

$$\left.\begin{aligned}a\xi+\beta_{,}\zeta+\beta_{,,}v+\gamma&=0,\\ a_{,}v+\beta_{,,}\xi+\beta\zeta+\gamma_{,}&=0,\\ a_{,,}\zeta+\beta v+\beta_{,}\xi+\gamma_{,,}&=0,\end{aligned}\right\} \quad \text{. . . . . (d)}$$

are the tangential equations of the poles of the coordinate planes of ZY, ZX, and XY respectively; and the combination of the three equations determines the three tangential coordinates of the polar plane of the origin.

From the preceding equations, eliminating υ and ζ, we may find the value of ξ the tangential coordinate, along the axis of X, of the polar plane of the origin, or

$$\xi_{\prime}=\frac{(\beta^2-a_{\prime}a_{\prime\prime})\gamma+(a_{\prime\prime}\beta_{\prime\prime}-\beta\beta_{\prime})\gamma_{\prime}+(a\beta_{\prime}-\beta\beta_{\prime\prime})\gamma_{\prime\prime}}{aa_{\prime}a_{\prime\prime}+2\beta\beta_{\prime}\beta_{\prime\prime}-a\beta^2-a_{\prime}\beta_{\prime}^2-a_{\prime\prime}\beta_{\prime\prime}^2}. \quad . \quad . \quad \text{(e)}$$

Like expressions may be found for $\upsilon_{\prime}$ and $\zeta_{\prime}$.

When

$$aa_{\prime}a_{\prime\prime}+2\beta\beta_{\prime}\beta_{\prime\prime}-a\beta^2-a_{\prime}\beta_{\prime}^2-a_{\prime\prime}\beta_{\prime\prime}^2=0, \quad . \quad . \quad . \quad \text{(f)}$$

the origin of coordinates is on the surface; for then the intercepts of the axes of coordinates cut off by the tangent plane to the surface, namely $\frac{1}{\xi}, \frac{1}{\upsilon}, \frac{1}{\zeta}$, are each equal to 0.

When $\gamma=0$, $\gamma_{\prime}=0$, $\gamma_{\prime\prime}=0$, it follows that $\xi_{\prime}=0$, $\upsilon_{\prime}=0$, $\zeta_{\prime}=0$. But when $\gamma=0, \gamma_{\prime}=0, \gamma_{\prime\prime}=0$, the origin is at the centre, and therefore the coordinates $\frac{1}{\xi_{\prime}}, \frac{1}{\upsilon_{\prime}}, \frac{1}{\zeta_{\prime}}$ of the polar plane of the centre are infinite.

78.] *To show that when the surface touches a coordinate plane, the coefficient of the square of the corresponding variable becomes* $=0$.

When the tangent plane to the surface becomes indefinitely near to the plane, suppose of XY, it cuts the axis of Z indefinitely near to the origin. Hence, when ultimately the plane of XY becomes a tangent plane to the surface, $\frac{1}{\zeta}$ becomes 0.

The general equation of the surface may be written

$$a_{\prime\prime}+(a\xi^2+a_{\prime}\upsilon^2+2\beta\upsilon\zeta+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi\upsilon+2\gamma\xi+2\gamma_{\prime}\upsilon+2\gamma_{\prime\prime}\zeta-1)\frac{1}{\zeta^2}=0. \text{ (a)}$$

But as the second member is $=0$, since $\frac{1}{\zeta}=0$, we must have $a_{\prime\prime}=0$. In like manner, when the surface touches the coordinate planes of ZX or ZY, $a_{\prime}=0$, or $a=0$. Hence, when the surface touches the three coordinate planes, the general equation of the surface becomes

$$2\beta\upsilon\zeta+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi\upsilon+2\gamma\xi+2\gamma_{\prime}\upsilon+2\gamma_{\prime\prime}\zeta=1. \quad . \quad . \quad \text{(b)}$$

79.] *When the surface touches one of the coordinate planes, to determine the equation of the point of contact.*

Let the plane of XY be the tangent plane, then $a_{\prime\prime}=0$, as shown in the preceding section, and the general equation may be written

$$(a\xi^2+a_{\prime}\upsilon^2+2\beta\xi\upsilon+2\gamma\xi+2\gamma_{\prime}\upsilon-1)\frac{1}{\zeta}+2(\beta\upsilon+\beta_{\prime}\xi+\gamma_{\prime\prime})=0. \quad \text{(a)}$$

But since $\frac{1}{\zeta}=0$, the first member of this equation is $=0$. Hence

$\beta v+\beta_{/}\xi+\gamma_{//}=0$ is the tangential equation of the point of contact of the plane of XY.

Consequently the tangential equations of the points of contact of the surface with the planes of XY, YZ, and XZ are

$$\beta v+\beta_{/}\xi+\gamma_{//}=0,\ \beta_{/}\zeta+\beta_{//}v+\gamma=0,\ \text{and}\ \beta_{//}\xi+\beta\zeta+\gamma_{/}=0. \qquad \text{(b)}$$

80.] *On the equation of the tangent plane to a curved surface whose tangential equation is* $\phi(\xi, v, \zeta)=0$,

Let the tangential coordinates of the tangent plane be conceived as receiving minute increments, consistently with the *permanence* of the projective coordinates of the point of contact. Let ξ, v, ζ be assumed as having received infinitesimal augments, and thus to be changed into $\xi+d\xi$, $v+dv$, $\zeta+d\zeta$.

Then, on the ordinary principles of differentiation,

$$\phi(\xi_{/}, v_{/}, \zeta_{/})=\phi(\xi, v, \zeta)+\frac{d\phi(\xi, v, \zeta)}{d\xi}d\xi+\frac{d\phi(\xi, v, \zeta)}{dv}dv+\frac{d\phi(\xi, v, \zeta)}{d\zeta}d\zeta.$$

But $\phi(\xi_{/}, v_{/}, \zeta_{/})=0$, $\phi(\xi, v, \zeta)=0$; and writing for brevity a capital Φ for $\phi(\xi, v, \zeta)$, we obtain the condition

$$\frac{d\Phi}{d\xi}d\xi+\frac{d\Phi}{dv}dv+\frac{d\Phi}{d\zeta}d\zeta=0. \quad \dots\dots \quad \text{(a)}$$

If we now, on the same assumed principle, differentiate the dual equation $x\xi+yv+z\zeta=1$, we shall find $xd\xi+ydv+zd\zeta=0$.

Equating the coefficients of these differentials, first introducing an equalizing factor Δ, we shall have

$$\Delta x=\frac{d\Phi}{d\xi},\ \Delta y=\frac{d\Phi}{dv},\ \Delta z=\frac{d\Phi}{d\zeta}.$$

Multiplying these expressions severally by ξ, v, ζ, and adding,

$$\Delta(x\xi+yv+z\zeta)=\frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{dv}v+\frac{d\Phi}{d\zeta}\zeta.$$

But as $x\xi+yv+z\zeta=1$, we shall have, finally,

$$\Delta=\frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{dv}v+\frac{d\Phi}{d\zeta}\zeta.$$

Hence

$$\left.x=\frac{\frac{d\Phi}{d\xi}}{\frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{dv}v+\frac{d\Phi}{d\zeta}\zeta}\right\} \quad \dots\dots\dots \quad \text{(b)}$$

Like expressions may be found for y and z.

These very general and beautiful *formulæ of transition*, as they may be called, reduce the passage from the projective to the tangential equation of a curve or curved surface, or reciprocally, to a mere mechanical operation as it were; and the problem is thus reduced in all cases to one of elimination.

The formulæ which exhibit the relations between the *projective* and *tangential* coordinates of the same curve or curved surface are simple and symmetrical.

Let $\Phi=\phi(\xi, \upsilon, \zeta)=0$ be the *tangential* equation of a curved surface. The projective coordinates x, y, z of the point of contact of the tangent plane may be found from the following expressions:—

$$x=\frac{\frac{d\Phi}{d\xi}}{\frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{d\upsilon}\upsilon+\frac{d\Phi}{d\zeta}\zeta},\qquad y=\frac{\frac{d\Phi}{d\upsilon}}{\frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{d\upsilon}\upsilon+\frac{d\Phi}{d\zeta}\zeta},\qquad x=\frac{\frac{d\Phi}{d\zeta}}{\frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{d\upsilon}\upsilon+\frac{d\Phi}{d\zeta}\zeta}.$$

Let $F=f(x, y, z)=0$ be the *projective* equation of a curved surface. The tangential coordinates ξ, υ, ζ of the tangent plane drawn through the point (xyz) may be found from the following expressions:—

$$\xi=\frac{\frac{dF}{dx}}{\frac{dF}{dx}x+\frac{dF}{dy}y+\frac{dF}{dz}z},\qquad \upsilon=\frac{\frac{dF}{dy}}{\frac{dF}{dx}x+\frac{dF}{dy}y+\frac{dF}{dz}z},\qquad \zeta=\frac{\frac{dF}{dz}}{\frac{dF}{dx}x+\frac{dF}{dy}y+\frac{dF}{dz}z}. \quad \text{(c)}$$

By the help of these groups of equations and the original equations $\Phi=\phi(\xi, \upsilon, \zeta)=0$, or $f(x, y, z)=0$, we may eliminate ξ, υ, ζ, or x, y, z, and obtain the final equations in x, y, z, or in ξ, υ, ζ.

CHAPTER VIII.

ON THE MAGNITUDE AND POSITION OF THE AXES OF A SURFACE OF THE SECOND ORDER.

81.] If we assume the tangential equation of a central surface of the second order, as given in (a), [74],

$$a^2\xi^2+b^2\upsilon^2+c^2\zeta^2=1, \quad \text{. . . . . . .} \quad \text{(a)}$$

and refer this surface to another system of rectangular coordinates, also passing through the centre, by the help of the formulæ of

Now the perpendicular to one of the planes makes angles with the axes of coordinates, whose cosines are $\frac{\xi_{\prime}}{\Pi}, \frac{v_{\prime}}{\Pi}, \frac{\zeta_{\prime}}{\Pi}$, where Π is the reciprocal of P.

Hence ω being the angle between this perpendicular and the straight line in the two planes, we shall have

$$\cos\omega=\frac{\xi_{\prime}v_{\prime\prime}\zeta_{\prime}-\xi_{\prime}v_{\prime}\zeta_{\prime\prime}+v_{\prime}\xi_{\prime}\zeta_{\prime\prime}-v_{\prime}\xi_{\prime\prime}\zeta_{\prime}+\zeta_{\prime}\xi_{\prime\prime}v_{\prime}-\zeta_{\prime}\xi_{\prime}v_{\prime\prime}}{\Delta\Pi},$$

the numerator of which is identically $=0$; hence ω is a right angle.

We should have found a like result had we multiplied by $\xi_{\prime\prime}, v_{\prime\prime}, \zeta_{\prime\prime}$ the tangential coordinates of the second intersecting plane.

70.] *To determine the conditions in order that a given straight line and a given plane may be parallel.*

Let $\xi_{\prime}, v_{\prime}, \zeta_{\prime}$ be the tangential coordinates of the given plane, $\xi=\mu\zeta+a$, $v=\nu\zeta+\beta$ the equations of the given straight line. Now a plane which is perpendicular to this straight line will also be perpendicular to the given plane.

Let the coordinates of the second plane which is to be perpendicular to the given straight line be $\xi_{\prime\prime}, v_{\prime\prime}, \zeta_{\prime\prime}$. This plane and the given straight line will be perpendicular, see (d), sec. [68], where

$$\mu\xi_{\prime\prime}+\nu v_{\prime\prime}+\zeta_{\prime\prime}=0 \text{ and } a\xi_{\prime\prime}+\beta v_{\prime\prime}=0;$$

hence

$$\xi_{\prime\prime}=-\frac{\beta}{a}v_{\prime\prime} \text{ and } \zeta_{\prime\prime}=\left(\frac{\beta\mu-a\nu}{a}\right)v_{\prime\prime};$$

but as these two planes must be at right angles one to the other, we must have

$$\xi_{\prime}\xi_{\prime\prime}+v_{\prime}v_{\prime\prime}+\zeta_{\prime}\zeta_{\prime\prime}=0.$$

Substituting the values of $\xi_{\prime\prime}$ and $\zeta_{\prime\prime}$, we get

$$\beta\xi_{\prime}-av_{\prime}-(\beta\mu-a\nu)\zeta_{\prime}=0. \quad . \quad . \quad . \quad . \quad . \qquad \text{(a)}$$

71.] *A straight line is parallel to a given straight line, to determine the relation between the constants.*

Let $\xi=\mu\zeta+a$, $v=\nu\zeta+\beta$ be the equations of one of these lines, and $\xi=\mu_{\prime}\zeta+a_{\prime}$, $v=\nu_{\prime}\zeta+\beta_{\prime}$ those of the other; then, as these straight lines are perpendicular to the same plane, we must have

$$\mu\xi_{\prime}+\nu v_{\prime}+\zeta_{\prime}=0, \qquad a\xi_{\prime}+\beta v_{\prime}=0,$$
$$\mu_{\prime}\xi_{\prime}+\nu_{\prime}v_{\prime}+\zeta_{\prime}=0, \qquad +a_{\prime}\xi_{\prime}+\beta_{\prime}v_{\prime}=0,$$

or

$$\frac{a}{\beta}=\frac{a_{\prime}}{\beta_{\prime}}=\frac{\mu-\mu_{\prime}}{\nu_{\prime}-\nu}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(e)}$$

72.] *To investigate an expression for the angle between two given straight lines whose equations are*

$$\xi=\mu\zeta+a, \quad v=\nu\zeta+\beta; \quad \xi=\mu_{\prime}\zeta+a_{\prime}, \quad v=\nu_{\prime}\zeta+\beta_{\prime}$$

The tangential equation of the ellipsoid is $a^2\xi^2+b^2v^2+c^2\zeta^2=1$. Hence $\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}=2a^2\xi$, and $\Delta=2$. Like expressions for the other variables v and ζ may be found. Hence

$$\mathrm{T}^2=\frac{(a^2-b^2)^2\xi^2v^2+(b^2-c^2)^2v^2\zeta^2+(a^2-c^2)^2\xi^2\zeta^2}{\xi^2+v^2+\zeta^2}. \quad . \quad \text{(d)}$$

By the help of this formula we may determine the magnitude and position of the axes of a surface.

83.] We may define the axis of a central surface as a line which, drawn from the origin at the centre, to the point of contact of a tangent plane, coincides with the perpendicular let fall from the origin on the same plane.

In this case $\mathrm{T}=0$; but as the numerator of the value of T^2 is the sum of three squares, in order that this expression may be $=0$, they must severally be equal to 0, or

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}v-\frac{\mathrm{d}\Phi}{\mathrm{d}v}\xi=0,\ \frac{\mathrm{d}\Phi}{\mathrm{d}v}\zeta-\frac{\mathrm{d}\Phi}{\mathrm{d}\zeta}v=0,\ \frac{\mathrm{d}\Phi}{\mathrm{d}\zeta}\xi-\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\zeta=0. \quad . \quad \text{(a)}$$

Now we have shown in [80] that $\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}=2x$, and $2x=2\mathrm{P}\cos\lambda$, since in this supposed case P is equal to and coincides with r. But

$$\cos\lambda=\mathrm{P}\xi;\ \text{hence}\ \frac{\mathrm{d}\Phi}{\mathrm{d}\xi}=2\mathrm{P}^2\xi;\ .\ .\ .\ .\ .\ .\ \text{(b)}$$

and like expressions for the other two variables may be found.

Let $$a\xi^2+a_{\prime}v^2+a_{\prime\prime}\zeta^2+2\beta v\zeta+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi v=1 \quad . \quad . \quad \text{(c)}$$

be the tangential equation of a surface of the second order referred to its centre, then

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}=2a\xi+2\beta_{\prime}\zeta+2\beta_{\prime\prime}v;\ \text{but}\ \frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\ \text{is also equal to}\ 2\mathrm{P}^2\xi.$$

Hence $a\xi+\beta_{\prime}\zeta+\beta_{\prime\prime}v=\mathrm{P}^2\xi$; consequently the three resulting equations are

$$\left.\begin{aligned}(a\ -\mathrm{P}^2)\xi+\beta_{\prime}\zeta+\beta_{\prime\prime}v&=0,\\(a_{\prime}-\mathrm{P}^2)v+\beta_{\prime\prime}\xi+\beta\zeta&=0,\\(a_{\prime\prime}-\mathrm{P}^2)\zeta+\beta v+\beta_{\prime}\xi&=0.\end{aligned}\right\}\ .\ .\ .\ .\ .\ \text{(d)}$$

Eliminating ξ, v, ζ from these three equations, which the absence of an absolute term enables us to do, we obtain an equation analogous to the well-known cubic equation for determining the three axes of the surface.

If we could by any means ascertain that particular position of the tangent plane which would make the perpendicular coincide with the diameter passing through the point of contact, it would follow that ξ, v, ζ would become known quantities, and we could thus calculate the value of P; but if instead of so doing we eliminate ξ, υ, ζ, we shall ascertain in how many ways P may become a semiaxis of the surface.

Eliminating ξ, υ, ζ from equations (d), we obtain

$$\left.\begin{array}{l}(a-P^2)(a_{\prime}-P^2)(a_{\prime\prime}-P^2)-\beta^2(a-P^2)-\beta_{\prime}^2(a_{\prime}-P^2)\\ \quad -\beta_{\prime\prime}^2(a_{\prime\prime}-P^2)+2\beta\beta_{\prime}\beta_{\prime\prime}=0,\end{array}\right\} \quad . \quad \text{(e)}$$

a cubic equation which gives us when solved the values of the squares of the three semiaxes. Reducing the preceding equation, and arranging by powers of P, we get

$$\left.\begin{array}{l}P^6-(a+a_{\prime}+a_{\prime\prime})P^4+[(a_{\prime}a_{\prime\prime}-\beta^2)+(aa_{\prime\prime}-\beta_{\prime}^2)+(a_{\prime}a-\beta_{\prime\prime}^2)]P^2\\ \quad +\beta^2a+\beta_{\prime}^2a_{\prime}+\beta_{\prime\prime}^2a_{\prime\prime}-aa_{\prime}a_{\prime\prime}-2\beta\beta_{\prime}\beta_{\prime\prime}=0.\end{array}\right\} \quad \text{(f)}$$

Hence we may at once infer that the sum of the squares of the semiaxes is equal to the sum of the coefficients of the squares of the variables.

We shall recur to this equation, which may be termed the "*tangential cubic equation of axes.*"

It may have been observed, by those conversant with the subject, that the concentric sphere, which has generally been made use of to determine the magnitude of the axes of the surface, has been dispensed with. The principle adopted in the text, that of defining the axes as those perpendiculars on a tangent plane which coincide with the diameters drawn through the points of contact of the tangent planes, possesses the advantage of determining those magnitudes from a consideration of the properties of the figure itself.

84]. *To determine the angles which one of the axes of the surface makes with the axes of coordinates.*

The cosine of the angle λ which the axis P makes with the axis of X is $P\xi$; and $x=P^2\xi$. But, as we have shown in [80], we shall have $P^2\xi=x=a\xi+\beta_{\prime}\zeta+\beta_{\prime\prime}\upsilon$; and finding like expressions for the other variables,

$$\left.\begin{array}{l}(a-P^2)\xi+\beta_{\prime}\zeta+\beta_{\prime\prime}\upsilon=0,\\ (a_{\prime}-P^2)\upsilon+\beta_{\prime\prime}\xi+\beta\zeta=0,\\ (a_{\prime\prime}-P^2)\zeta+\beta\upsilon+\beta_{\prime}\xi=0.\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let the three roots of the cubic equation resulting from the elimination of ξ, υ, ζ between these equations, and which determines the axes, be τ^2, $\tau_{\prime}^2$, $\tau_{\prime\prime}^2$, and let

$$a-\tau^2=\delta, \quad a_{\prime}-\tau^2=\delta_{\prime}, \quad a_{\prime\prime}-\tau^2=\delta_{\prime\prime}; \quad . \quad . \quad . \quad \text{(b)}$$

also let $\zeta=m\xi$, $\upsilon=n\xi$; and the preceding equations become

$$\left.\begin{array}{l}\delta+\beta_{\prime}m+\beta_{\prime\prime}n=0,\\ \delta_{\prime}n+\beta_{\prime\prime}+\beta m=0,\\ \delta_{\prime\prime}m+\beta n+\beta_{\prime}=0;\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

from these equations we get

$$m=\frac{\beta\beta_{\prime}-\delta_{\prime\prime}\beta_{\prime\prime}}{\delta_{\prime}\delta_{\prime\prime}-\beta^2},\quad n=\frac{\beta\beta_{\prime\prime}-\beta_{\prime}\delta_{\prime}}{\delta_{\prime}\delta_{\prime\prime}-\beta^2};$$

squaring these expressions, and adding,

$$\left.\begin{array}{l}(m^2+n^2+1)(\beta^2-\delta_{\prime}\delta_{\prime\prime})^2=\beta^2\beta_{\prime}^2+\beta_{\prime\prime}^2\delta_{\prime\prime}^2+\beta^2\beta_{\prime\prime}^2+\beta_{\prime}^2\delta_{\prime}^2\\ \quad -2\beta\beta_{\prime}\beta_{\prime\prime}\delta_{\prime}-2\beta\beta_{\prime}\beta_{\prime\prime}\delta_{\prime\prime}+(\beta^2-\delta_{\prime}\delta_{\prime\prime})^2.\end{array}\right\} \quad . \quad (d)$$

We may combine this result with the cubic equation (e) in the preceding section, and which may be written in the form

$$\delta\delta_{\prime}\delta_{\prime\prime}-\beta^2\delta-\beta_{\prime}^2\delta_{\prime}-\beta_{\prime\prime}^2\delta_{\prime\prime}+2\beta\beta_{\prime}\beta_{\prime\prime}=0. \quad . \quad . \quad (e)$$

Let this expression be successively multiplied by $\delta_{\prime}$ and $\delta_{\prime\prime}$. The resulting expressions become

$$\left.\begin{array}{l}\delta\delta_{\prime\prime}\delta_{\prime}^2-\beta^2\delta\delta_{\prime}-\beta_{\prime}^2\delta_{\prime}^2-\beta_{\prime\prime}^2\delta\delta_{\prime\prime}+2\beta\beta_{\prime}\beta_{\prime\prime}\delta_{\prime} \;=0,\\ \delta\delta_{\prime}\delta_{\prime\prime}^2-\beta^2\delta\delta_{\prime\prime}-\beta_{\prime}^2\delta\delta_{\prime\prime}-\beta_{\prime\prime}^2\delta_{\prime\prime}^2+2\beta\beta_{\prime}\beta_{\prime\prime}\delta_{\prime\prime}=0.\end{array}\right\} \quad . \quad . \quad (f)$$

Adding these two expressions (f) to (e) and eliminating, we shall obtain a result that will allow of the dividing out of the common factor $(\beta^2-\delta_{\prime}\delta_{\prime\prime})$. Hence

$$(m^2+n^2+1)=\frac{(\beta^2-\delta_{\prime}\delta_{\prime\prime})+(\beta_{\prime}^2-\delta\delta_{\prime\prime})+(\beta_{\prime\prime}^2-\delta\delta_{\prime})}{(\beta^2-\delta_{\prime}\delta_{\prime\prime})}. \quad . \quad (g)$$

Now, as $\zeta=m\xi$, $v=n\xi$,

$$m^2+n^2+1=\frac{\xi^2+v^2+\zeta^2}{\xi^2}=\frac{\Pi^2}{\xi^2}=\frac{1}{\mathrm{P}^2\xi^2}=\frac{1}{\cos^2\lambda}.$$

Consequently

$$\left.\begin{array}{l}\cos^2\lambda=\dfrac{\beta^2-\delta_{\prime}\delta_{\prime\prime}}{(\beta^2-\delta_{\prime}\delta_{\prime\prime})+(\beta_{\prime}^2-\delta\delta_{\prime\prime})+(\beta_{\prime\prime}^2-\delta\delta_{\prime})},\\ \cos^2\mu=\dfrac{\beta_{\prime}^2-\delta\delta_{\prime\prime}}{(\beta^2-\delta_{\prime}\delta_{\prime\prime})+(\beta_{\prime}^2-\delta\delta_{\prime\prime})+(\beta_{\prime\prime}^2-\delta\delta_{\prime})},\\ \cos^2\nu=\dfrac{\beta_{\prime\prime}^2-\delta\delta_{\prime}}{(\beta^2-\delta_{\prime}\delta_{\prime\prime})+(\beta_{\prime}-\delta\delta_{\prime\prime})+(\beta_{\prime\prime}^2-\delta\delta_{\prime})}.\end{array}\right\} \quad . \quad . \quad (h)$$

We may thus find similar expressions for the other two semiaxes. Hence the problem is completely solved, as we can express the squares of the three semiaxes and the cosines of the nine angles in terms of the coefficients of the given tangential equation.

On the particular case when the surface is one of revolution.

85.] In this case two of the axes are equal; and therefore two of the roots of the cubic equation of axes (f) in [83] become equal. But it is a well known property of algebraical equations that when two of the roots of an equation are equal, one of these will be found in the limiting or derived equation. Writing ρ for P^2 in the equation of axes, and differentiating, we find

$$3\rho^2-2(a+a_{\prime}+a_{\prime\prime})\rho-[(\beta^2-a_{\prime}a_{\prime\prime})+(\beta_{\prime}^2-aa_{\prime\prime})+(\beta_{\prime\prime}^2-aa_{\prime})]; \quad (a)$$

or writing R for $a+a_{\prime}+a_{\prime\prime}$, and

$$Q \text{ for } (\beta^2-a_{\prime}a_{\prime\prime})+(\beta_{\prime}^2-aa_{\prime\prime})+(\beta_{\prime\prime}^2-aa_{\prime}),$$

the limiting equation becomes

$$3\rho^2-2R\rho-Q=0, \text{ or } \rho=\frac{R\pm\sqrt{R^2+3Q}}{3}. \quad . \quad . \quad \text{(b)}$$

One of these values of ρ is the square of one of the equal semiaxes.

We may obtain the value of the equal semiaxes from other considerations.

If we refer to the formulæ which determine the inclinations of the axes of the surface to the axes of coordinates, as given in (h), sec. [84], it is evident that, when they become unlimited in number, the values of the cosines of the angles which these equal axes make with the axes of coordinates must be indeterminate also. If we refer to the expressions for these cosines as given in (h), they will become indeterminate, or of the form $\frac{0}{0}$, if we put

$$\beta^2-\delta_{\prime}\delta_{\prime\prime}=0,\quad \beta_{\prime}^2-\delta\delta_{\prime\prime}=0,\quad \beta_{\prime\prime}^2-\delta\delta_{\prime}=0. \quad . \quad . \quad . \quad \text{(c)}$$

Now reverting to sec. [84] (b), it has been shown that, if τ^2 be one of the roots of the cubic equation of axes,

$$a_{\prime}-\tau^2=\delta_{\prime},\quad a_{\prime\prime}-\tau^2=\delta_{\prime\prime},$$

or

$$a_{\prime}a_{\prime\prime}-(a_{\prime}+a_{\prime\prime})\tau^2+\tau^4=\delta_{\prime}\delta_{\prime\prime}=\beta^2 \quad . \quad . \quad . \quad . \quad \text{(d)}$$

on the assumption in (c).

Consequently $\tau^4-(a_{\prime}+a_{\prime\prime})\tau^2=\beta^2-a_{\prime}a_{\prime\prime}$.

Finding similar expressions for the two other axes of coordinates, and adding the resulting equations, we shall have

$$3\tau^4-2(a+a_{\prime}+a_{\prime\prime})\tau^2=\beta^2-a_{\prime}a_{\prime\prime}+\beta_{\prime}^2-aa_{\prime\prime}+\beta_{\prime\prime}^2-aa_{\prime}),$$

or

$$3\tau^4-2R\tau^2-Q=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

precisely the same equation as (b) derived from the discussion of the limiting equation.

It may easily be shown that, when two of the roots of the cubic equation of axes are equal, the three roots are

$$\frac{R-\sqrt{R^2+3Q}}{3},\quad \frac{R-\sqrt{R^2+3Q}}{3}, \text{ and } \frac{R+2\sqrt{R^2+3Q}}{3}. \quad \text{(f)}$$

We know that if r, $r_{\prime}$, $r_{\prime\prime}$ be the three roots of a cubic equation, the sum of the products of the roots, taken two by two, is equal to the coefficient of the third term,

$$\text{or } rr_{\prime}+r_{\prime}r_{\prime\prime}+rr_{\prime\prime}=-Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

Now, if we make $r=\frac{R+2\sqrt{R^2+3Q}}{3}$, $r_{\prime}$ and $r_{\prime\prime}$ each equal to $\frac{R-\sqrt{R^2+3Q}}{3}$, and substitute these values in the preceding expression, the resulting equation becomes identical[*].

CHAPTER IX.

ON THE TANGENTIAL EQUATIONS OF THE PARABOLOIDS.

86.] Let the axis of the surface be taken as the axis of z, the tangents to the vertices of the principal sections as the axes of x and y. Now we may conceive the elliptic paraboloid as generated by the parallel motion of the plane of a variable ellipse whose centre moves along the axis of Z, and whose vertices always rest on two guiding parabolas in the planes of ZX and ZY, having a common axis Z. Let $4k$ and $4k_{\prime}$ be the parameters of the principal sections passing through the axis of z. Let the paraboloid be cut by a plane parallel to the plane of xy at the distance z. Then, if

---

[*] If we take the *cubic equation of axes* given in most books on surfaces of the second order, as treated by the method of projective coordinates (see Leroy, 'Analyse appliquée à la Géométrie des trois dimensions,' p. 198), we shall find

$$(s-A)(s-A_{\prime})(s-A_{\prime\prime})-B^2(s-A)-B_{\prime}^2(s-A_{\prime})-B_{\prime\prime}^2(s-A_{\prime\prime})-2BB_{\prime}B_{\prime\prime}=0. \quad (a)$$

Now, if R be a root of this equation, and we put

$$A-R=D,\quad A_{\prime}-R=D_{\prime},\quad A_{\prime\prime}-R=D_{\prime\prime},$$

and if we follow the course indicated in the text, we shall find for the cosines of the angles α, β, γ, which the semiaxis R makes with the three axes,

$$\cos^2\alpha=\frac{(B^2-D_{\prime}D_{\prime\prime})}{(B^2-D_{\prime}D_{\prime\prime})+(B_{\prime}^2-D_{\prime\prime}D)+(B_{\prime\prime}^2-DD_{\prime})}, \quad .\ .\ .\ .\ (b)$$

and like expressions for $\cos^2\beta$, $\cos^2\gamma$.

We do not remember to have seen these formulæ in any treatise on this subject.

When the surface is one of revolution, two of the axes must be equal, hence two of the roots of the cubic equation of axes must be equal, hence the *limiting equation* to (a) must contain at least one of the equal roots. Differentiating this equation, we shall find the following for the limiting equation

$$[(s-A_{\prime})(s-A_{\prime\prime})-B^2]+[(s-A_{\prime\prime})(s-A)-B_{\prime}^2]+[(s-A)(s-A_{\prime})-B_{\prime\prime}^2]=0, \quad (c)$$

and this derivative, as also the original equation, are satisfied by putting

$$D_{\prime}D_{\prime\prime}-B^2=0,\quad D_{\prime\prime}D-B_{\prime}^2=0,\quad DD_{\prime}-B_{\prime\prime}^2=0,$$

and the expressions for the direction cosines become

$$\cos\alpha=\frac{0}{0},\ \cos\beta=\frac{0}{0},\ \cos\gamma=\frac{0}{0},$$

as evidently should be the case.

X and Y be the semiaxes of the ellipse in which the paraboloid is cut by the parallel plane, we shall have $X^2=4kz$, $Y^2=4k_{,}z$; hence

$$\frac{x^2}{4kz}+\frac{y^2}{4k_{,}z}=1, \text{ or } k_{,}x^2+ky^2=4kk_{,}z, \quad . \quad . \quad . \quad \text{(a)}$$

is the projective equation of the elliptic paraboloid.

$$\left.\begin{array}{l} \text{Now } \dfrac{dF}{dx}=2k_{,}x,\ \dfrac{dF}{dy}=2ky,\ \dfrac{dF}{dz}=-4kk_{,}, \\ \text{and} \\ \dfrac{dF}{dx}x+\dfrac{dF}{dy}y+\dfrac{dF}{dz}z=4kk_{,}z. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Hence

$$\xi=\frac{k_{,}x}{2kk_{,}z}=\frac{x}{2kz},\ \ \upsilon=\frac{y}{2k_{,}z},\ \ \zeta=\frac{1}{z}.$$

Substituting these values of x, y, and z in the dual equation

$$x\xi+y\upsilon+z\zeta=1,$$

we shall obtain for the tangential equation of the paraboloid,

$$k\xi^2+k_{,}\upsilon^2+\zeta=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

87.] *On the transformation of the axes of coordinates in the case of the paraboloids.*

Assuming the equations given in sec. [56], namely

$$\xi=\xi_{,}\cos\lambda+\upsilon_{,}\cos\lambda_{,}+\zeta_{,}\cos\lambda_{,,},$$
$$\upsilon=\xi_{,}\cos\mu+\upsilon_{,}\cos\mu_{,}+\zeta_{,}\cos\mu_{,,},$$
$$\zeta=\xi_{,}\cos\nu+\upsilon_{,}\cos\nu_{,}+\zeta_{,}\cos\nu_{,,},$$

or writing $l\ m\ n$, $l_{,}\ m_{,}\ n_{,}$, $l_{,,}\ m_{,,}\ n_{,,}$ for the cosines,

$$\xi=l\xi_{,}\ +l_{,}\upsilon_{,}\ +l_{,,}\zeta_{,},$$
$$\upsilon=m\xi_{,}+m_{,}\upsilon_{,}+m_{,,}\zeta',$$
$$\zeta=n\xi_{,}\ +n_{,}\upsilon_{,}\ +n_{,,}\zeta_{,},$$

hence

$$k\xi^2=k[l^2\xi_{,}^2+l_{,}^2\upsilon_{,}^2+l_{,,}^2\zeta_{,}^2+2ll_{,}\xi_{,}\upsilon_{,}+2ll_{,,}\xi_{,}\zeta_{,}+2l_{,}l_{,,}\upsilon_{,}\zeta_{,}],$$
$$k_{,}\upsilon^2=k_{,}[m^2\xi_{,}^2+m_{,}^2\upsilon_{,}^2+m_{,,}^2\zeta_{,}^2+2mm_{,}\xi_{,}\upsilon_{,}+2mm_{,,}\xi_{,}\zeta_{,}+2m_{,}m_{,,}\upsilon_{,}\zeta_{,}],$$
$$\zeta=n\xi_{,}+n_{,}\upsilon_{,}+n_{,,}\zeta_{,}.$$

Arranging according to powers of ξ, υ, and ζ, and omitting traits to ξ, υ, ζ as no longer necessary, and adding,

$$[kl^2+k_{,}m^2]\xi^2+[kl_{,}^2+k_{,}m_{,}^2]\upsilon^2+[kl_{,,}^2+k_{,}m_{,,}^2]\zeta^2+2[kl_{,}l_{,,}+k_{,}m_{,}m_{,,}]\zeta\upsilon$$
$$+2[kll_{,,}+k_{,}mm_{,,}]\xi\zeta+2[kll_{,}+k_{,,}mm_{,}]\xi\upsilon+n\xi+n_{,}\upsilon+n_{,,}\zeta=0. \quad \text{(d)}$$

When the paraboloid is a surface of revolution and the origin placed at the focus of the surface, its equation becomes

$$k(\xi^2+\upsilon^2+\zeta^2)+\zeta=0. \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

We may now translate the origin to another point, the direction of the coordinate planes continuing the same.

Let p, q, r be the projective coordinates of the vertex of the surface on the new coordinate planes; then, assuming

$$\xi = \frac{\xi_{\prime}}{1 - p\xi_{\prime} - q\upsilon_{\prime} - r\zeta_{\prime}}, \text{ see (e), sec. [58]},$$

like expressions may be found for υ and ζ.

It is plain that the only terms that will be affected by the translation of the coordinates will be the linear terms; and these will become

$$\begin{aligned} &n\xi - pn\xi^2 - qn\xi\upsilon - rn\xi\zeta, \\ &n_{\prime}\upsilon - qn_{\prime}\upsilon^2 - pn_{\prime}\xi\upsilon - rn_{\prime}\upsilon\zeta, \\ &n_{\prime\prime}\zeta - pn_{\prime\prime}\xi\zeta - qn_{\prime\prime}\upsilon\zeta - n_{\prime\prime}r\zeta^2. \end{aligned}$$

Hence the general equation of the paraboloid referred to any rectangular axes in space is

$$\left.\begin{aligned} &[kl^2 + k_{\prime}m^2 + np]\xi^2 + [kl_{\prime}^2 + k_{\prime}m_{\prime}^2 + n_{\prime}q]\upsilon^2 + [kl_{\prime\prime}^2 + k_{\prime}m_{\prime\prime}^2 + n_{\prime\prime}r]\zeta^2 \\ &\quad + [2kll_{\prime\prime} + 2k_{\prime}mm_{\prime\prime} + n_{\prime}r + n_{\prime\prime}q]\zeta\upsilon \\ &\quad + [2kll_{\prime\prime} + 2kmm_{\prime\prime} + nr + n_{\prime\prime}p]\xi\zeta \\ &\quad + [2kll_{\prime} + 2k_{\prime}mm_{\prime} + nq + n_{\prime}p]\xi\upsilon + n\xi + n_{\prime}\upsilon + n_{\prime\prime}\zeta = 0, \end{aligned}\right\} \text{(f)}$$

omitting the traits as no longer necessary.

88.] *Given the general equation of the paraboloid to any set of rectangular axes passing through the vertex, to determine the magnitude and position of the parameters.*

We have shown in the preceding section that the equation of the paraboloid, referred to its vertex as origin, will be

$$\left.\begin{aligned} &[kl^2 + k_{\prime}m^2]\xi^2 + [kl_{\prime}^2 + k_{\prime}m_{\prime}^2]\upsilon^2 + (kl_{\prime\prime}^2 + k_{\prime}m_{\prime\prime}^2)\zeta^2 \\ &\quad + 2[kl_{\prime}l_{\prime\prime} + k_{\prime}m_{\prime}m_{\prime\prime}]\zeta\upsilon + 2[kll_{\prime\prime} + k_{\prime}mm_{\prime\prime}]\xi\zeta \\ &\quad + 2[kll_{\prime} + k_{\prime}mm_{\prime}]\xi\upsilon + n\xi + n_{\prime}\upsilon + n_{\prime\prime}\zeta = 0. \end{aligned}\right\} \quad . \quad . \quad \text{(a)}$$

Let the general equation of the paraboloid be

$$f\xi^2 + f_{\prime}\upsilon^2 + f_{\prime\prime}\zeta^2 + 2g\upsilon\zeta + 2g_{\prime}\xi\zeta + 2g_{\prime\prime}\xi\upsilon + h\xi + h_{\prime}\upsilon + h_{\prime\prime}\zeta = 0. \quad . \quad \text{(b)}$$

Comparing the coefficients of these equations, term by term, we shall have

$$\left.\begin{aligned} &kl^2 + k_{\prime}m^2 = f, \quad kl_{\prime}^2 + k_{\prime}m_{\prime}^2 = f_{\prime}, \quad kl_{\prime\prime}^2 + k_{\prime}m_{\prime\prime}^2 = f_{\prime\prime}, \\ &kl_{\prime}l_{\prime\prime} + k_{\prime}m_{\prime}m_{\prime\prime} = g, \quad kll_{\prime\prime} + k_{\prime}mm_{\prime\prime} = g_{\prime}, \quad kll_{\prime} + k_{\prime}mm_{\prime} = g_{\prime\prime}; \end{aligned}\right\} \text{(c)}$$

hence

$$\left.\begin{aligned} &k^2[l^2l_{\prime}^2 + l_{\prime}^2l_{\prime\prime}^2 + l_{\prime\prime}^2l^2] + k_{\prime}^2[m^2m_{\prime}^2 + m_{\prime}^2m_{\prime\prime}^2 + m_{\prime\prime}^2m^2] \\ &\quad + kk_{\prime}[l_{\prime}^2m^2 + l^2m_{\prime}^2 + l_{\prime\prime}^2m_{\prime}^2 + l_{\prime}^2m_{\prime\prime}^2 + l_{\prime\prime}^2m^2 + l^2m_{\prime\prime}^2 \\ &\qquad = ff_{\prime} + f_{\prime}f_{\prime\prime} + f_{\prime\prime}f, \end{aligned}\right\} \quad . \quad \text{(d)}$$

and

$$\left.\begin{aligned} g^2+g_{\prime}^2+g_{\prime\prime}^2=k^2[l^2l_{\prime}^2+l_{\prime}^2l_{\prime\prime}^2+l_{\prime\prime}^2l^2] \\ +k_{\prime}^2[m^2m_{\prime}^2+m_{\prime}^2m_{\prime\prime}^2+m_{\prime\prime}^2m^2] \\ +2kk_{\prime}[ll_{\prime}mm_{\prime}+l_{\prime}l_{\prime\prime}m_{\prime}m_{\prime\prime}+ll_{\prime\prime}m_{\prime\prime}m]. \end{aligned}\right\} . \quad (e)$$

If we now subtract this equation from the preceding, we shall have

$$\left.\begin{aligned} ff_{\prime}+f_{\prime}f_{\prime\prime}+f_{\prime\prime}f-g^2-g_{\prime}^2-g_{\prime\prime}^2 \\ =kk_{\prime}[(l_{\prime}m-lm_{\prime})^2+(l_{\prime\prime}m_{\prime}-l_{\prime}m_{\prime\prime})^2+(lm_{\prime\prime}-l_{\prime\prime}m)^2]. \end{aligned}\right\} . \quad (f)$$

Assume

$$l_{\prime}m-lm_{\prime}=\cos\phi,\ l_{\prime\prime}m_{\prime}-l_{\prime}m_{\prime\prime}=\cos\chi, \text{ and } lm_{\prime\prime}-l_{\prime\prime}m=\cos\psi. \quad (g)$$

Multiply the first by $m_{\prime\prime}$, the second by m, and the third by $m_{\prime}$; then we shall have

$$\left.\begin{aligned} l_{\prime}mm_{\prime\prime}-lm_{\prime}m_{\prime\prime}=m_{\prime\prime}\cos\phi, \\ l_{\prime\prime}m_{\prime}m-l_{\prime}m_{\prime\prime}m=m\cos\chi, \\ lm_{\prime\prime}m_{\prime}-l_{\prime\prime}mm_{\prime}=m_{\prime}\cos\psi; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (h)$$

adding, we shall have $0=m_{\prime\prime}\cos\phi+m\cos\chi+m_{\prime}\cos\psi$.

But as m, $m_{\prime}$, $m_{\prime\prime}$ are the cosines of the angles which a certain straight line makes with a system of rectangular axes, ϕ, χ, ψ must be the angles which a straight line at right angles to the former makes with the same axes; hence

$$(l_{\prime}m-lm_{\prime})^2+(l_{\prime\prime}m_{\prime}-l_{\prime}m_{\prime\prime})^2+(lm_{\prime\prime}-l_{\prime\prime}m)^2=\cos^2\phi+\cos^2\chi+\cos^2\psi=1,$$

and the preceding equation now becomes

$$ff_{\prime}+f_{\prime}f_{\prime\prime}+f_{\prime\prime}f-(g^2+g_{\prime}^2+g_{\prime\prime}^2)=kk_{\prime}. \quad . \quad . \quad . \quad (i)$$

If we add the expressions in (c), we get

$$k+k_{\prime}=f+f_{\prime}+f_{\prime\prime}.$$

Hence, if ρ be put for one fourth of one of the principal parameters of the paraboloid, we shall obtain its value from the quadratic equation

$$\rho^2-[f+f_{\prime}+f_{\prime\prime}]\rho+[ff_{\prime}+f_{\prime}f_{\prime\prime}+f_{\prime\prime}f-(g^2+g_{\prime}^2+g_{\prime\prime}^2)]=0. \quad (j)$$

On the Hyperbolic Paraboloid.

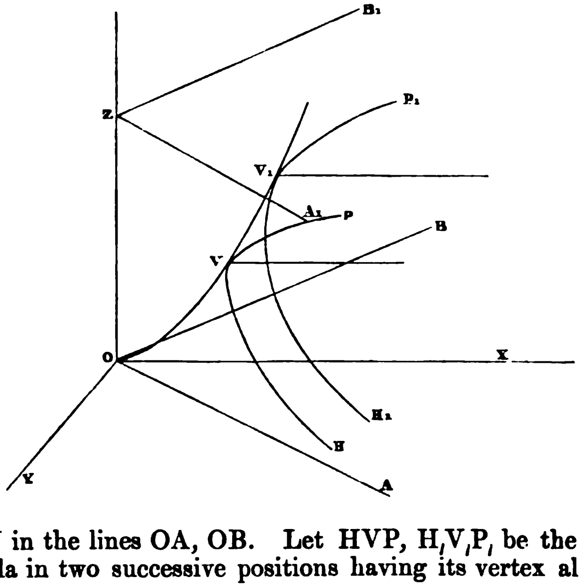
89.] Of all the surfaces of the second order this is the most difficult to form clear conceptions about as to its configuration and limits. Its sheets extending to infinity in different directions, the application of the usual methods of algebraical investigation becomes somewhat vague and indistinct. This surface admits of no circular sections, nor can it be cut by a plane in a closed curve. It is its own reciprocal polar, and is one of those surfaces called *gauche* by the French mathematicians.

Its genesis and form may best be conceived by the help of the following mode of generation.

Let three rectangular axes be assumed. In the plane of XZ, suppose, let a parabola be described, having its vertex at the origin, and its axis of figure coinciding with the axis of Z. Let its equation be $x^2=4kz$. Through the axis of Z let two planes be conceived to pass, equally inclined by the common angle θ to the plane of XZ, the plane of the above mentioned parabola. These planes may be called the *asymptotic planes* of the surface. Now let the plane of a variable hyperbola, having its centre always on the axis of Z, be conceived to move parallel to the plane of XY, and let this moving horizontal plane—assuming the plane of XY to be horizontal—cut the vertical asymptotic planes in two straight lines meeting in the axis of Z. Let these lines be the asymptotes of the moving hyperbola, which shall have its vertex on a point of the guiding parabola. Then the ordinate A of this parabola will be half the transverse axis of the hyperbola, and the other semiaxis B will be A tan θ, 2θ being the angle between the asymptotic planes.

Let $OVV_{,}$ be the guiding parabola in the plane of ZX. Let $OAZA_{,}$ and $OBZB_{,}$ be the two asymptotic planes cutting the plane

Fig. 18.



of XOY in the lines OA, OB. Let HVP, $H_{,}V_{,}P_{,}$ be the moving hyperbola in two successive positions having its vertex always on the parabola $OVV_{,}$.

It is to be observed that while the elliptic paraboloid may be generated by the parallel motion of the plane of a variable ellipse guided by two parabolas whose planes are at right angles, and having a common axis, that of Z, the hyperbolic paraboloid is generated by the parallel motion of the plane of a variable hyperbola guided by one parabola and by a pair of asymptotic planes.

90.] From the foregoing construction it will be evident that the entire of this surface, while its sheets extend to infinity in opposite directions, will be wholly confined between the asymptotic planes, $AOZA_{\prime}$ and $BOZB_{\prime}$ inclined one to the other by the angle 2θ. As the surface approaches the plane of the guiding parabola or the plane of XZ, it will thin down to an edge; and this edge will be the guiding parabola.

It will also be shown further on, that while *above* the plane of XY the surface is confined between the two asymptotic planes whose angle of intersection is 2θ, *below* the horizontal plane the surface will be developed between the same asymptotic planes, but in the supplemental angle $\pi-2\theta$.

91.] The equation of the guiding parabola being $x^2=4kz$, let the equation of the moving hyperbola be $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$.

Now, A being an ordinate of the parabola, $A^2=4kz$; and as $B=A\tan\theta$, $B^2=A^2t^2=4t^2kz$, if $\tan\theta$ be put equal to t.

Hence the projective equation of the paraboloid becomes

$$t^2x^2-y^2-4t^2kz=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

To obtain the tangential equation of this surface.

Putting F for this expression, and taking its partial differentials,

$$\text{and} \quad \left.\begin{array}{l} \frac{dF}{dx}=2t^2x,\ \frac{dF}{dy}=-2y,\ \frac{dF}{dz}=-4t^2k, \\ \\ \frac{dF}{dx}x+\frac{dF}{dy}y+\frac{dF}{dz}z=4t^2kz, \end{array}\right\} \quad . \quad . \quad . \quad . \quad (b)$$

hence, as in sec. [80], (c),

$$\xi=\frac{x}{2kz},\ \upsilon=\frac{-y}{2t^2kz},\ \text{and}\ z=-\frac{1}{\zeta}.$$

Substituting these values of x, y, and z in the projective equation (a), we obtain for the required tangential equation of the surface,

$$k\xi^2-kt^2\upsilon^2+\zeta=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

It will be found that the system of tangential coordinates affords peculiar facilities for the investigation of the properties of this surface.

92.] *To ascertain whether in a tangent plane to this surface there can exist any linear generatrices.*

Let the tangential equation of the straight line which we shall assume as in a tangent plane, and a generatrix of the surface, be

$$\xi=\mu\zeta+a\,; \quad \ldots\ldots \quad \text{(a)}$$

and as this line is to be in a tangent plane, it must satisfy the tangential equation of the surface, $k\xi^2-kt^2v+\zeta=0$. Eliminating ζ between these equations, we get

$$\xi^2+\frac{\xi}{\mu k}-\frac{a}{\mu k}=t^2v^2. \quad \ldots\ldots \quad \text{(b)}$$

Now in order that this equation may break up into factors representing straight lines on the surface, we must have

$$\frac{-4a}{\mu k}=\frac{1}{\mu^2k^2} \text{ or } \frac{1}{\mu k}=-4a. \quad \ldots\ldots \quad \text{(c)}$$

Making these substitutions, and taking the square roots, we shall have as the resulting equation,

$$\xi=\pm tv+2a. \quad \ldots\ldots \quad \text{(d)}$$

This is a very remarkable result. As t the coefficient of v is independent of μ and a the constants in the given tangential equation of the straight line, it will follow that no matter where the point on the plane XZ may be, through which the assumed line passes, the projection of this line on the plane of XY will always be parallel to one of the asymptotic planes; and as this projecting plane is also a tangent plane to the surface, it will follow that all vertical planes drawn parallel to the asymptotic planes are not only tangent planes to the surface, but they also cut it in linear generatrices.

93.] To ascertain whether any other tangent planes can be drawn through these linear generatrices besides the vertical planes parallel to the asymptotic planes.

The equation of the paraboloid being

$$k\xi^2-kt^2v^2+\zeta=0, \quad \ldots\ldots \quad \text{(a)}$$

let

$$\xi=\lambda v+\gamma, \quad \ldots\ldots \quad \text{(b)}$$

be the tangential equation of the straight line in the plane of XY.

If we eliminate v from these equations, the result becomes

$$k(t^2-\lambda^2)\xi^2-2k\gamma t^2\xi+kt^2\gamma^2-\lambda^2\zeta=0. \quad \ldots\ldots \quad \text{(c)}$$

Now this equation cannot be made to represent the equation of a straight line unless $\lambda=t$, and the preceding equation (b) thus becomes

$$\xi=tv+\gamma,$$

or, in other words, the line in the plane of XY must be parallel to the asymptotic planes as before.

On this supposition the equation (c) becomes $\xi=\frac{-\zeta}{2k\gamma}+\frac{\gamma}{2}$, which is identical with (a), sec. [92], if we make $\gamma=2a$, for then $\frac{-1}{2k\gamma}=\mu$.

No two successive generatrices of this surface can meet, or be in the same plane.

They cannot meet, for they lie all in planes which are parallel to one of the asymptotic planes; nor can they be ever in the same plane, for the projection of these generatrices intersect when they are projected on any plane but that of XY.

If we do not impose on a rectilinear generatrix the condition that a tangent plane shall pass through it, its equations in the three coordinate planes may easily be found.

The equation in the plane of XY being $\xi=\pm tv+2a$, the equation in the plane of XZ will be $\xi=\frac{-\zeta}{4ka}+a$, while the equation in the plane of ZY will become

$$v=\frac{\zeta}{4kat}+\frac{a}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(d)}$$

94.] If we draw a series of tangent planes to the surface, all parallel to the axis of Y, we shall have $v=0$, and the equation of the surface becomes $k\xi^2+\zeta=0$, the tangential equation of the guiding parabola, or, in other words, a cylinder whose axis is parallel to the axis of Y circumscribes the surface along the guiding parabola.

If we make $\xi=0$, the general equation becomes $kt^2v^2-\zeta=0$, which is the equation of a parabola in the plane of ZY and situated *below* the horizontal plane.

When $\zeta=0$, $k\xi^2-kt^2v^2=0$, or $\xi=\pm tv$, or the tangent plane that coincides with the axis of Z cuts the surface in the plane of XY along the sections of the asymptotic planes with the plane of XY.

Had we assumed the point in the plane of YZ and taken the tangential equation of the right line in that plane $v=\nu\zeta+\beta$, and eliminated ζ from this equation and from that of the surface, we should have found

$$\xi=tv-2\beta t.$$

Or, in other words, the projection of this line on the plane of XY must be parallel to one of the asymptotic planes.

In the general equation, if we make $t=1$, or the moving hyperbola an equilateral hyperbola, and take the asymptotic planes as the coordinate planes of ZX and ZY, the tangential equation of the hyperbolic paraboloid assumes the very simple form

$$2k\xi v+\zeta=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(e)}$$

CHAPTER X.

ON THE APPLICATION OF ALGEBRA TO THE THEORY OF RECIPROCAL POLARS.

95.] *Let a cone, the projective coordinates of whose vertex are p, q, r, envelope a surface of the second order; the curve of contact is a plane curve.*

The general tangential equation of a central surface of the second order, referred to rectangular axes passing through the centre, is

$$a\xi^2+a_{\prime}\upsilon^2+a_{\prime\prime}\zeta^2+2\beta\upsilon\zeta+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi\upsilon-1=0=\Phi. \quad . \quad . \quad \text{(a)}$$

Then, as this is a homogeneous function in ξ, υ, ζ, we shall have

$$\Delta=\frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{d\upsilon}\upsilon+\frac{d\Phi}{d\zeta}=2. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Now $x=\dfrac{\frac{d\Phi}{d\xi}}{\Delta}=\frac{1}{2}\dfrac{d\Phi}{d\xi}$; hence, as shown in (b), sec. [80],

$$\left.\begin{aligned}x&=a\xi+\beta_{\prime}\zeta+\beta_{\prime\prime}\upsilon,\\ y&=a_{\prime}\upsilon+\beta_{\prime\prime}\xi+\beta\zeta,\\ z&=a_{\prime\prime}\zeta+\beta\upsilon+\beta_{\prime}\xi,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

x, y, and z being the projective coordinates of the point on the surface touched by the limiting tangent plane.

Now as these expressions are linear, we may find the values of ξ, υ, ζ in terms of x, y, z, and the resulting equations will also be linear. Hence

$$\left.\begin{aligned}\xi&=Lx+My+Nz,\\ \upsilon&=L_{\prime}x+M_{\prime}y+N_{\prime}z,\\ \zeta&=L_{\prime\prime}x+M_{\prime\prime}y+N_{\prime\prime}z;\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

and as the tangent planes to the surface must all pass through the vertex of the circumscribing cone, of which the projective coordinates are p, q, r, these tangential coordinates ξ, υ, ζ must satisfy the equation $p\xi+q\upsilon+r\zeta=1$. Introducing the values of ξ, υ, and ζ given above, we obtain the resulting equation,

$$(Lp+L_{\prime}q+L_{\prime\prime}r)x+(Mp+M_{\prime}q+M_{\prime\prime}r)y+(Np+N_{\prime}q+N_{\prime\prime}r)z=1, \quad \text{(e)}$$

the projective equation of a plane, whose projective coordinates are x, y, z, and whose coefficients L, M, N, &c. are functions of the coefficients of the given tangential equation of the surface (a).

Now let $\xi_{\prime}$, $\upsilon_{\prime}$, and $\zeta_{\prime}$ be the three tangential coordinates of this plane, we shall have

$$\left.\begin{aligned}\xi_{\prime}&=Lp+L_{\prime}q+L_{\prime\prime}r,\\ \upsilon_{\prime}&=Mp+M_{\prime}q+M_{\prime\prime}r,\\ \zeta_{\prime}&=Np+N_{\prime}q+N_{\prime\prime}r.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

These are the remarkable relations that exist between p, q, r, the projective coordinates of the vertex of the circumscribing cone, and the tangential coordinates $\xi_{\prime}, \upsilon_{\prime}\ \zeta_{\prime}$ of its plane of contact; or, in other words, if p, q, r are the projective coordinates of the pole, $\xi_{\prime}, \upsilon_{\prime}, \zeta_{\prime}$ are the tangential coordinates of the polar plane.

96.] We may illustrate this theory.

Let the surface of the second order be an ellipsoid, referred to its axes as axes of coordinates; its tangential equation is as in sec. [74],

$$a^2\xi^2+b^2\upsilon^2+c^2\zeta^2=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Now referring to the group of formulæ (e) in the preceding section, we shall have

$$\mathrm{L}=\frac{1}{a^2},\ \mathrm{M}_{\prime}=\frac{1}{b^2},\ \mathrm{N}_{\prime\prime}=\frac{1}{c^2},\ \text{and } \mathrm{L}_{\prime},\ \mathrm{L}_{\prime\prime},\ \mathrm{M},\ \mathrm{M}_{\prime\prime},\ \mathrm{N},\ \mathrm{N}_{\prime} \text{ each } =0.$$

Hence $$p=a^2\xi_{\prime},\ q=b^2\upsilon_{\prime},\ r=c^2\zeta_{\prime}, \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Substituting these values of p, q, r, in the dual equation $p\xi+q\upsilon+r\zeta=1$, we shall have

$$a^2\xi_{\prime}\xi+b^2\upsilon_{\prime}\upsilon+c^2\zeta_{\prime}\zeta=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

We may therefore conclude that if an ellipsoid referred to its axes, and whose tangential equation is $a^2\xi^2+b^2\upsilon^2+c^2\zeta^2=1$, be cut by a plane whose tangential coordinates are $\xi_{\prime}, \upsilon_{\prime}, \zeta_{\prime}$, the equation

$$a^2\xi_{\prime}\xi+b^2\upsilon_{\prime}\upsilon+c^2\zeta_{\prime}\zeta=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

will be the tangential equation of the pole of this plane.

97.] Instead of p, q, r, which for the sake of clearness we have hitherto used as the projective coordinates of the vertex of the cone, we may write the more common symbols x, y, and z for p, q, r.

These equations now become

$$\left.\begin{aligned}\xi_{\prime}&=\mathrm{L}x+\mathrm{L}_{\prime}y+\mathrm{L}_{\prime\prime}z,\\ \upsilon_{\prime}&=\mathrm{M}x+\mathrm{M}_{\prime}y+\mathrm{M}_{\prime\prime}z,\\ \xi_{\prime}&=\mathrm{N}x+\mathrm{N}_{\prime}y+\mathrm{N}_{\prime\prime}z,\end{aligned}\right\},\ \text{or}\quad \left.\begin{aligned}\xi_{\prime}&=\mathrm{X},\\ \upsilon_{\prime}&=\mathrm{Y},\\ \xi_{\prime}&=\mathrm{Z}.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (a)$$

From these equations, which on the ground of their importance we shall call *The Polar Equations* of surfaces of the second order, may the whole theory of *reciprocal polars* be derived by the application of the elementary principles of common algebra.

Thus if the polar plane be fixed, $\xi_{\prime}, \upsilon_{\prime}$, and $\zeta_{\prime}$ are constants, hence x, y, and z are constants, or the pole is fixed. When $\xi_{\prime}, \upsilon_{\prime}$, and $\zeta_{\prime}$ are connected by two linear equations, so also are X, Y, Z; or if the polar plane pass through a fixed straight line, the pole will also traverse a fixed straight line.

When $\xi_{\prime}, \upsilon_{\prime}, \zeta_{\prime}$ are connected by one linear equation, so also are X, Y, Z; or when the polar plane passes through a fixed point, the pole traverses a fixed plane.

When $\xi_{,}, v_{,}, \zeta_{,}$ are connected by an equation of the second degree, so also are X, Y, Z; or if the *polar plane* envelopes a surface of the second order, the *pole* traverses a surface of the second degree.

Generally, if the tangential equation of a surface be $\phi(\xi, v, \zeta)=0$, the projective equation of its reciprocal polar will be $\phi(X, Y, Z)=0$.

Should for simplicity the directrix or polarizing surface be a sphere of radius equal to unity, the tangential equation of a surface being $\phi(\xi, v, \zeta)=0$, its reciprocal polar will be $\phi(x, y, z)=0$, and conversely.

By the aid of this very remarkable theorem, and of the properties of tangential equations already discussed, we may reduce the whole theory of reciprocal polars under the dominion of common algebra, with the utmost simplicity. The following are a few of the most obvious subordinate relations that may be derived from this cardinal theorem.

Given the projective equation of a surface $f(x, y, z)=0$, or the tangential equation of the same surface $\phi(\xi, v, \zeta)=0$, we may write down the tangential or the projective equations of its reciprocal polar by simply interchanging the letters x, y, z with ξ, v, ζ.

Let us conceive a figure composed of points, straight lines, planes, curves of single or double curvature, and curved surfaces, a surface of the second order being taken as the auxiliary or polarizing surface. We may imagine another figure constructed, whose points, straight lines, and planes shall be the poles, conjugate polars, and polar planes of the planes, straight lines, and points of the original figure. These two figures may be called reciprocal polars*, one of the other.

From the reciprocal relations between the two equations

$$\phi(\xi, v, \zeta)=0, \qquad \phi(x, y, z)=0,$$

we may infer the following conclusions:—

1.

A plane is the reciprocal of a point.

2.

A straight line is the reciprocal of a straight line.

---

* Let a point be assumed on a surface (S), and the polar plane of this point taken relative to a surface of the second order (C); as the assumed point varies on the surface (S), the polar plane envelopes a surface (Σ), which is called the *reciprocal polar* of the given surface (S).—Annales de Mathématiques, par Gergonne, tom. viii. p. 201.

Curves and curved surfaces, so related, may with propriety be called *reciprocal polars*, because it is obviously a matter of indifference whether the polar plane of the pole on (S) envelopes the surface (Σ), or the pole of the plane enveloping (Σ) describes (S). [Note to first edition.]

3.	n planes are the reciprocals of n points.	
4.	n planes passing through a straight line	Are the reiprocals of n points in a straight line.
5.	n straight lines in a plane.	n straight lines meeting in a point.
6.	The point in which a plane is pierced by a given straight line.	The plane which passes through a point and a given straight line.
7.	A cone whose vertex is at a given point.	A curve lying in a given plane.
8.	A polygon of n sides in a plane.	A pyramid of n sides passing through a given point.
9.	A point on a curved surface.	A tangent plane to a curved surface.
10.	The point of contact of a tangent plane.	The tangent plane drawn through a given point.
11.	A cone circumscribing a given surface.	A plane section of the reciprocal surface.
12.	A number of surfaces inscribed in the same cone.	As many reciprocal surfaces intersecting each other in the same plane.
13.	A chord joining two points of a surface.	The intersections of two tangent planes to the reciprocal surface.

14.	A polyhedron of *n* faces inscribed in a surface.	A polyhedron of *n* solid angles circumscribed to the reciprocal surface.
15.	A polyhedron of *n* edges inscribed in a surface.	A polyhedron of *n* edges circumscribed to the reciprocal surface.
16.	A number of surfaces meeting in a point.	The same number of reciprocal surfaces touching the same tangent plane.
17.	A surface passing through *n* given points.	The reciprocal surface touching *n* given planes.
18.	A number of parallel straight lines.	The same number of straight lines all lying in a plane passing through the origin.
19.	A curve in a plane passing through the centre of polarizing sphere.	A cylinder whose axis is perpendicular to the plane of the given curve.
20.	A number of straight lines in a plane passing through the centre of the polarizing sphere.	As many straight lines perpendicular to the plane passing through the centre of the polarizing sphere.
21.	A polygon of *n* sides inscribed in a curve.	A polygon of *n* sides circumscribed to the reciprocal curve.
22.	The sides of a polygon inscribed in a curve meet two by two on a straight line.	The lines which join the corresponding angles of the reciprocal curve meet two by two in a point.

23.	The vertices of a triangle move along three fixed lines.	The sides of the reciprocal triangle will pass through three fixed points.
24.	A conic section.	A conic section.
25.	A plane at infinity.	The centre of the polarizing sphere.
26.	A straight line through the centre of the polarizing sphere.	A straight line at infinity.
27.	A plane through the centre of the polarizing sphere.	A point at infinity in the perpendicular to the plane.
28.	*n* points given on a curve of double curvature.	*n* tangent planes to the same developable surface.
29.	A plane intersects a curve of double curvature in *n* points.	*n* tangent planes through a point to the same developable surface.
30.	*n* points common to two or more curves of double curvature.	*n* tangent planes common to two or more developable surfaces.
31.	A series of tangent planes drawn to a curved surface through a point on it.	As many points of contact of a curved surface with a tangent plane.
32.	A cusp on the surface of the one.	A curve of contact with a tangent plane to the other.
33.	A surface is generated by straight lines.	Its reciprocal polar is generated by straight lines.

As the reciprocal polar of a surface of the second order is also a surface of the same degree, a great variety of the properties of these surfaces may be derived in this manner, and thus a duality exists between the properties of curves and surfaces of the second degree, which in the general case is found only between curves and surfaces, and their reciprocal polars.

98.] We shall proceed to illustrate the foregoing principles by their application to a few examples.

The sum of the perpendiculars let fall from n given points on a plane is constant. To determine the envelope of the plane.

Let the sum of the perpendiculars be nk; and let p, q, r be the projective coordinates of one of the points, P the perpendicular from this point on the plane whose tangential coordinates are ξ, υ, and ζ.

Now $P=\dfrac{1-p\xi-q\upsilon-r\zeta}{\sqrt{\xi^2+\upsilon^2+\zeta^2}}$, $P_{\prime}=\dfrac{1-p_{\prime}\xi-q_{\prime}\upsilon-r_{\prime}\zeta}{\sqrt{\xi^2+\upsilon^2+\zeta^2}}$, &c., see (c), sec. [57].

Taking the sum of all the perpendiculars, we shall have

$$n+-(p+p_{\prime}+p_{\prime\prime}+p_{\prime\prime\prime}\ \&c.)\xi-(q+q_{\prime}+q_{\prime\prime}+q_{\prime\prime\prime}\ \&c.)\upsilon$$
$$-(r+r_{\prime}+r_{\prime\prime}+r_{\prime\prime\prime}\ \&c.)\zeta=nk\sqrt{\xi^2+\upsilon^2+\zeta^2}.$$

Let a, b, c be the coordinates of the centre of gravity of the n points, then

$$p+p_{\prime}+p_{\prime\prime}\ \&c.=na,\quad q+q_{\prime}+q_{\prime\prime}\ \&c.=nb,\quad r+r_{\prime}+r_{\prime\prime}\ \&c.=nc,$$

and the equation of the locus now becomes

$$1-a\xi-b\upsilon-c\zeta=k\sqrt{\xi^2+\upsilon^2+\zeta^2};$$

or, squaring and adding,

$$\left.\begin{aligned}(k^2-a^2)\xi^2+(k^2-b^2)\upsilon^2+(k^2-c^2)\zeta^2-2ab\xi\upsilon-2ac\xi\zeta-2bc\upsilon\zeta\\ +2a\xi+2b\upsilon+2c\zeta=1.\end{aligned}\right\}\quad \text{(a)}$$

The tangential equation of a sphere whose centre is at the centre of gravity of the n given points, as may be inferred from (d) [75].

When the sum of the perpendiculars is $=0$, or $k=0$, the equation becomes

$$a\xi+b\upsilon+c\zeta=1,$$

the tangential equation of a point.

99.] *The sum of the squares of the perpendiculars let fall from n given points on a plane whose tangential coordinates are* ξ, υ, ζ *is constant, and equal to* nk^2; *the plane envelopes a surface of the second order.*

Let the projective coordinates of the given points on the coordinate planes be p, q, r, $p_{\prime}$, $q_{\prime}$, $r_{\prime}$, &c., then, as

$$P=\frac{1-p\xi-q\upsilon-r\zeta}{\sqrt{\xi^2+\upsilon^2+\zeta^2}},\quad P_{\prime}=\frac{1-p_{\prime}\xi-q_{\prime}\upsilon-r_{\prime}\zeta\ \&c.}{\sqrt{\xi^2+\upsilon^2+\zeta^2}},\ \&c.,$$

we shall have

$$\left.\begin{array}{l}(p^2+p_{\prime}^2+p_{\prime\prime}^2 \text{ \&c.})\xi^2+(q^2+q_{\prime}^2+q_{\prime\prime}^2\text{\&c.})\upsilon^2 \\ \quad +(r^2+r_{\prime}^2+r_{\prime\prime}^2 \text{ \&c.})\zeta^2+2(pq+p_{\prime}q_{\prime}+p_{\prime\prime}q_{\prime\prime} \text{ \&c.})\xi\upsilon \\ \quad +2(pr+p_{\prime}r_{\prime}+p_{\prime\prime}r_{\prime\prime} \text{ \&c.})\xi\zeta+2(qr+q_{\prime}r_{\prime}+q_{\prime\prime}r_{\prime\prime} \text{ \&c.})\upsilon\zeta \\ \quad -2(p+p_{\prime}+p_{\prime\prime} \text{ \&c.})\xi-2(q+q_{\prime}+q_{\prime\prime} \text{ \&c.})\upsilon \\ \quad -2(r+r_{\prime}+r_{\prime\prime} \text{ \&c.})\zeta+n=nk^2(\xi^2+\upsilon^2+\zeta^2).\end{array}\right\} \quad (\text{a})$$

Now let the centre of gravity of the n points be taken as the origin of coordinates, and let a, b, c be the radii of gyration round the three principal axes of the system of n points, taken as axes of coordinates, then $p^2+p_{\prime}^2+p_{\prime\prime}^2+\text{\&c.}=na^2$, $q^2+q_{\prime}^2+q_{\prime\prime}^2+\text{\&c.}=nb^2$, $r^2+r_{\prime}^2+r_{\prime\prime}^2+\text{\&c.}=nc^2$.

We shall have also $p+p_{\prime}+p_{\prime\prime}+\text{\&c.}=0$, $q+q_{\prime}+q_{\prime\prime}+\text{\&c.}=0$, $r+r_{\prime}+r_{\prime\prime}+\text{\&c.}=0$, $pq+p_{\prime}q_{\prime}+p_{\prime\prime}q_{\prime\prime}+\text{\&c.}=0$, $pr+p_{\prime}r_{\prime}+p_{\prime\prime}r_{\prime\prime}+\text{\&c.}=0$, $qr+q_{\prime}r_{\prime}+q_{\prime\prime}r_{\prime\prime}+\text{\&c.}=0$; hence the equation (a) now becomes

$$(k^2-a^2)\xi^2+(k^2-b^2)\upsilon^2+(k^2-c^2)\zeta^2=1, \quad . \quad . \quad . \quad (\text{b})$$

the tangential equation of an ellipsoid.

The distances of the foci of the principal sections of this surface from the centre are independent of k.

Now k depends on the magnitude of the sum of the squares of the perpendiculars, while a, b, c depend on the relative positions of the n given points. Hence we may infer that if from the same n given points there be let fall different groups of perpendiculars on different planes, the sums of whose squares shall be nk^2, $nk_{\prime}^2$, &c., the several tangent planes will envelope as many confocal ellipsoids.

100.] *A series of surfaces of the second order touch seven fixed planes; the poles of any given plane relative to these surfaces are also on a fixed plane.*

Let the given plane be taken as that of xy, and let the equation of one of the surfaces be

$$\alpha\xi^2+\alpha_{\prime}\upsilon^2+\alpha_{\prime\prime}\zeta^2+2\beta\upsilon\zeta+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi\upsilon+2\gamma\xi+2\gamma_{\prime}\upsilon+2\gamma_{\prime\prime}\zeta=1; \quad (\text{a})$$

the tangential equation of the pole of the plane of xy relative to this surface is

$$\alpha_{\prime\prime}\zeta+\beta\upsilon+\beta_{\prime}\xi+\gamma_{\prime\prime}=0, \text{ see (d), sec. [77].} \quad . \quad . \quad (\text{b})$$

Now as there are seven linear equations to determine the nine unknown coefficients of (a), we may eliminate any six, and connect the three remaining by an equation which also will be linear. Eliminating then α, $\alpha_{\prime}$, $\beta_{\prime\prime}$, γ, $\gamma_{\prime}$, $\gamma_{\prime\prime}$, and α, $\alpha_{\prime}$, $\alpha_{\prime\prime}$, $\beta_{\prime\prime}$, γ, $\gamma_{\prime}$, successively, we shall obtain

$$\left.\begin{array}{l}\alpha_{\prime\prime}=\mathrm{L}\beta+\mathrm{M}\beta_{\prime}+\mathrm{N}, \\ \gamma_{\prime\prime}=\mathrm{L}_{\prime}\beta+\mathrm{M}_{\prime}\beta_{\prime}+\mathrm{N}_{\prime};\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{c})$$

L, M, N, $\mathrm{L}_{\prime}$, $\mathrm{M}_{\prime}$, $\mathrm{N}_{\prime}$ being determinate functions of the twenty-one constant tangential coordinates of the seven fixed planes.

Substituting these values in (b) we get the equation

$$(L\zeta+\upsilon+L_{\prime})\beta+(M\zeta+\xi+M_{\prime})\beta_{\prime}+(N\zeta+N_{\prime})=0. \quad . \quad (d)$$

Now this equation is satisfied, leaving β and $\beta_{\prime}$ indeterminate, by putting each of the three factors in (d) $=0$; solving these equations, we find

$$\xi=\text{constant}, \quad \upsilon=\text{constant}, \quad \zeta=\text{constant};$$

the three tangential coordinates of a fixed plane.

When there are eight fixed planes, it may be shown, in like manner, *that the locus of the poles of any given plane relative to those surfaces is a right line.*

101.] *A surface of the second order touches seven given planes, to find the locus of its centre.*

Let the tangential equation of the given surface be

$$a\xi^2+a_{\prime}\upsilon^2+a_{\prime\prime}\zeta^2+2\beta\upsilon\zeta+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi\zeta+2\gamma\xi+2\gamma_{\prime}\upsilon+2\gamma_{\prime\prime}\zeta=1,$$

and let the twenty-one coordinates of the seven given planes be $\xi_{\prime}$, $\upsilon_{\prime}$, $\zeta_{\prime}$; $\zeta_{\prime\prime}$, $\upsilon_{\prime\prime}$, $\zeta_{\prime\prime}$; $\xi_{\prime\prime\prime}$, $\upsilon_{\prime\prime\prime}$, $\zeta_{\prime\prime\prime}$, &c. Substituting these values successively in the preceding equation, we shall have seven linear equations by which we may eliminate the six quantities a, $a_{\prime}$, $a_{\prime\prime}$; β, $\beta_{\prime}$, $\beta_{\prime\prime}$. The resulting equation will also be linear, and of the form

$$L\gamma+M\gamma_{\prime}+N\gamma_{\prime\prime}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

which is the projective equation of a plane. Now γ, $\gamma_{\prime}$, $\gamma_{\prime\prime}$, as has been shown, are the projective coordinates of the centre of the surface. Hence the centre of the surface moves along a plane. When there are eight planes, we may then eliminate γ or $\gamma_{\prime}$, and the two resulting equations will become

$$L\gamma+M\gamma_{\prime}-1=0, \quad L_{\prime}\gamma+N\gamma_{\prime\prime}-1=0,$$

or the centre will move along a right line.

102.] *If two surfaces of the second order are enveloped by a cone, they may also be enveloped by a second cone.*

Let the vertex of the cone be taken as the origin of coordinates, and let the tangential equations of the surfaces be

$$\left.\begin{array}{l} a\xi^2+a_{\prime}\upsilon^2+a_{\prime\prime}\zeta^2+2\beta\upsilon\zeta+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi\upsilon+2\gamma\xi+2\gamma_{\prime}\upsilon+2\gamma_{\prime\prime}\zeta=1, \\ a\xi^2+a_{\prime}\upsilon^2+a_{\prime\prime}\zeta^2+2b\upsilon\zeta+2b_{\prime}\xi\zeta+2b_{\prime\prime}\xi\upsilon+2c\xi+2c_{\prime}\upsilon+2c_{\prime\prime}\zeta=1; \end{array}\right\} \quad (a)$$

and as the common tangent planes must pass through the origin, ξ, υ, ζ, are the same in the equations of the two surfaces; but at the origin $\frac{1}{\xi}=0$, $\frac{1}{\upsilon}=0$, $\frac{1}{\zeta}=0$. At this point let $\xi=\phi\zeta$, $\upsilon=\psi\zeta$. Substituting these values in the preceding equations and dividing by $\zeta=\infty$,

$$\left.\begin{array}{l} a\phi^2+a_{\prime}\psi^2+a_{\prime\prime}+2\beta\psi+2\beta_{\prime}\phi+2\beta_{\prime\prime}\phi\psi=0, \\ a\phi^2+a_{\prime}\psi^2+a_{\prime\prime}+2b\psi+2b_{\prime}\phi+2b_{\prime\prime}\phi\psi=0; \end{array}\right\} \quad . \quad . \quad (b)$$

and as these equations represent the same tangent plane, they must be identical. Hence we shall have, introducing an equalizing factor λ,

$$a=\lambda a,\ \ a_{\prime}=\lambda a_{\prime},\ \ a_{\prime\prime}=\lambda a_{\prime\prime},\ \ b=\lambda\beta,\ \ b_{\prime}=\lambda\beta_{\prime},\ \ b_{\prime\prime}=\lambda\beta_{\prime\prime}.$$

Making these substitutions in the preceding equations, they become

$$a\xi^2+a_{\prime}\upsilon^2+a_{\prime\prime}\zeta^2+2\beta\zeta\upsilon+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi\upsilon+2\gamma\xi+2\gamma_{\prime}\upsilon+2\gamma_{\prime\prime}\zeta=1,$$

$$\lambda a\xi^2+\lambda a_{\prime}\upsilon^2+\lambda a_{\prime\prime}\zeta^2+2\lambda\beta\zeta\upsilon+2\lambda\beta_{\prime}\xi\zeta+2\lambda\beta_{\prime\prime}\xi\upsilon+2c\xi+2c_{\prime}\upsilon+2c_{\prime\prime}\zeta=1.$$

Multiplying the former equation by λ, and subtracting from it the latter, we get

$$2(\lambda\gamma-c)\xi+2(\lambda\gamma_{\prime}-c_{\prime})\upsilon+2(\lambda\gamma_{\prime\prime}-c_{\prime\prime})\zeta=\lambda-1, \quad . \ . \quad (c)$$

the tangential equation of a point which is the vertex of the second enveloping cone.

The *projective* coordinates of this point are

$$x=\frac{2(\lambda\gamma-c)}{\lambda-1},\quad y=\frac{2(\lambda\gamma_{\prime}-c_{\prime})}{\lambda-1},\quad z=\frac{2(\lambda\gamma_{\prime\prime}-c_{\prime\prime})}{\lambda-1}.$$

103.] *Let a plane cut off from three fixed rectangular axes segments, the sum of which, multiplied by a constant area, shall be equal to the tetrahedron whose faces are the three coordinate planes and the limiting tangent plane; to determine the surface enveloped by this latter plane.*

Let the three coordinates of the variable plane be ξ, υ, ζ; then the volume of the pyramid is $=\frac{1}{6\xi\upsilon\zeta}$. Let the constant area be $\frac{a^2}{3}$, consequently

$$\left(\frac{1}{\xi}+\frac{1}{\upsilon}+\frac{1}{\zeta}\right)\frac{a^2}{3}=\frac{1}{6\xi\upsilon\zeta},\ \text{ or }\ 2a^2(\zeta\upsilon+\xi\zeta+\xi\upsilon)=1 \quad . \ . \quad (a)$$

is the equation of the envelope of the tangent plane, the tangential equation of a surface of the second order.

To find the axes of this surface. Comparing the above with the general tangential equation of a surface of the second order,

$$a\xi^2+a_{\prime}\upsilon^2+a_{\prime\prime}\zeta^2+2\beta\upsilon\zeta+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi\upsilon+2\gamma\xi+2\gamma_{\prime}\upsilon+2\gamma_{\prime\prime}\zeta=1,$$

we shall have

$$a=a_{\prime}=a_{\prime\prime}=0,\quad \gamma=\gamma_{\prime}=\gamma_{\prime\prime}=0,\quad \beta=\beta_{\prime}=\beta_{\prime\prime}=a^2. \quad . \quad (b)$$

Substituting these values in the cubic equation which determines the magnitude of the axes, sec. [83], (f), we get

$$\tau^6-3a^4\tau^2-2a^6=0. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (c)$$

Taking its first derivative, we shall have $\tau^4-a^4=0$.

Now putting (c) in the form

$$\tau^2(\tau^4-a^4)-2a^4(\tau^2+a^2)=0,$$

we find that this equation and its first derivative are satisfied by the root $\tau^2=-a^2$.

Hence two of the roots are each equal to $-a^2$, and the remaining root is $=2a^2$. This we might antecedently have inferred from the absence of the second term in (c).

Consequently as two of the roots are negative and one positive, the surface is a discontinuous hyperboloid or one of two sheets.

Since the linear terms do not appear in (a), the centre of the surface is at the origin.

To determine the position of the axes. In equations (h), sec. [84], substituting for $\delta_{,}\delta_{,,}-\beta^2$ its value $3a^2$, and finding the same values for the other like expressions, we obtain

$$\cos^2\lambda=\frac{1}{3},\quad \cos^2\mu=\frac{1}{3},\quad \cos^2\nu=\frac{1}{3}\,; \quad . \quad . \quad . \quad \text{(d)}$$

hence the positive axis of the surface is equally inclined to the axes of coordinates.

If we were to make in the same formula the necessary substitutions to obtain the position of the two other axes, we should find $\cos\lambda_{,}=\frac{0}{0}$, and so for the other angles also. This evidently should be the case, as the two equal axes have no definite position.

To transform this equation into one that shall contain the squares of the variables.

Let the cosines of the angles which the new axes of coordinates OX, OY, OZ make with the original axes be

$$\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ l,\ m,\ n,\ l_{,},\ m_{,},\ n_{,},$$

as shown in (d) sec. [56]. Then

$$\xi=\frac{\xi_{,}}{\sqrt{3}}+l\upsilon_{,}+l_{,}\zeta_{,},\quad \upsilon=\frac{\xi_{,}}{\sqrt{3}}+m\upsilon_{,}+m_{,}\zeta_{,},\quad \zeta=\frac{\xi_{,}}{\sqrt{3}}+n\upsilon_{,}+n_{,}\zeta_{,}.$$

Substitute these values in (a), and we shall find

$$\left.\begin{aligned} 2a^2\xi_{,}^2+2a^2[ml+nl+mn]\,\upsilon_{,}^2+2a^2[m_{,}l_{,}+n_{,}l_{,}+m_{,}n_{,}]\,\zeta_{,}^2 \\ +2a^2[ml_{,}+m_{,}l+nl_{,}+n_{,}l+m_{,}n+n_{,}m]\,\upsilon_{,}\zeta_{,} \\ +2a^2\frac{(l_{,}+m_{,}+n_{,})}{\sqrt{3}}\,\xi_{,}\zeta_{,}+2a^2\,\frac{(l+m+n)}{\sqrt{3}}\,\xi_{,}\upsilon_{,}=1. \end{aligned}\right\} \quad . \quad . \quad \text{(e)}$$

As the two systems of axes are rectangular, we must consequently have, as shown in sec. [56], $l+m+n=0$, $l_{,}+m_{,}+n_{,}=0$, and

$$ml_{,}+m_{,}l+nl_{,}+n_{,}l+m_{,}n+n_{,}m=0.$$

But this last expression may be written

$$m\,(l+n)+n_{\prime}(l+m)+l_{\prime}(m+n)=0,$$

but $\quad l+n=-m,\ l+m=-n,\ m+n=-l,$

hence $\quad ll_{\prime}+mm_{\prime}+nn_{\prime}=0.$

Now as $l+m+n=0$, the square of this expression must be $=0$, or

$$2lm+2ln+2mn=-(l^2+m^2+n^2)=-1\,;$$

and as

$$l_{\prime}+m_{\prime}+n_{\prime}=0,\ 2l_{\prime}m_{\prime}+2l_{\prime}n_{\prime}+2m_{\prime}n_{\prime}=-(l_{\prime}^2+m_{\prime}^2+n_{\prime}^2)=-1.$$

The equation (e) from these substitutions, now becomes

$$a^2[2\xi^2-\upsilon^2-\zeta^2]=1, \quad \ldots\ldots\ldots \quad \text{(f)}$$

the equation of the discontinuous hyperboloid referred to its axes as axes of coordinates.

104.] *If a series of planes retrench from a cone of the second degree a constant volume, they will envelope a discontinuous hyperboloid, or one of two sheets.*

In the first place, it may be shown, that if the projective equation of a conic section be

$$\mathrm{A}x^2+\mathrm{A}_{\prime}y^2+2\mathrm{B}xy+2\mathrm{C}x+2\mathrm{C}_{\prime}y=1, \quad \ldots \quad \text{(a)}$$

its area S will be

$$\frac{\pi[(\mathrm{A}+\mathrm{C}^2)(\mathrm{A}_{\prime}+\mathrm{C}_{\prime}^2)-(\mathrm{B}+\mathrm{C}\mathrm{C}_{\prime})^2]}{(\mathrm{A}\mathrm{A}_{\prime}-\mathrm{B}^2)^{\frac{3}{2}}}[*]. \quad \ldots \quad \text{(b)}$$

---

[*] This proposition may be proved as follows:—

Let $\quad \mathrm{A}x^2+\mathrm{A}_{\prime}y^2+2\mathrm{B}xy=1, \quad \ldots\ldots\ldots \quad \text{(a)}$

be the equation of a conic section; any rectangular axes passing through the centre being taken as axes of coordinates, we may determine the axes of the section by the following method,

We shall have $\dfrac{d\mathrm{F}}{dx}y-\dfrac{d\mathrm{F}}{dy}x=0$, or $\dfrac{d\mathrm{F}}{x.dx}=\dfrac{d\mathrm{F}}{y.dy}$, as in (33), sec. [22].

Now $\dfrac{d\mathrm{F}}{dx}=\dfrac{\cos\lambda}{\mathrm{P}}$, hence $\dfrac{d\mathrm{F}}{x.dx}=\dfrac{\cos\lambda}{\mathrm{P}x}=\dfrac{\mathrm{P}\cos\lambda}{\mathrm{P}^2x}=\dfrac{1}{\mathrm{P}^2}$, since P coincides with r.

But $\dfrac{2d\mathrm{F}}{xdx}=\dfrac{2}{\mathrm{P}^2}=2\mathrm{A}+2\mathrm{B}\dfrac{y}{x}$, by differentiation; or $\dfrac{1}{\mathrm{P}^2}=\mathrm{A}+\mathrm{B}\dfrac{y}{x}$.

We have also $\dfrac{1}{\mathrm{P}^2}=\mathrm{A}_{\prime}+\mathrm{B}\dfrac{x}{y}$; hence $\left(\dfrac{1}{\mathrm{P}^2}-\mathrm{A}\right)\left(\dfrac{1}{\mathrm{P}^2}-\mathrm{A}_{\prime}\right)=\mathrm{B}^2$,

or $\quad \dfrac{1}{\mathrm{P}^4}-(\mathrm{A}+\mathrm{A}_{\prime})\dfrac{1}{\mathrm{P}^2}+\mathrm{A}\mathrm{A}_{\prime}-\mathrm{B}^2=0. \quad \ldots\ldots\ldots \quad \text{(b)}$

This equation determines the axes of the section. Hence the product of the squares of the reciprocals of the semiaxes is

$$\mathrm{A}\mathrm{A}_{\prime}-\mathrm{B}^2, \quad \ldots\ldots\ldots\ldots \quad \text{(c)}$$

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$, be the equation of the cone, c being its real, a and b its imaginary axes, and let $x\xi+y\upsilon+z\zeta=1$, be the equation of the secant plane.

Eliminating z between these equations, we get the equation, on the plane of xy, of the projection of the section made by the secant plane cutting the cone. This equation is

$$\left(\frac{c^2\zeta^2}{a^2}-\xi^2\right)x^2+\left(\frac{c^2\zeta^2}{b^2}-\upsilon^2\right)y^2-2\xi\upsilon xy+2\xi x+2\upsilon y=1. \quad . \quad \text{(c)}$$

Substituting in the preceding expression for the area, the coefficients of this equation, we get for S, the area of the projection of the section on the plane of XY,

$$S=\frac{\pi abc\zeta}{[c^2\zeta^2-a^2\xi^2-b^2\upsilon^2]^{\frac{3}{2}}}, \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

S being the area of this section, then S sec θ will be the area of the section of the cone made by the secant plane, and if P be the perpendicular from the origin on this plane, the volume of the cone will be equal to S sec θ.P, but $P=\frac{\cos\theta}{\zeta}$, hence the volume $=\frac{S}{\zeta}$.

---

and consequently the area

$$S=\frac{\pi}{\sqrt{AA_{,}-B^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Now let

$$Ax^2+A_{,}y^2+2Bxy+2Cx+2C_{,}y=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

be the general equation of a conic section, and let the origin of coordinates be translated to a point $x=x_{,}+a$, $y=y_{,}+b$, and make the resulting coefficients of $x_{,}$ and $y_{,}$ severally$=0$; the equation of the curve referred to its centre will be

$$Ax^2+A_{,}y^2+2Bxy=1-Ca-C_{,}b,$$

or putting for a and b their values, $a=\frac{BC_{,}-A_{,}C}{AA_{,}-B^2}$, $b=\frac{BC-AC_{,}}{AA_{,}-B^2}$, equation (e) becomes

$$Ax^2+A_{,}y^2+2Bxy=\frac{(A+C^2)(A_{,}+C_{,}^2)-(B+CC_{,})^2}{AA_{,}-B^2}.$$

Dividing by the absolute term, and writing $\overline{A}$, $\overline{A}$, and $\overline{B}$ for the new coefficients of x^2, y^2, and xy, in order to reduce the absolute term to 1,

$$\overline{A}=\frac{A(AA_{,}-B^2)}{(A+C^2)(A_{,}+C_{,}^2)-(B+CC_{,})^2}, \quad \overline{A}_{,}=\frac{A_{,}(AA_{,}-B^2)}{(A+C^2)(A_{,}+C_{,}^2)-(B+CC_{,})^2},$$

$$\overline{B}=\frac{B(AA_{,}-B^2)}{(A+C^2)(A_{,}+C_{,}^2)-(B+CC_{,})^2}.$$

Consequently as the area is $=\frac{\pi}{\sqrt{\overline{A}\overline{A}_{,}-\overline{B}^2}}$, the area of any conic section in terms of its general coefficients is

$$S=\frac{\pi[(A+C^2)(A_{,}+C_{,}^2)-(B+CC_{,})^2]}{[(AA_{,}-B^2)]^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

Let this volume be also $=\pi abc$, suppose, hence substituting in the preceding expression this value of S, the resulting expression becomes

$$c^2\zeta^2-a^2\xi^2-b^2\upsilon^2=1, \quad . \; . \; . \; . \; . \; . \; . \; . \qquad \text{(e)}$$

the tangential equation of a discontinuous hyperboloid.

In nearly the same way it may be shown, that if a series of planes retrench from any surface of the second order a constant volume, the enveloped surface is a surface concentric similar and similarly situated.

105.] *On the cubic equation of axes, when the surface is one of revolution, and the origin at a focus.*

In the general tangential equation of a surface of the second order,

$$a\xi^2+a_{\prime}\upsilon^2+a_{\prime\prime}\zeta^2+2\beta\upsilon\zeta+2\beta_{\prime}\xi\zeta+2\beta_{\prime\prime}\xi\upsilon+2\gamma\xi+2\gamma_{\prime}\upsilon+2\gamma_{\prime\prime}\zeta=1. \quad \text{(a)}$$

Let $a=a_{\prime}=a_{\prime\prime}$, and $\beta=\beta_{\prime}=\beta_{\prime\prime}=0$, then the preceding equation becomes

$$a\xi^2+a\upsilon^2+a\zeta^2+2\gamma\xi+2\gamma_{\prime}\upsilon+2\gamma_{\prime\prime}\zeta=1. \quad . \; . \; . \qquad \text{(b)}$$

If we now translate the origin of coordinates to the centre of the surface, in parallel directions, using the formula given in sec. [75], (d), the equation, referred to the central axes of coordinates, becomes

$$\left.\begin{array}{r}(a+\gamma^2)\xi^2+(a+\gamma_{\prime}^2)\upsilon^2+(a+\gamma_{\prime\prime}^2)\zeta^2+2\gamma_{\prime}\gamma_{\prime\prime}\upsilon\zeta\\ +2\gamma_{\prime\prime}\gamma\xi\zeta+2\gamma_{\prime}\gamma\xi\upsilon=1.\end{array}\right\} \; . \; . \; . \qquad \text{(c)}$$

Now if we write

$$\left.\begin{array}{r}(a+\gamma^2)=\bar{a}, \;\; (a+\gamma_{\prime}^2)=\bar{a}, \;\; (a+\gamma_{\prime\prime}^2)=\bar{a}_{\prime\prime},\\ \gamma_{\prime}\gamma_{\prime\prime}=\bar{\beta}, \;\; \gamma_{\prime\prime}\gamma=\bar{\beta}_{\prime}, \;\; \gamma\gamma_{\prime}=\bar{\beta}_{\prime\prime},\end{array}\right\} \; . \; . \; . \; . \qquad \text{(d)}$$

the cubic equation of axes, see sec. [83], (f), will become

$$\left.\begin{array}{r}\mathrm{P}^6-(\bar{a}+\bar{a}_{\prime}+\bar{a}_{\prime\prime})\mathrm{P}^4+[(\bar{a}_{\prime}\bar{a}_{\prime\prime}-\bar{\beta}^2)+(\bar{a}_{\prime\prime}\bar{a}-\bar{\beta}_{\prime}^2)+(\bar{a}_{\prime}\bar{a}-\bar{\beta}_{\prime\prime}^2)]\mathrm{P}^2\\ +\bar{\beta}^2\bar{a}+\bar{\beta}_{\prime}^2\bar{a}_{\prime}+\bar{\beta}_{\prime\prime}^2\bar{a}_{\prime\prime}-\bar{a}\bar{a}_{\prime}\bar{a}_{\prime\prime}-2\bar{\beta}\bar{\beta}_{\prime}\bar{\beta}_{\prime\prime}=0.\end{array}\right\} \qquad \text{(e)}$$

Or substituting for these coefficients their values as given in the preceding equations (d), and putting τ for P^2, the cubic equation now becomes

$$\left.\begin{array}{r}\tau^3-[3a+\gamma^2+\gamma_{\prime}^2+\gamma_{\prime\prime}^2]\tau^2+[3a^2+2a(\gamma^2+\gamma_{\prime}^2+\gamma_{\prime\prime}^2)]\tau\\ -[a^3+a^2(\gamma^2+\gamma_{\prime}^2+\gamma_{\prime\prime}^2)]=0.\end{array}\right\} \; . \qquad \text{(f)}$$

Let $$a+\gamma^2+\gamma_{\prime}^2+\gamma_{\prime\prime}^2=k, \quad . \; . \; . \; . \; . \; . \qquad \text{(g)}$$

and the preceding equation may now be written

$$\tau^3-(2a+k)\tau^2+(a^2+2ak)\tau-a^2k=0, \quad . \; . \; . \qquad \text{(h)}$$

of which equation the three roots are manifestly *a*, *a*, and *k*.

Let *a* and *b* be the semiaxes of this surface, then as

$$a^2=a+\gamma^2+\gamma_{\prime}^2+\gamma_{\prime\prime}^2, \text{ and } b^2=a, \;\; a^2-b^2=\gamma^2+\gamma_{\prime}^2+\gamma_{\prime\prime}^2, \; . \qquad \text{(i)}$$

or the eccentric distance is equal to the distance of the origin from the centre.

If we now turn to the formulæ in sec. [84], (h), by which the positions of the axes are determined, bearing in mind that

$$(\bar{a}-\tau^2)=\delta, \quad (\bar{a}_{,}-\tau^2)=\delta_{,},$$

or in this case

$$(a+\gamma^2-\tau^2)=\delta, \quad (a+\gamma_{,}^2-\tau^2)=\delta_{,};$$

and putting for τ^2 the value k, the square of the greater semiaxis of the surface, since

$$\left.\begin{array}{ll} & k=a+\gamma^2+\gamma_{,}^2+\gamma_{,,}^2, \quad \delta=-(\gamma_{,}^2+\gamma_{,,}^2), \quad \delta_{,}=-(\gamma_{,,}^2+\gamma^2), \\ \text{we shall have} & \delta\delta_{,}-\beta_{,,}^2=\gamma_{,,}^2(\gamma^2+\gamma_{,}^2+\gamma_{,,}^2), \\ \text{and} & (\delta\delta_{,}-\beta_{,,}^2)+(\delta_{,}\delta_{,,}-\beta^2)+(\delta_{,,}\delta-\beta_{,}^2)=(\gamma^2+\gamma_{,}^2+\gamma_{,,}^2)^2. \end{array}\right\} \quad (\mathrm{j})$$

Consequently the formulæ for the determination of the inclination of the major axis to the axes of coordinates are

$$\left.\begin{array}{ll} \cos\lambda=\dfrac{\gamma}{\sqrt{\gamma^2+\gamma_{,}^2+\gamma_{,,}^2}}, & \cos\mu=\dfrac{\gamma_{,}}{\sqrt{\gamma^2+\gamma_{,}^2+\gamma_{,,}^2}}, \\ \cos\nu=\dfrac{\gamma_{,,}}{\sqrt{\gamma^2+\gamma_{,}^2+\gamma_{,,}^2}}, & \end{array}\right\} \quad . \quad (\mathrm{k})$$

equations which determine the position of the semiaxis $\sqrt{k}$. But the line drawn from the origin to the centre of the surface makes the same angles with the axes of coordinates, therefore this line coincides with the semiaxis $\sqrt{k}$; hence its origin is on this axis, and $k-a=\gamma^2+\gamma_{,}^2+\gamma_{,,}^2$, therefore the eccentric distance is equal to the distance of the origin from the centre, or the focus is at the origin.

Had we substituted a instead of k in the formulæ, sec. [84], (h), which determine the inclination of the axes of the surface to the axes of coordinates, we should have found

$$\cos\lambda=\frac{0}{0}, \quad \cos\mu=\frac{0}{0}, \quad \cos\nu=\frac{0}{0}.$$

We may therefore infer that when in the general tangential equation of a surface of the second order, the coefficients of the squares of the variables are all equal, and the coefficients of the rectangles are each $=0$, the equation represents a surface of revolution, the origin of coordinates being at one of the foci, while the coefficients of the linear terms are twice the projective coordinates of the centre of the surface.

106.] *Three straight lines, constituting a right-angled trihedral angle, revolve round a fixed point in space, meeting a surface of the second order* (S) *in three points. The plane which passes through these three points envelopes a surface of revolution* (Σ) *of the second* *order, whose focus is at the given point, and whose directrix plane relative to this focus is the polar plane of the fixed point relative to the given surface* (S).

Let the fixed point be taken as origin, and let the projective equation of (S) be

$$\left.\begin{array}{r} Ax^2+A_{\prime}y^2+A_{\prime\prime}z^2+2Byz+2B_{\prime}xz+2B_{\prime\prime}xy \\ +2Cx+2C_{\prime}y+2C_{\prime\prime}z=1. \end{array}\right\} \quad . \quad . \quad \text{(a)}$$

Now it may easily be shown that the projective equation of the polar plane of the origin is

$$Cx+C_{\prime}y+C_{\prime\prime}z=1^{*}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Let r be the length of one of the revolving lines, and let λ, μ, ν be the angles it makes with the axes of coordinates, then

$$x=r\cos\lambda, \quad y=r\cos\mu, \quad \text{and} \quad z=r\cos\nu.$$

Substitute these values of x, y, and z in the dual equation $x\xi+y\upsilon+z\zeta=1$, and it becomes

$$\xi\cos\lambda+\upsilon\cos\mu+\zeta\cos\nu=\frac{1}{r},$$

or writing l, m, n for $\cos\lambda$, $\cos\mu$, $\cos\nu$ the preceding formula becomes

$$l\xi+m\upsilon+n\zeta=\frac{1}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

In the general equation (a), writing lr, mr, nr for x, y, z, it becomes

$$\left.\begin{array}{l} Al^2+A_{\prime}m^2+A_{\prime\prime}n^2+2Bmn+2B_{\prime}ln+2B_{\prime\prime}lm \\ +2(Cl+C_{\prime}m+C_{\prime\prime}n)\,\dfrac{1}{r}=\dfrac{1}{r^2}. \end{array}\right\} \quad . \quad . \quad . \quad \text{(d)}$$

Eliminating r between these equations (c) and (d), we obtain the resulting expression

$$\left.\begin{array}{l} Al^2+A_{\prime}m^2+A_{\prime\prime}n^2+2Bmn+2B_{\prime}ln+2B_{\prime\prime}lm \\ \quad +2(Cl^2\xi+C_{\prime}m^2\upsilon+C_{\prime\prime}n^2\zeta)+2Cl(m\upsilon+n\zeta) \\ \quad +2C_{\prime}m(l\xi+n\zeta)+2C_{\prime\prime}n(l\xi+m\upsilon) \\ \quad =l^2\xi^2+m^2\upsilon^2+n^2\zeta^2+2mn\upsilon\zeta+2ln\xi\zeta+2lm\xi\upsilon. \end{array}\right\} \quad . \quad . \quad \text{(e)}$$

---

[*] Let the projective equation of a surface of the second order be

$$Ax^2+A_{\prime}y^2+A_{\prime\prime}z^2+2Byz+2B_{\prime}xz+2B_{\prime\prime}xy+2Cx+2C_{\prime}y+2C_{\prime\prime}z=1.$$

Then it may be shown that the locus of the middle points of all the chords of this surface passing through the origin of coordinates is the surface whose equation is

$$Ax^2+A_{\prime}y^2+A_{\prime\prime}z^2+2Byz+2B_{\prime}xz+2B_{\prime\prime}xy+Cx+C_{\prime}y+Cz \;=0;$$

and if we subtract this latter equation from the former, we shall find the locus of the intersection of these two surfaces. Subtracting the latter from the former, the equation of the locus is

$$Cx+C_{\prime}y+C_{\prime\prime}z=1,$$

the projective equation of a plane.

If we now find similar expressions for the other two revolving lines $r_{\prime}$ and $r_{\prime\prime}$, adding all three equations together, and introducing the six relations of the nine direction cosines given in (e), sec. [56], we shall obtain for the tangential equation of the enveloped surface (Σ),

$$\xi^2+v^2+\zeta^2-2C\xi-2C_{\prime}v-2C_{\prime\prime}\zeta=A+A_{\prime}+A_{\prime\prime}. \quad . \quad . \quad (f)$$

If we substitute $\quad \frac{1}{a}$ for $A+A_{\prime}+A_{\prime\prime} \quad . \quad . \quad . \quad . \quad . \quad (g)$

the preceding equation will be transformed into

$$a(\xi^2+v^2+\zeta^2)-2aC\xi-2aC_{\prime}v-2aC_{\prime\prime}\zeta=1. \quad . \quad . \quad . \quad (h)$$

Now as the coefficients of the squares of the variables are equal, and the rectangles vanish, (h) is the tangential equation of a surface of revolution whose focus is at the origin, as has been already shown in sec. [105].

The projective coordinates of the centre of (Σ) are aC, $aC_{\prime}$, and $aC_{\prime\prime}$. The cosines of the angles which the major axis of (Σ) makes with the axes of coordinates are

$$\frac{aC}{\sqrt{a^2C^2+a^2C_{\prime}^2+a^2C_{\prime\prime}^2}}=\frac{C}{\sqrt{C^2+C_{\prime}^2+C_{\prime\prime}^2}}, \text{ as also}$$

$$\frac{C_{\prime}}{\sqrt{C^2+C_{\prime}^2+C_{\prime\prime}^2}} \text{ and } \frac{C_{\prime\prime}}{\sqrt{C^2+C_{\prime}^2+C_{\prime\prime}^2}};$$

but these are the cosines of the angles which a perpendicular P from the fixed point the origin, on the polar plane of this point relative to (S) makes with the axes of coordinates; hence the major axis of (Σ) coincides with this perpendicular.

This plane is a directrix plane of (Σ); for if a and b be the semi-axis of this surface,

$$b^2=a \text{ and } a^2=a+a^2(C^2+C_{\prime}^2+C_{\prime\prime}^2), \text{ see sec. [105], (i);}$$

hence $\frac{a^2-b^2}{b^4}=C^2+C_{\prime}^2+C_{\prime\prime}^2=\frac{1}{P^2}$ or $P=\frac{a}{e}-ae$, or P is the distance between the focus and the directrix plane.

107.] *To show that the Continuous Hyperboloid admits of linear generatrices.*

The tangential equation of the continuous hyperboloid $a^2\xi^2+b^2v^2-c^2\zeta^2=1$ may be written

$$(a\xi+c\zeta)(a\xi-c\zeta)=(1+bv)(1-bv); \quad . \quad . \quad . \quad (a)$$

and if we assume the equation of a straight line in the plane of XY,

$$p\xi+qv=1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

eliminating v between these equations, the resulting expression becomes

$$(a^2q^2+b^2p^2)\xi^2-2b^2p\xi+b^2-q^2=c^2q^2\zeta^2. \quad . \quad . \quad . \quad (c)$$

Now this is the tangential equation of the curve in which the tangent plane through the point (p, q) on the plane of XY cuts the plane of ZX; and in order that this intersection may be a straight line each side of the preceding equation must be a complete square, or

$$(a^2q^2+b^2p^2)(b^2-q^2)=b^4p^2,$$

or

$$a^2q^2+b^2p^2=a^2b^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{d})$$

or p and q must be ordinates of the principal section of the surface in the plane of XY.

If we introduce this value of $a^2q^2+b^2p^2$ into the preceding equation, the result becomes

$$a^2\xi^2-2p\xi+\frac{p^2}{a^2}=\frac{c^2}{b^2}q^2\zeta^2,$$

or taking the square root,

$$a\xi-\frac{p}{a}=\pm\frac{c}{b}q\zeta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{e})$$

It is obvious that if either c^2 were negative, or b^2 negative, the square root would be imaginary.

Hence no surface, the squares of whose axes are all positive, or one positive and two negative, can admit of rectilinear generatrices.

The preceding equation may be written in the usual form,

$$\frac{a^2}{p}.\xi\pm\frac{acq}{bp}.\zeta=1.$$

As p and q are always ordinates of the principal ellipse in the plane of XY, it follows that every rectilinear generatrix to the continuous hyperboloid must always pass through a point on the principal elliptic section of the surface.

108.] *Let a surface of the second order be cut by a given secant plane; to determine the tangential equation of the section of the surface made by this plane.*

Let the tangential equation of the surface be

$$a^2\xi^2+b^2v^2+c^2\zeta^2=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{a})$$

Let the coordinates of the fixed plane which cuts the surface in the section whose equation is required be $\xi_{\prime}$, $v_{\prime}$, $\zeta_{\prime}$, and let $\xi=m\zeta+a$, $v=n\zeta+\beta$ be the equations of a straight line in space. Let this straight line be in the plane whose coordinates are $\xi_{\prime}$, $v_{\prime}$, $\zeta_{\prime}$, then these variables must satisfy the given equations, and we shall have

$$\xi-\xi_{\prime}=m(\zeta-\zeta_{\prime}), \text{ and } v-v_{\prime}=n(\zeta-\zeta_{\prime}).$$

Substituting these values of ξ and v in the equation of the curved surface, we shall have two resulting values of ζ, which are the reciprocals of the intercepts of the axis of Z made by two tangent planes passing through the straight line in the plane whose coordinates are $\xi_{\prime}$, $v_{\prime}$, $\zeta_{\prime}$.

When this line becomes a tangent to the section of the surface the two tangent planes coincide; hence the two values of ζ become equal, and they are given by a quadratic equation whose roots must be equal. Now

$$\left.\begin{aligned}\xi^2&=\xi_{\prime}^2+2m\xi_{\prime}(\zeta-\zeta_{\prime})+m^2(\zeta-\zeta_{\prime})^2,\\ \upsilon^2&=\upsilon_{\prime}^2+2n\upsilon_{\prime}(\zeta-\zeta_{\prime})+n^2(\zeta-\zeta_{\prime})^2,\\ \zeta^2&=\zeta_{\prime}^2+2\zeta_{\prime}(\zeta-\zeta_{\prime})+(\zeta-\zeta_{\prime})^2.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Multiply the first by a^2, the second by b^2, the last by c^2, and add together these expressions. The result becomes

$$\begin{aligned}0=a^2\xi_{\prime}^2+b^2\upsilon_{\prime}^2+c^2\zeta_{\prime}^2-1+2[a^2m\xi_{\prime}+b^2n\upsilon_{\prime}+c^2\zeta_{\prime}](\zeta-\zeta_{\prime})\\ +(m^2a^2+n^2b^2+c^2)(\zeta-\zeta_{\prime})^2.\end{aligned}$$

Now as this equation must have two equal roots, we shall have

$$(a^2\xi_{\prime}^2+b^2\upsilon_{\prime}^2+c^2\zeta_{\prime}^2-1)(m^2a^2+n^2b^2+c^2)=[a^2m\xi_{\prime}+b^2n\upsilon_{\prime}+c^2\zeta_{\prime}]^2, \quad \text{(c)}$$

and this may be reduced to the form

$$a^2b^2(n\xi_{\prime}-m\upsilon_{\prime})^2+b^2c^2(\upsilon_{\prime}-n\zeta_{\prime})^2+a^2c^2(m\zeta_{\prime}-\xi_{\prime})^2=m^2a^2+n^2b^2+c^2;$$

and if we substitute in this equation for m and n their values

$$m=\frac{\xi-\xi_{\prime}}{\zeta-\zeta_{\prime}}, \quad n=\frac{\upsilon-\upsilon_{\prime}}{\zeta-\zeta_{\prime}},$$

the preceding expression becomes

$$\left.\begin{aligned}a^2b^2(\xi_{\prime}\upsilon-\xi\upsilon_{\prime})^2+b^2c^2(\upsilon_{\prime}\zeta-\upsilon\zeta_{\prime})^2+a^2c^2(\zeta_{\prime}\xi-\zeta\xi_{\prime})^2\\ =a^2(\xi-\xi_{\prime})^2+b^2(\upsilon-\upsilon_{\prime})^2+c^2(\zeta-\zeta_{\prime})^2.\end{aligned}\right\} \quad . \quad . \quad \text{(d)}$$

This expression may still further be reduced to the form

$$\left.\begin{aligned}[a^2\xi_{\prime}^2+b^2\upsilon_{\prime}^2+c^2\zeta_{\prime}^2-1]\,[a^2\xi^2+b^2\upsilon^2+c^2\zeta^2-1]\\ =(a^2\xi_{\prime}\xi+b^2\upsilon_{\prime}\upsilon+c^2\zeta_{\prime}\zeta-1)^2.\end{aligned}\right\} \quad . \quad . \quad \text{(e)}$$

The projective equation of a cone circumscribing a given surface of the second order is

$$\left.\begin{aligned}[a^2x_{\prime}^2+b^2y_{\prime}^2+c^2z_{\prime}^2-1]\,[a^2x^2+b^2y^2+c^2z^2-1]\\ =[a^2x_{\prime}x+b^2y_{\prime}y+c^2z_{\prime}z-1]^2,\end{aligned}\right\} \quad . \quad . \quad \text{(f)}$$

$x_{\prime}$, $y_{\prime}$, and $z_{\prime}$ being the coordinates of the vertex of the cone. The duality of the two equations (e) and (f) is manifest.

109.] *The reciprocal polar of any surface of the second order, the centre of the directrix surface being on the given surface, is a paraboloid.*

The directrix surface being for simplicity a sphere whose radius is unity, at whose centre the origin of coordinates is placed, let the projective equation of the given surface be

$$\mathrm{A}x^2+\mathrm{A}_{\prime}y^2+\mathrm{A}_{\prime\prime}z^2+2\mathrm{B}yz+2\mathrm{B}_{\prime}xz+2\mathrm{B}_{\prime\prime}xy+2\mathrm{C}x+2\mathrm{C}_{\prime}y+2\mathrm{C}_{\prime\prime}z=0,$$

the tangential equation of its reciprocal polar is (see (d), sec. [87])

$$\mathrm{A}\xi^2+\mathrm{A}_{\prime}\upsilon^2+\mathrm{A}_{\prime\prime}\zeta^2+2\mathrm{B}\upsilon\zeta+2\mathrm{B}_{\prime}\xi\zeta+2\mathrm{B}_{\prime\prime}\xi\upsilon+2\mathrm{C}\xi+2\mathrm{C}_{\prime}\upsilon+2\mathrm{C}_{\prime\prime}\zeta=0,$$

the tangential equation of a paraboloid.

CHAPTER XI.

ON CONCYCLIC SURFACES OF THE SECOND ORDER*.

110.] The properties of confocal surfaces of the second order, or surfaces whose principal sections have the same foci, are discussed at considerable length in various publications, especially on the Continent, devoted to the cultivation of mathematical science; while the dual properties of their reciprocal surfaces have not been at all noticed, so far as I am aware. It is true that M. Chasles and other geometers who followed him have discussed the properties of cyclic cones; but the theory admits of much wider application.

It is known, and is very easily proved, that every umbilical surface of the second order may be described by the parallel motion of a variable circle whose centre moves along a fixed line.

Concyclic surfaces may therefore be defined as concentric surfaces of the second order, having their axes coincident, and the planes of their circular sections parallel.

Concyclic surfaces are the reciprocal polars of confocal surfaces.

Let a, b, c be the semiaxes of the original confocal surfaces in the order of magnitude. Let $a^2-b^2=h^2$, $a^2-c^2=k^2$. In confocal surfaces a, b, and c are supposed to vary, while h and k are constant.

Let $c_{,}$, $b_{,}$, $a_{,}$ be the semiaxes, in the order of magnitude, of the derived surface.

The radius of the circular section of the derived ellipsoid which passes through the centre is $b_{,}$; $b_{,}$ will be a semidiameter of the principal section which contains the greatest and least axis, hence $b_{,}$ is a semidiameter of the principal section whose equation is $\frac{x^2}{a_{,}^2}+\frac{z^2}{c_{,}^2}=1$. Let θ be the angle which $b_{,}$ makes with the greatest axis, then $\frac{1}{b_{,}^2}=\frac{\cos^2\theta}{c_{,}^2}+\frac{\sin^2\theta}{a_{,}^2}$, and hence

$$\tan^2\theta=\frac{\frac{1}{b_{,}^2}-\frac{1}{c_{,}^2}}{\frac{1}{a_{,}^2}-\frac{1}{c_{,}^2}}=\frac{a_{,}^2(c_{,}^2-b_{,}^2)}{b_{,}^2(c_{,}^2-a_{,}^2)}. \quad \ldots\ldots \quad \text{(a)}$$

---

* It is strange how the properties of systems of concentric surfaces of the second order having coincident circular sections or, as they may be more briefly termed, *concyclic surfaces*, have hitherto almost wholly, at least so far as the author is aware, escaped the observation of geometers; it is the more remarkable as the theorems connected with the subject are numerous, and many of great elegance.

So far indeed as the properties of such cones are concerned, and of spherical conics thence derived, M. Chasles has discussed them in two memoirs of singular simplicity and beauty, published in the 'Brussels Transactions,' tome v. 1829. [Note to first edition.]

This is the angle which the plane of the circular section makes with the plane of the greatest and mean axis of the derived surface.

Now in confocal surfaces let $b^2-c^2=h^2$, $a^2-c^2=k^2$; hence

$$\frac{h^2}{k^2}=\frac{b^2-c^2}{a^2-c^2}=\frac{\dfrac{1}{b_{\prime}^2}-\dfrac{1}{c_{\prime}^2}}{\dfrac{1}{a_{\prime}^2}-\dfrac{1}{c_{\prime}^2}}=\frac{a_{\prime}(c_{\prime}^2-b_{\prime}^2)}{b_{\prime}^2(c_{\prime}^2-a_{\prime}^2)}. \quad . \quad . \quad . \quad \text{(b)}$$

Comparing this with the preceding expression, we get $\tan\theta=\frac{h}{k}$; but h and k are constant, hence $\tan\theta$ is constant, or all reciprocal polars of confocal surfaces have their circular sections parallel.

$$\left.\begin{array}{l}\text{In confocal ellipsoids } b^2=c^2+h^2,\ a^2=c^2+k^2.\\ \text{In concyclic ellipsoids } \dfrac{1}{b^2}=\dfrac{1}{c^2}+\dfrac{1}{f^2},\ \dfrac{1}{a^2}=\dfrac{1}{c^2}+\dfrac{1}{g^2}.\end{array}\right\} \quad . \quad \text{(c)}$$

111.] We shall give a few examples of the analogies between these surfaces.

If parallel tangent planes be drawn to a series of confocal surfaces, and perpendiculars from the centre be let fall upon them, the differences of the squares of these perpendiculars will be independent of their direction. Thus, let one of the perpendiculars be $P^2=a^2\cos^2\lambda+b\cos^2\mu+c^2\cos^2\nu$; now $b^2=c^2+h^2$, $a^2=c^2+k^2$, hence

$$P^2=c^2+h^2\cos^2\mu+k^2\cos^2\lambda;$$

for any other perpendicular on a parallel plane

$$\left.\begin{array}{ll} & P_{\prime}^2=c_{\prime}^2+h^2\cos^2\mu+k^2\cos^2\lambda;\\ \text{hence} & P^2-P_{\prime}^2=c^2-c_{\prime}^2.\end{array}\right\} \quad . \quad . \quad . \quad . \quad \text{(a)}$$

In like manner, if there be any number of concyclic ellipsoids having coincident diameters, the differences of the reciprocals of the squares of these diameters are independent of their position.

Let

$$\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}=\frac{1}{\rho^2};$$

but

$$\frac{1}{b^2}=\frac{1}{c^2}+\frac{1}{f^2},\quad \frac{1}{a^2}=\frac{1}{c^2}+\frac{1}{g^2};$$

hence

$$\frac{1}{\rho^2}=\frac{1}{c^2}+\frac{\cos^2\alpha}{g^2}+\frac{\cos^2\beta}{f^2},\ \text{and also}\ \frac{1}{\rho_{\prime}^2}=\frac{1}{c_{\prime}^2}+\frac{\cos^2\alpha}{g^2}+\frac{\cos^2\beta}{f^2}.$$

Consequently

$$\frac{1}{\rho^2}-\frac{1}{\rho_{\prime}^2}=\frac{1}{c^2}-\frac{1}{c_{\prime}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

These are simple theorems, but they serve to illustrate the complete identity that exists in the analytical investigation of the principles of duality in all their diversified forms.

112.] *Through a given point three central confocal surfaces may be described: an ellipsoid, a continuous and a discontinuous hyperboloid,* or as they are named by French writers, *l'hyperboloïde à une nappe* and *l'hyperboloïde à deux nappes.*

Let α, β, γ be the projective coordinates of the point. Let $a^2=c^2+k^2$, $b^2=c^2+h^2$, $a_{\prime}^2=c_{\prime}^2+k^2$, $b_{\prime}^2=c_{\prime}^2+h^2$; $a_{\prime\prime}^2=c_{\prime\prime}^2+k^2$, $b_{\prime\prime}^2=c_{\prime\prime}^2+h^2$, then the equation of the surface becomes

$$\frac{\alpha^2}{c^2+k^2}+\frac{\beta^2}{c^2+h^2}+\frac{\gamma^2}{c^2}=1, \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

or reducing,

$$\left.\begin{array}{r} c^6-c^4(\alpha^2+\beta^2+\gamma^2-h^2-k^2)-c^2(\alpha^2h^2+\beta^2k^2+\gamma^2h^2+\gamma^2k^2-k^2h^2) \\ -\gamma^2h^2k^2=0. \end{array}\right\}\text{(b)}$$

Now as there are two permanencies and but one variation of sign in this equation, we shall have, by the theory of equations, two of the roots negative and one positive. The product of the squares of the three coincident axes is $\gamma^2\, h^2\, k^2$.

Let c^2, $c_{\prime}^2$, $c_{\prime\prime}^2$ be the three roots of this equation, then

$$c^2+c_{\prime}^2+c_{\prime\prime}^2=\alpha^2+\beta^2+\gamma^2-h^2-k^2,$$

or as

$$a^2=c^2+k^2, \quad b_{\prime}^2=c_{\prime}^2+h^2,$$

we find that

$$a^2+b_{\prime}^2+c_{\prime\prime}^2=\alpha^2+\beta^2+\gamma^2,$$

or the sum of the square of the major axis of the first surface + the square of the mean axis of the second surface + the square of the least semiaxis of the third surface is equal to four times the square of the distance of the given point from the origin.

Had the equation been solved for b^2, we should have found two of the values of b^2 positive and one negative; while if the equation had been solved for a^2, the three values of a^2 would have been found positive. Thus the three confocal surfaces passing through a point are an ellipsoid, a discontinuous hyperboloid, and a continuous hyperboloid.

If the two hyperboloids be assumed as constant and the ellipsoid as variable, the successive points in which the variable ellipsoids meet the curve in which the hyperboloids intersect are "*corresponding points*" in the theory of the attraction of ellipsoids; for since $c^2\, c_{\prime}^2\, c_{\prime\prime}^2=h^2\, k^2\, \gamma^2$, while $c_{\prime}^2$ and $c_{\prime\prime}^2$ are constant, c varies as γ.

Had the equation (a) been solved for a^2, we should have

$$\begin{array}{l} a^6-a^4[\alpha^2+\beta^2+\gamma^2+h^2+k^2]+a^2[\alpha^2h^2+\alpha^2k^2+h^2\gamma^2+\beta^2k^2+h^2k^2] \\ \qquad -\alpha^2h^2k^2=0, \end{array}$$

in which there are three variations and no permanence of sign. Hence the three values of a^2 are positive.

Since $a^2+a_{\prime}^2+a_{\prime\prime}^2=a^2+\beta^2+\gamma^2+h^2+k^2$,

$$a^2+(a_{\prime}^2-h^2)+(a_{\prime\prime}^2-k^2) \text{ or } a^2+b_{\prime}^2+c_{\prime\prime}^2=a^2+\beta^2+\gamma^2.$$

113.] *A series of concyclic surfaces of the second order touch a given plane whose coordinates are ξ, v, ζ. To determine the equation of the axes of these surfaces.*

Let $a^2=\frac{1}{\alpha^2}$, $b^2=\frac{1}{\beta^2}$, $c^2=\frac{1}{\gamma^2}$, $h^2=\frac{1}{\lambda^2}$, $k^2=\frac{1}{\kappa^2}$,

then $$\beta^2=\alpha^2-\lambda^2,\ \gamma^2=\alpha^2-\kappa^2,$$

and the tangential equation of a surface touching the fixed plane is

$$\frac{\xi^2}{\alpha^2}+\frac{v^2}{\alpha^2-\lambda^2}+\frac{\zeta^2}{\alpha^2-\kappa^2}=1. \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Reducing this equation, and arranging by powers of α,

$$\left.\begin{aligned}\alpha^6-\alpha^4[\varpi^2+\kappa^2+\lambda^2]+\alpha^2[\lambda^2\kappa^2+(\lambda^2+\kappa^2)\xi^2+\lambda^2\zeta^2+\kappa^2v^2]\\ -\xi^2\lambda^2\kappa^2=0.\end{aligned}\right\} \quad \text{(b)}$$

Hence, as there are three variations of sign and no permanence in this equation, the roots are all positive.

It may be shown that as the three confocal surfaces which pass through a given point are each one of the three central surfaces of the second order, so of the three concyclic surfaces which touch a given plane one is an ellipsoid, the second a continuous hyperboloid, and the third a discontinuous hyperboloid.

114.] *A common tangent plane is drawn to three concyclic surfaces of the second order, the three points of contact two by two subtend right angles at the centre.*

Let $$a^2\xi^2+b^2v^2+c^2\zeta^2=1, \text{ and } a_{\prime}^2\xi^2+b_{\prime}^2v^2+c_{\prime}^2\zeta^2=1, \quad . \quad \text{(a)}$$

be the tangential equations of two concyclic surfaces of the second order.

Subtracting these equations one from the other, there results

$$(a^2-a_{\prime}^2)\xi^2+(b^2-b_{\prime}^2)v^2+(c^2-c_{\prime}^2)\zeta^2=0. \quad . \quad . \quad . \quad \text{(b)}$$

But as the surfaces are concyclic,

$$\frac{1}{a_{\prime}^2}=\frac{1}{a^2}+\frac{1}{k^2},\ \frac{1}{b_{\prime}^2}=\frac{1}{b^2}+\frac{1}{k^2},\ \frac{1}{c_{\prime}^2}=\frac{1}{c^2}+\frac{1}{k^2}; \quad . \quad . \quad . \quad \text{(c)}$$

hence $a^2-a_{\prime}^2=\frac{a^2a_{\prime}^2}{k^2}$, and like expressions for the other axes. The preceding equation (b) may be transformed into

$$a^2a_{\prime}^2\xi^2+b^2b_{\prime}^2v^2+c^2c_{\prime}^2\zeta^2=0. \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Now it has been shown (sec. [96]) that if (x, y, z) be a point of contact of a tangent plane,

$$x=a^2\xi,\ y=b^2v,\ z=c^2\zeta; \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

and if ϕ, χ, ψ are the angles which the diameter $2r$ through the point of contact makes with the axes,

$$\cos\phi=\frac{x}{r}=\frac{a^2\xi}{r},\ \cos\chi=\frac{b^2v}{r},\ \cos\psi=\frac{c^2\zeta}{r};$$

in like manner for the second point of contact,

$$\cos\phi_{,}=\frac{a_{,}^2\xi}{r_{,}},\ \cos\chi_{,}=\frac{b_{,}^2v}{r_{,}},\ \cos\psi_{,}=\frac{c_{,}^2\zeta}{r_{,}}.$$

Making the substitutions suggested by these transformations, the preceding equation (d) becomes

$$rr_{,}(\cos\phi\cos\phi_{,}+\cos\chi\cos\chi_{,}+\cos\psi\cos\psi_{,})=0, \quad . \quad . \quad \text{(f)}$$

or r and $r_{,}$ are at right angles.

115.] *Let there be two concyclic ellipsoids, and any point on the external one be assumed as the vertex of a cone enveloping the other, the plane of contact will meet the tangent plane to the first surface through the vertex of the cone in a straight line, such that the diametral plane passing through this line will be at right angles to the diameter which passes through the vertex of the cone.*

Let A, B, C be the semiaxes of the external ellipsoid, a, b, c those of the internal ellipsoid.

Let $$\frac{1}{A^2}=\frac{1}{a^2}-\frac{1}{h^2},\ \frac{1}{B^2}=\frac{1}{b^2}-\frac{1}{h^2},\ \frac{1}{C^2}=\frac{1}{c^2}-\frac{1}{h^2}, \quad . \quad . \quad . \quad \text{(a)}$$

and let p, q, r be the projective coordinates of the vertex of the cone.

Now as p, q, r are the projective coordinates of the point on the external surface through which the tangent plane is drawn, we shall have

$$p=A^2\xi_{,},\ q=B^2v_{,},\ r=C^2\zeta_{,}, \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

$\xi_{,}$, $v_{,}$, and $\zeta_{,}$ being the tangential coordinates of the *tangent* plane to the external surface through the vertex of the cone.

Again, as p, q, r are the projective coordinates of the vertex of the cone circumscribing the interior surface, the tangential coordinates of the *polar* plane of this point will be given by the equations

$$p=a^2\xi_{,,},\ q=b^2v_{,,},\ r=c^2\zeta_{,,}, \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

$\xi_{,,}$, $v_{,,}$, $\zeta_{,,}$ being the tangential coordinates of the polar plane of the vertex of the cone with respect to the interior surface.

Now the equations of the right line which is the intersection of the two planes whose coordinates are $\xi_{,}$, $v_{,}$, $\zeta_{,}$ and $\xi_{,,}$, $v_{,,}$, $\zeta_{,,}$, are

$$\xi-\xi_{,}=\frac{\xi_{,}-\xi_{,,}}{\zeta_{,}-\zeta_{,,}}(\zeta-\zeta_{,}),\ \text{and}\ v-v_{,}=\frac{v_{,}-v_{,,}}{\zeta_{,}-\zeta_{,,}}(\zeta-\zeta_{,}):\ \text{see sec. [64].}$$

But $\xi_{,}=\frac{p}{A^2}$, $\xi_{,,}=\frac{p}{a^2}$, hence $\xi_{,,}-\xi_{,}=p\left(\frac{1}{a^2}-\frac{1}{A^2}\right)=\frac{p}{h^2}$; in like manner $\zeta_{,,}-\zeta_{,}=\frac{r}{h^2}$, and $\upsilon_{,,}-\upsilon_{,}=\frac{q}{h^2}$.

Consequently the equations of the right line now become

$$\xi-\frac{p}{A^2}=\frac{p}{r}\left(\zeta-\frac{r}{C^2}\right),$$

or $$\xi=\frac{p}{r}\zeta+p\left(\frac{1}{A^2}-\frac{1}{C^2}\right);\ \text{also}\ \upsilon=\frac{q}{r}\zeta+q\left(\frac{1}{B^2}-\frac{1}{C^2}\right). \quad . \quad (d)$$

The general equations of a straight line in space are as in sec. [61],

$$\xi=\mu\zeta+\alpha,\quad \upsilon=\nu\zeta+\beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Equating the coefficients of (d) with those of (e),

$$\mu=\frac{p}{r},\ \nu=\frac{q}{r},\ \alpha=p\left(\frac{1}{A^2}-\frac{1}{C^2}\right),\ \beta=q\left(\frac{1}{B^2}-\frac{1}{C^2}\right). \quad . \quad (f)$$

It has been shown in sec. [62], (f), that if a straight line be drawn through the origin parallel to the straight line whose equations are (e), the direction cosines which this line makes are

$$\cos lOX=\frac{\beta}{\Delta},\quad \cos lOY=\frac{-\alpha}{\Delta},\quad \cos lOZ=\frac{-(\beta\mu-\alpha\nu)}{\Delta}. \quad . \quad . \quad (g)$$

If in these equations we substitute for μ, ν, α, β their values, we shall have

$$\left.\begin{aligned}&\cos lOX=\frac{q\left(\frac{1}{B^2}-\frac{1}{C^2}\right)}{\Delta},\quad \cos lOY=-p\left(\frac{1}{A^2}-\frac{1}{C^2}\right),\\ &\cos lOZ=\frac{pq}{r}\left(\frac{1}{A^2}-\frac{1}{B^2}\right).\end{aligned}\right\} \quad . \quad . \quad (h)$$

Now the cosines of the angles which $D=\sqrt{p^2+q^2+r^2}$ the diameter drawn through the vertex of the cone makes with the axes, are $\frac{p}{D}$, $\frac{q}{D}$, $\frac{r}{D}$; multiplying these expressions two by two, we shall have

$$\frac{p\cos lOX+q\cos lOY+r\cos lOZ}{D\Delta}=0, \quad . \quad . \quad . \quad (i)$$

or the diameter through the vertex of the cone, on the surface of the exterior ellipsoid, is perpendicular to the plane passing through the origin and the straight line in which the tangent plane to the exterior ellipsoid drawn through the vertex of the cone, and the polar plane of the vertex of the cone with reference to the interior ellipsoid intersect.

116.] *Let a cone envelope an ellipsoid, so that the plane of contact shall touch a second surface confocal with the former. The line* *drawn from the vertex of the cone to the point of contact of this tangent plane will be at right angles to it.*

Let the tangential equations of the surfaces be

$$A^2\xi^2+B^2\upsilon^2+C^2\zeta^2=1, \text{ and } a^2\xi^2+b^2\upsilon^2+c^2\zeta^2=1. \quad . \quad \text{(a)}$$

Let

$$A^2=a^2+f^2,\ B^2=b^2+f^2,\ C^2=c^2+f^2. \quad . \quad . \quad . \quad \text{(b)}$$

Let p, q, r be the projective coordinates of the vertex of the circumscribing cone, and let $x_\prime$, $y_\prime$, $z_\prime$ be the projective coordinates of the point of contact of the polar plane with the interior ellipsoid, and let $\xi_\prime$, $\upsilon_\prime$, $\zeta_\prime$ be the tangential coordinates of the polar plane. Hence, as in sec. [115],

$$\left.\begin{array}{l} p=A^2\xi_\prime,\ q=B^2\upsilon_\prime,\ r=C^2\zeta_\prime, \\ x_\prime=a^2\xi_\prime,\ y_\prime=b^2\upsilon_\prime,\ z_\prime=c^2\zeta_\prime. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Consequently

$$p-x_\prime=(A^2-a^2)\xi_\prime \text{ or } p-x_\prime=f^2\xi_\prime;$$

so also

$$\left. q-y_\prime=f^2\upsilon_\prime,\ r-z_\prime=f^2\zeta_\prime. \right\} \quad . \quad . \quad . \quad \text{(d)}$$

Now the cosines of the angles which the line drawn from the vertex of the circumscribing cone to the point of contact of the tangent plane being $\lambda_\prime$, $\mu_\prime$, $\nu_\prime$, we shall have

$$\frac{\cos\lambda_\prime}{\cos\nu_\prime}=\frac{p-x_\prime}{r-z_\prime}=\frac{\xi_\prime}{\zeta_\prime}, \text{ and } \frac{\cos\mu_\prime}{\cos\nu_\prime}=\frac{q-y_\prime}{r-z_\prime}=\frac{\upsilon_\prime}{\zeta_\prime}. \quad . \quad . \quad . \quad \text{(e)}$$

But if we let fall on this tangent plane to the interior surface a perpendicular P from the centre making the angles λ, μ, ν with the axes, we shall have

$$\frac{\cos\lambda}{\cos\nu}=\frac{P\xi_\prime}{P}=\frac{\xi_\prime}{\zeta_\prime};\ \frac{\cos\mu}{\cos\nu}=\frac{\upsilon_\prime}{\zeta_\prime},$$

or

$$\lambda=\lambda_\prime,\quad \mu=\mu_\prime,\quad \nu=\nu_\prime,$$

or the line from the vertex of the cone to the point of contact of the interior tangent plane is parallel to the perpendicular from the centre on the same tangent plane, and is therefore perpendicular to this latter.

To determine the length $P_\prime$ of the line from the vertex of the cone to the point of contact of the tangent plane to the interior surface.

Since $\quad P_\prime^2=(p-x_\prime)^2+(q-y_\prime)^2+(r-z_\prime)^2,$

$$P_\prime^2=f^4(\xi_\prime^2+\upsilon_\prime^2+\zeta_\prime^2)=\frac{f^4}{P^2} \text{ or } P_\prime P=f^2. \quad . \quad . \quad . \quad \text{(f)}$$

Hence the product of the perpendiculars from the centre and from the vertex of the cone on the interior tangent plane is constant.

$P_{,}$ is the normal passing through the point of contact of the tangent plane to the interior surface; to determine the coordinates of the point in which it will meet the plane of XY, suppose. The projective equations of this normal are

$$x-x_{,}=\frac{\xi_{,}}{\zeta_{,}}(z-z_{,}),\quad y-y_{,}=\frac{v_{,}}{\zeta_{,}}(z-z_{,}).$$

To determine the point where this normal meets the plane of XY, we must put $z=0$, and the preceding equations become

$$x=x_{,}-\frac{\xi_{,}}{\zeta_{,}}z_{,},\quad y=y_{,}-\frac{v_{,}}{\zeta_{,}}z;$$

but

$$x_{,}=a^2\xi_{,},\quad y_{,}=b^2v_{,},\quad z_{,}=c^2\zeta_{,}.$$

Hence

$$x=(a^2-c^2)\xi_{,},\quad y=(b^2-c^2)v_{,}.$$

Now $\xi_{,}$ and $v_{,}$ are the reciprocals of the segments cut off the axes of X and Y by the trace on the plane of XY made by the tangent plane, and x and y are the coordinates of the foot of the normal. Hence this curious but well-known relation, that if we construct the ellipse the squares of whose semiaxes are a^2-c^2 and b^2-c^2, the foot of the normal and the trace of the tangent plane on the plane of XY will be pole and polar with respect to this section.

To determine the length of the normal N from the vertex of the cone to the plane of XY,

$$N^2=(p-x)^2+(q-y)^2+r^2.$$

But we have shown that $x=(a^2-c^2)\xi_{,}$ and $y=(b^2-c^2)v_{,}$.

Hence $N^2=(A^2-a^2+c^2)^2\xi_{,}^2+(B^2-b^2+c^2)^2v_{,}^2+C^4\zeta_{,}^2$.

It may easily be shown that $A^2-a^2+c^2=C^2$, and that $B^2-b^2+c^2$ is also equal to C^2.

Hence $N^2=C^4(\xi_{,}^2+v_{,}^2+\zeta_{,}^2)=\frac{C^4}{P^2}$, or $NP=C^2$. But we have shown that $P_{,}P=f^2$, subtracting, $P(N-P_{,})=C^2-f^2=c^2$.

117.] *Parallel planes are drawn to a series of confocal ellipsoids; to determine the locus of the points of contact.* Let

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

be the equation of one of the ellipsoids; and as they are confocal, let

$$a^2=c^2+k^2,\quad b^2=c^2+h^2. \quad . \; . \; . \; . \; . \; . \; . \quad (b)$$

Let ξ, v, ζ be the tangential coordinates of one of the parallel planes, then

$$x=a^2\xi,\quad y=b^2v,\quad z=c^2\zeta; \quad . \; . \; . \; . \; . \; . \quad (c)$$

and as these planes must be parallel, let

$$\xi=m\zeta,\quad v=n\zeta. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (d)$$

From these eight equations we must eliminate a, b, c, ξ, υ, ζ, and this elimination will afford us a double locus for the point x, y, z.

Now $x=a^2\xi=(c^2+k^2)m\zeta$, $y=(c^2+h^2)n\zeta$, and $z=c^2\zeta$; from these three equations, eliminating c and ζ, we get

$$nh^2x-mk^2y+mn(k^2-h^2)z=0, \quad . \quad . \quad . \quad . \quad \text{(e)}$$

the equation of a diametral plane of the surface. Now as

$$a^2=\frac{x}{\xi}=\frac{x}{m\zeta}=\frac{c^2x}{mz} \text{ and } c^2=a^2-k^2,$$

we find

$$a^2=\frac{k^2x}{x-mz},\ b^2=\frac{h^2y}{y-nz}, \text{ and } c^2=a^2-k^2=\frac{mk^2z}{x-mz}.$$

Substituting these values of a^2, b^2, c^2 in (a), we get, after some reductions,

$$\frac{x^2}{k^2}+\frac{y^2}{h^2}-\frac{z^2}{k^2}-\frac{nyz}{h^2}+\frac{(1-m^2)xz}{mk^2}=1. \quad . \quad . \quad . \quad . \quad \text{(f)}$$

This is the equation of a one sheet or continuous hyperboloid whose centre is at the centre of the confocal ellipsoids.

Hence the locus of the points of contact of the parallel tangent planes with the confocal ellipsoids is a curve, the intersection of the diametral plane (e) with the discontinuous hyperboloid (f).

118.] *To a series of concyclic surfaces tangent planes are drawn touching the surfaces in the points where they are pierced by a common diameter; to find the surface enveloped by these tangent planes.*

Let $$a^2\xi^2+b^2\upsilon^2+c^2\zeta^2=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

be the tangential equation of one of the surfaces, c being the greatest axis, and as they are concyclic, we shall have

$$\frac{1}{a^2}=\frac{1}{c^2}+\frac{1}{k^2}, \qquad \frac{1}{b^2}=\frac{1}{c^2}+\frac{1}{h^2}. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Let ξ, υ, ζ be the tangential coordinates of one of the tangent planes, and let x, y, z be the projective coordinates of the point of contact of one of these planes; then let $x=mz$, $y=nz$; . (c)

also let $$x=a^2\xi,\ y=b^2\upsilon,\ z=c^2\zeta; \quad . \quad . \quad . \quad . \quad \text{(d)}$$

now between the eight equations (a), (b), (c), (d), we have to eliminate the six quantities x, y, z, a, b, c.

Since $x=a^2\xi$, $\xi=\frac{x}{a^2}=\frac{mz}{a^2}=mz\left(\frac{1}{c^2}+\frac{1}{k^2}\right)=mc^2\zeta\left(\frac{1}{c^4}+\frac{1}{k^2}\right)$.

Hence

$$\xi=m\zeta+\frac{mc^2\zeta}{k^2}, \text{ or } \frac{1}{c^2}=\frac{m\zeta}{k^2(\xi-m\zeta)};$$

but $\frac{1}{c^2}=\frac{n\zeta}{h^2(\upsilon-n\zeta)}$; equating these values of $\frac{1}{c^2}$, we get

$$(k^2-h^2)mn\zeta+mh^2\upsilon-nk^2\xi=0, \quad . \quad . \quad . \quad . \quad . \qquad \text{(e)}$$

the tangential equation of a point at infinity.

Since $\frac{1}{a^2}=\frac{\xi}{x}=\frac{\xi}{nz}=\frac{\xi}{nc^2\zeta}=\frac{\xi}{n\zeta}\left(\frac{1}{a^2}-\frac{1}{k^2}\right)$, we shall have

$$a^2=\frac{k^2(\xi-m\zeta)}{\xi}; \text{ also } b^2=\frac{h^2(\upsilon-n\zeta)}{\upsilon},\ c^2=\frac{k^2(\xi-m\zeta)}{m\zeta}. \quad . \quad . \qquad \text{(f)}$$

Substituting these values of a^2, b^2, c^2 in the tangential equation of the surface, we get

$$k^2\xi^2+h^2\upsilon^2-k^2\zeta^2-nh^2\upsilon\zeta+\frac{(1-m^2)}{m}\xi\zeta=1, \quad . \quad . \qquad \text{(g)}$$

the tangential equation of a continuous hyperboloid. Hence, as the plane envelopes a hyperboloid and passes through a point situated at infinity, the locus is an hyperbolic cylinder.

The reader will doubtless have remarked the duality that exists between several of the foregoing problems. In the two latter especially, the diametral plane in the confocal surfaces is the polar plane of the point at infinity, the common direction of the axes of all the cylinders which envelope the concyclic surfaces; while these cylinders are themselves the polars of the several curves in which the diametral plane cuts the confocal surfaces.

It will be shown, as we proceed with these investigations, that every surface of the second order that admits of circular sections has four directrix planes parallel to the planes of the circular sections, two by two. In the case of the elliptic paraboloid two of the directrix planes are at infinity. These planes may be called the *umbilical directrix planes.* It will also be shown that in every such surface there are *four foci*, the poles of the umbilical directrix planes, and distinct from the foci of the principal sections of the surface. By the help of these umbilical directrix planes and corresponding foci may all the properties of spheres and surfaces of revolution of the second order be transformed and transferred to surfaces with three unequal axes.

CHAPTER XII.

ON THE SURFACE OF THE CENTRES OF CURVATURE OF AN ELLIPSOID*.

119.] It is well known to geometers that the lines of greatest and least curvature at any point on the surface of an ellipsoid are at right angles to each other, and that they may be constructed by the intersections of two confocal hyperbolas, one continuous, the other discontinuous, or as they are usually called, a single-sheet one and a double-sheet one. It is also known that these three surfaces are reciprocally orthogonal, or that any two of them cut the third along its lines of curvature where the three intersect in a point. If we fix on the ellipsoid as the surface whose lines of curvature are in question, and normals be drawn to the surface of the ellipsoid along any given line of curvature, the radii of curvature will not only lie on these normals at the successive points, but they will all, taken indefinitely near to each other, constitute a developable surface, and the line of centres of curvature will constitute its edge of regression. Hence if we draw tangent planes to the two hyperboloids at this point, they will intersect in the normal to the ellipsoid, and will also be tangent planes to the above developable surface.

Let the equation of the ellipsoid referred to its axes and passing through the point $(x_{,}y_{,}z_{,})$ be

$$\frac{x_{,}^2}{a^2}+\frac{y_{,}^2}{b^2}+\frac{z_{,}^2}{c^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

then the equation of the tangent plane passing through the point $(x_{,}y_{,}z_{,})$ will be

$$\frac{xx_{,}}{a^2}+\frac{yy_{,}}{b^2}+\frac{zz_{,}}{c^2}=1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

and the tangential coordinates of this tangent plane will be

$$\xi_{,}=\frac{x_{,}}{a^2},\ \nu_{,}=\frac{y_{,}}{b^2},\ \zeta_{,}=\frac{z_{,}}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (c)$$

Let the equation of the tangent plane to one of the hyperboloids passing through the point $(x_{,}y_{,}z_{,})$ be

$$\frac{xx_{,}}{a_{,}^2}+\frac{yy_{,}}{b_{,}^2}+\frac{zz_{,}}{c_{,}^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (d)$$

---

* The consideration of this surface, first imagined by Monge, but not discussed by him, will be found to throw some light on the nature of the *umbilici*, and of the lines of curvature passing through them, relative to which there has been some diversity of opinion. On this subject see Monge, 'Application de l'Analyse à la Géométrie'; Dupin, 'Développements de Géométrie,' pp. 173–187; Poisson, 'Journal de l'Ecole Polytechnique,' 21ᵉ cahier, p. 205.—[Note to first edition.]

or, as the surfaces are confocal, we may put

$$\left.\begin{aligned} &a^2-b^2=a_{\prime}^2-b_{\prime}^2=h^2;\ a^2-c^2=a_{\prime}^2-c_{\prime}^2=k^2. \\ \text{Hence}\quad &b_{\prime}^2=a_{\prime}^2+b^2-a^2,\ c_{\prime}^2=a_{\prime}^2+c^2-a^2, \end{aligned}\right\} \quad . \ . \ . \ \text{(e)}$$

and the preceding equation may be written

$$\frac{xx_{\prime}}{a_{\prime}^2}+\frac{yy_{\prime}}{a_{\prime}^2+b^2-a^2}+\frac{zz_{\prime}}{a_{\prime}^2+c^2-a^2}=1. \quad . \ . \ . \ . \ . \ \text{(f)}$$

The tangential equation of the hyperboloid passing through the point $(x_{\prime}\ y_{\prime}\ z_{\prime})$ is therefore

$$a_{\prime}^2\xi^2+(a_{\prime}^2+b^2-a^2)v^2+(a_{\prime}^2+c^2-a^2)\zeta^2=1. \quad . \ . \ \text{(g)}$$

Hence we have $\quad x_{\prime}=a_{\prime}^2\xi,\ y_{\prime}=b_{\prime}^2v,\ z_{\prime}=c_{\prime}^2\zeta.$ . . . . . (h)

But as the tangent planes are at right angles, one to the other, we must have

$$\xi_{\prime}\xi+v_{\prime}v+\zeta_{\prime}\zeta=0; \quad . \ . \ . \ . \ . \ . \ . \ . \ \text{(i)}$$

or, substituting for $\xi_{\prime}$, $v_{\prime}$, $\zeta_{\prime}$ their values as given in (c), the preceding equation becomes

$$\frac{x_{\prime}}{a^2}\xi+\frac{y_{\prime}}{b^2}v+\frac{z_{\prime}}{c^2}\zeta=0; \quad . \ . \ . \ . \ . \ . \ \text{(j)}$$

substituting for $x_{\prime}$, $y_{\prime}$, $z_{\prime}$ their values given in (h), we obtain

$$\frac{a_{\prime}^2}{a^2}\xi^2+\frac{(a_{\prime}^2+b^2-a^2)}{b^2}v^2+\frac{(a_{\prime}^2+c^2-a^2)}{c^2}\zeta^2=0, \quad . \ . \ \text{(k)}$$

or

$$\xi^2+v^2+\zeta^2=(a^2-a_{\prime}^2)\left\{\frac{\xi^2}{a^2}+\frac{v^2}{b^2}+\frac{\zeta^2}{c^2}\right\}. \quad . \ . \ . \ . \ \text{(l)}$$

If we now refer to the tangential equation of the hyperboloid (g) we shall find

$$a_{\prime}^2\xi^2+(a_{\prime}^2+b^2-a^2)v^2+(a_{\prime}^2+c^2-a^2)\zeta^2=1,$$

or

$$(\xi^2+v^2+\zeta^2)(a^2-a_{\prime}^2)=a^2\xi^2+b^2v^2+c^2\zeta^2-1. \quad . \ . \ \text{(m)}$$

Comparing this equation with the preceding, we may eliminate $(a^2-a_{\prime}^2)$, and obtain as the tangential equation of the "surface of centres"

$$(\xi^2+v^2+\zeta^2)^2=\left\{\frac{\xi^2}{a^2}+\frac{v^2}{b^2}+\frac{\zeta^2}{c^2}\right\}(a^2\xi^2+b^2v^2+c^2\zeta^2-1). \quad . \ \text{(n)}$$

This is the tangential equation of the "*surface of centres of curvature*," or, as it may for brevity be called, *the surface of centres*.

120.] This surface consists of two sheets—one generated by the successive normals to the surface along one line of curvature, the second by the successive normals along the corresponding line of curvature. Let a perpendicular P on a tangent plane to the surface of centres make the angles λ, μ, v with the axes of coordinates, then as $P\xi=\cos\lambda$, $Pv=\cos\mu$, $P\zeta=\cos v$, the last equation may be written

$$\left[\frac{\cos^2\lambda}{a^2}+\frac{\cos^2\mu}{b^2}+\frac{\cos^2\nu}{c^2}\right](a^2\cos^2\lambda+b^2\cos^2\mu+c^2\cos^2\nu-\mathrm{P}^2)=1. \quad \text{(a)}$$

Now the first member of this equation represents $\frac{1}{\mathrm{R}^2}$, the inverse semidiameter squared of the original ellipsoid, making the angles λ, μ, ν with the axes; and $a^2\cos^2\lambda+b^2\cos^2\mu+c^2\cos^2\nu=\mathrm{P}_{\prime}^2$ is the square of the perpendicular on a tangent plane to the ellipsoid parallel to the tangent plane to the surface of centres. Hence

$$\mathrm{R}^2=\mathrm{P}_{\prime}^2-\mathrm{P}^2, \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(b)}$$

whence we have this remarkable property of the surface of centres:—

Any two parallel tangent planes being drawn to the surface of centres and to the ellipsoid, the difference of the squares of the coincident perpendiculars let fall upon them from the centre is always equal to the square of the coinciding semidiameter of the ellipsoid.

Assume the original equation (n) written in the form

$$\left.\begin{array}{l} b^2c^2\xi^2+a^2c^2v^2+a^2b^2\zeta^2=(b^2-c^2)^2a^2\zeta^2v^2+(a^2-b^2)^2c^2\xi^2v^2 \\ \quad +(a^2-c^2)^2b^2\xi^2\zeta^2 \end{array}\right\}. \quad \text{(c)}$$

Then, by giving to ζ a constant value, we might determine the tangential equation of the section made in the plane of XY by the cone whose vertex is in the axis of z, and which envelopes the surface of centres.

But it will be better to determine the sections of the surface made by the principal planes; and this may be effected by putting ζ, v, ξ successively equal to ∞ and 0. Hence we shall have in the planes of YZ, XZ, XY, the sections whose tangential equations are

$$\left.\begin{array}{ll} \left.\begin{array}{l}(a^2-b^2)^2c^2v^2+(a^2-c^2)^2b^2\zeta^2=b^2c^2 \\ c^2v^2+b^2\zeta^2=(b^2-c^2)^2\zeta^2v^2\end{array}\right\} & \text{in the plane of YZ,} \\ \left.\begin{array}{l}(b^2-c^2)^2a^2\zeta^2+(a^2-b^2)^2c^2\xi^2=a^2c^2 \\ a^2\zeta^2+c^2\xi^2=(a^2-c^2)^2\xi^2\zeta^2\end{array}\right\} & \text{in the plane of XZ,} \\ \left.\begin{array}{l}(a^2-c^2)^2b^2\xi^2+(b^2-c^2)^2a^2v^2=a^2b^2 \\ b^2\xi^2+a^2v^2=(a^2-b^2)^2\xi^2v^2\end{array}\right\} & \text{in the plane of XY.} \end{array}\right\} \text{(d)}$$

Consequently the sections of the surface of centres in the principal planes are two in each; one an ellipse, the other the evolute of an ellipse.

It may easily be shown that the tangential equation of the evolute of the ellipse whose projective equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will be

$$a^2\xi^2+b^2v^2=(a^2-b^2)^2\xi^2v^2. \; . \; . \; . \; . \; . \; . \quad \text{(e)}$$

The projective equation of the evolute of an ellipse of which the semiaxes are a and b is $(ay)^{\frac{2}{3}}+(bx)^{\frac{2}{3}}-(a^2-b^2)^{\frac{2}{3}}=0$; . . . . (f) taking the partial differentials of this expression, we shall have

$$\frac{dF}{dx}=\frac{2}{3}\frac{b}{(bx)^{\frac{1}{3}}},\quad \frac{dF}{dy}=\frac{2}{3}\frac{a}{(ax)^{\frac{1}{3}}},\quad \frac{dF}{dx}x+\frac{dF}{dy}y=\frac{2}{3}(a^2-b^2)^{\frac{2}{3}}.$$

But, as we have shown in sec. [22],

$$\xi=\frac{\frac{dF}{dx}}{\frac{dF}{dx}x+\frac{dF}{dy}y},\qquad \upsilon=\frac{\frac{dF}{dy}}{\frac{dF}{dx}x+\frac{dF}{dy}y}.$$

Consequently $$x\xi=\frac{b^2}{\xi^2(a^2-b^2)^2},\quad y\upsilon=\frac{a^2}{\upsilon^2(a^2-b^2)^2}.$$

Substituting in the dual equation $x\xi+y\upsilon=1$, the values of these expressions, we obtain the tangential equation of the evolute of an ellipse,

$$a^2\xi^2+b^2\upsilon^2=(a^2-b^2)^2\xi^2\upsilon^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

If in the equations (f) and (g) we make $\upsilon=\infty$ or $y=0$, or $\xi=\infty$, or $x=0$, we shall have for the semiaxes of the evolute in the axes of X and Y,

$$\mathrm{A}=\frac{a^2-b^2}{b},\quad \mathrm{B}=\frac{a^2-b^2}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(h)}$$

Hence the semiaxes of the evolute are inversely proportional to the semiaxes of the ellipse.

It is worthy of notice that, while the axes of the elliptic sections of the surface of centres in the three principal planes are functions of the three axes of the ellipsoid, the axes of the evolutes are functions only of the axes of the principal planes in which they happen to be. Thus c does not appear among the constants of the evolute in the plane of XY.

From the foregoing investigation it follows that the sections of the surface of centres in the principal planes are two in each—one an ellipse, the other the evolute of an ellipse.

On the umbilical lines of Curvature.

121.] Among the French mathematicians there has been much difference of opinion as to the nature of the lines of curvature which pass through an *umbilicus* of the ellipsoid. Some hold, with Monge and Dupin, that the two lines of curvature which everywhere else on the surface are at right angles to each other, here merge into one. This is such a violation of the law of continuity, that others adhere to the opinion of Poisson and Leroy, to the effect that at an umbilicus the radii of curvature are all equal, and that there is an infinite number of rectangular systems of lines of equal curvature all passing through the umbilicus.

An examination of the surface of centres will demonstratively show that the latter opinion is the correct one.

For this purpose let a tangent plane to the surface of centres be drawn through the umbilical normal. Now the *projective* coordinates of the umbilicus are

$$x_{,}=a\sqrt{\frac{a^2-b^2}{a^2-c^2}},\quad y_{,}=0,\quad z_{,}=c\sqrt{\frac{b^2-c^2}{a^2-c^2}};\quad .\ .\ \text{(a)}$$

and the segments of the axes of x and z cut off by the normal are

$$\bar{z}=\frac{\sqrt{(b^2-c^2)(a^2-c^2)}}{c},\quad \bar{x}=\frac{\sqrt{(a^2-c^2)(a^2-b^2)}}{a}.\ .\ \text{(b)}$$

Hence the tangential coordinates of the normal in the plane of xz are

$$\zeta^2=\frac{c^2}{(a^2-c^2)(b^2-c^2)},\quad \xi^2=\frac{a^2}{(a^2-c^2)(a^2-b^2)}.\quad .\ .\ \text{(c)}$$

Now, substituting these values of ζ and ξ in the equation (c), sec. [120], of the surface of centres, we shall have for the value of v^2 the following expression:—

$$(a^2-b^2)(b^2-c^2)v^2=\frac{b^2[(b^2-c^2)+(c^2-a^2)+(a^2-b^2)]}{[(b^2-c^2)+(c^2-a^2)+(a^2-b^2)]},\text{ or } v=\frac{0}{0}.\quad \text{(d)}$$

Or an infinite number of tangent planes may therefore be drawn through the umbilical normal to the surface of centres.

The principal sections of the surface of centres in the mean plane, or in the plane of XZ, of the greatest and least axes, possess some very curious properties.

The ellipse and the evolute in the plane of the greatest and least axes of the ellipsoid have four points of contact; and in these four points they are severally touched by the four umbilical normals; and these normals are the radii of curvature of the umbilici; and their value is severally $=\frac{b^3}{ac}$.

To show this, let the ellipse and evolute in the plane of xz be supposed to have the same tangent. Their equations are

$$\left.\begin{aligned} &(b^2-c^2)^2a^2\zeta^2+(a^2-b^2)^2c^2\xi^2=a^2c^2,\\ \text{and}\qquad &a^2\zeta^2+c^2\xi^2=(a^2-c^2)^2\xi^2\zeta^2.\end{aligned}\right\}\ .\ .\ .\ .\ .\ \text{(e)}$$

Let ξ and ζ be the same in both equations. Reducing on this supposition, we shall find

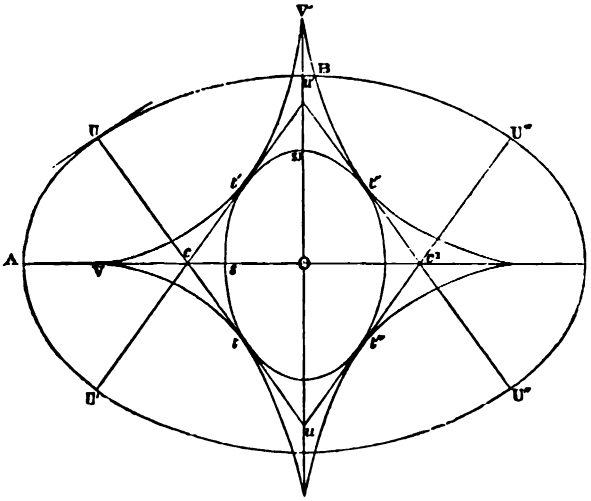
$$(a^2-c^2)(b^2-c^2)\zeta^2=c^2,$$

hence

$$\zeta^2=\frac{c^2}{(a^2-c^2)(b^2-c^2)}\text{ and }\xi^2=\frac{a^2}{(a^2-c^2)(a^2-b^2)};\ .\ \ .\ \text{(f)}$$

but these are also the tangential coordinates of the umbilical normal found in (c). Hence the four umbilical normals form an equilateral parallelogram or lozenge, and touch *externally* the ellipse, and internally the evolute, at the same four common points; and the common distances of these four points from the umbilici are severally $=\frac{b^3}{ac}$, or they are the centres of curvature of the umbilici.

Fig. 19.



These points may be called the *centres of umbilical curvature.*

122.] We may write the tangential equation of the central ellipse

$$\frac{(a^2-b^2)^2}{a^2}\xi^2+\frac{(b^2-c^2)^2}{c^2}\zeta^2=1$$

in the form

$$\frac{(a^2-b^2)^2}{a^2}\xi.\xi+\frac{(b^2-c^2)^2}{c^2}\zeta.\zeta=1. \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

Let $\bar{x}$ and $\bar{z}$ be the projective coordinates of the centre of umbilical curvature.

Then $\bar{x}=\frac{(a^2-b^2)^2}{a^2}\xi$, $\bar{z}=\frac{(b^2-c^2)^2}{c^2}\zeta$.

Substituting for ξ and ζ their values as given in (f), sec. [121], we shall find

$$\bar{x}^2=\frac{(a^2-b^2)^3}{a^2(a^2-c^2)},\ \bar{z}^2=\frac{(b^2-c^2)^3}{c^2(a^2-c^2)}. \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

The length of the side of the quadrilateral made by the intersections of the four umbilical normals is $\frac{b}{ac}(a^2-c^2)$; and it is divided at the common point of contact of this umbilical normal with the ellipse and evolute into the segments $\frac{b}{ac}(b^2-c^2)$ and $\frac{b}{ac}(a^2-b^2)$, so that the length of this line and its segments made by the point of contact are respectively

$$\frac{b}{ac}(a^2-c^2),\ \frac{b}{ac}(b^2-c^2),\ \text{and}\ \frac{b}{ac}(a^2-b^2).\ .\ .\ .\quad (c)$$

123. *The areas of the umbilical parallelogram, of the ellipse, and of the evolute circumscribed by and inscribed in the four umbilical normals, have certain reciprocal relations which are independent of the axes of the ellipsoid.*

The area of the ellipse $=\pi\frac{(a^2-b^2)(b^2-c^2)}{ac}$.

The area of the parallelogram $=2\frac{(a^2-c^2)\sqrt{(a^2-b^2)(b^2-c^2)}}{ac}$.

The area of the evolute $=\frac{3\pi}{8}\frac{(a^2-c^2)^2}{ac}$.

$$\text{Hence}\qquad \frac{\text{area of ellipse}\times\text{area of evolute}}{(\text{area of parallelogram})^2}=\frac{3}{8}\left(\frac{\pi}{2}\right)^2,\ .\ .\quad (a)$$

an abstract number independent of the axes of the ellipsoid.

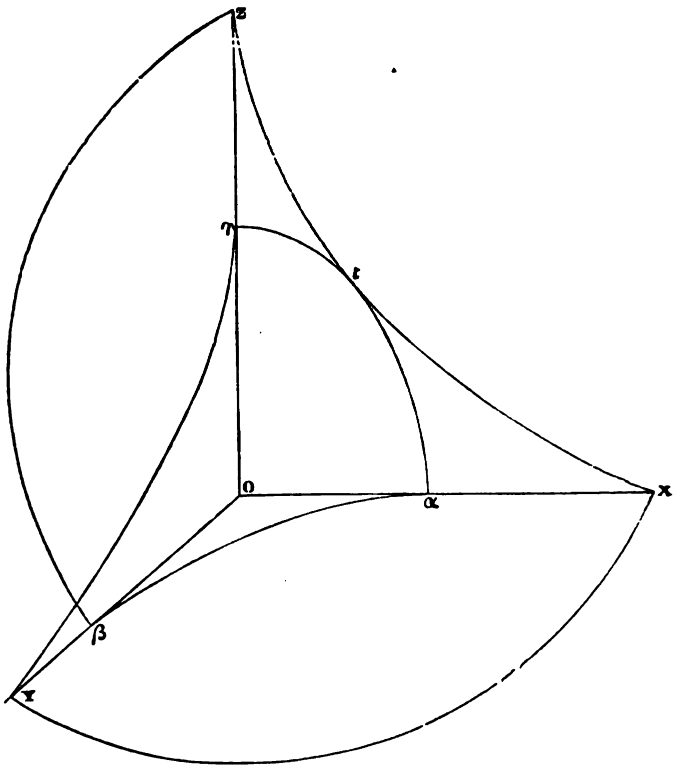
When the mean semiaxis b of the ellipsoid is a mean proportional between the semiaxes a and c, or $b^2=ac$, the umbilical ellipse becomes a circle whose radius is $a-c$.

If we examine the ellipse and correlative evolute in either of the other principal planes, we shall find that they have no common point of contact or intersection.

An inspection of the expressions for the axes of the ellipses and evolutes in the three principal planes will show that these axes are so related that the axes of the central ellipse in any one of the three coordinate planes are the axes of the evolutes in the two other planes, one in each evolute. Thus, if the axes of the ellipses in the three coordinate planes be hk, $h_{,}k_{,}$, and $h_{,,}k_{,,}$, the axes of the evolutes in the same coordinate planes will be $h_{,}k_{,,}$, $h_{,,}k$, and $hk_{,}$; so that the twelve axes of the three ellipses and three evolutes are reduced to six.

Thus, as in figure 20, we pass from Z to β along the elliptic quadrant, from β to a along the quadrant of the evolute, from a to γ along the elliptic arc, from γ to Y along the quadrant of the evolute, from Y to X along a quadrant of the ellipse, and from X to Z along a quadrant of the evolute.

Fig. 20.



Thus in our course along the "*edges of regression,*" setting out from Z, we traverse

the plane of ZY along an ellipse, whose quadrant is $Z\beta$,
the plane of YX along an evolute, „ „ is $\beta\alpha$,
the plane of XZ along an ellipse, „ „ is $\alpha\gamma$,
the plane of ZY along an evolute, „ „ is γY,
the plane of YX along an ellipse, „ „ is YX,
the plane of XZ along an evolute, back again to Z.

From the previous discussion of the properties of the centre of surfaces it follows that it consists of two distinct sheets which touch only in four points, the centres of umbilical curvature; for it is only in these four points that the radii of curvature of any point on the surface of an ellipsoid are equal.

On the projective equation of the surface of centres of an ellipsoid.

124.] Let

$$\frac{\xi^2}{a^2}+\frac{\upsilon^2}{b^2}+\frac{\zeta^2}{c^2}=\mathrm{P},\ a^2\xi^2+b^2\upsilon^2+c^2\zeta^2-1=\mathrm{Q},\ \text{and}\ \xi^2+\upsilon^2+\zeta^2=\mathrm{R}, \quad \text{(a)}$$

then the surface of the centres of curvature may be written

$$\mathrm{PQ}=\mathrm{R}^2,\ \text{or}\ \Phi=\mathrm{PQR}^{-2}-1=0. \quad .\ .\ .\ .\ .\ \text{(b)}$$

Let the partial differentials of this expression be taken successively with respect to ξ, υ, and ζ; hence

$$\mathrm{d}\Phi=\frac{\mathrm{dP.QR+dQ.PR-2PQ.dR}}{\mathrm{R}^3}. \quad .\ .\ .\ .\ \text{(c)}$$

But

$$\frac{\mathrm{dP}}{\mathrm{d}\xi}=\frac{2\xi}{a^2},\quad \frac{\mathrm{dQ}}{\mathrm{d}\xi}=2a^2\xi,\quad \frac{\mathrm{dR}}{\mathrm{d}\xi}=2\xi,$$

consequently

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}=2\left[\frac{\mathrm{QR}}{a^2}+\mathrm{PR}a^2-2\mathrm{PQ}\right]\xi\mathrm{R}^{-3}; \quad .\ .\ .\ \text{(d)}$$

or, since $\mathrm{PQ}=\mathrm{R}^2$, we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}=2\xi\left[\frac{\mathrm{Q}}{a^2}+\mathrm{P}a^2-2\mathrm{R}\right]\mathrm{R}^{-2}. \quad .\ .\ .\ .\ .\ .\ \text{(e)}$$

We have now to find the value of the expression

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\xi+\frac{\mathrm{d}\Phi}{\mathrm{d}\upsilon}\upsilon+\frac{\mathrm{d}\Phi}{\mathrm{d}\zeta}\zeta=\Delta.$$

Making the necessary substitutions,

$$\Delta=2\left[\mathrm{Q}\left(\frac{\xi^2}{a^2}+\frac{\upsilon^2}{b^2}+\frac{\zeta^2}{c^2}\right)+\mathrm{P}(a^2\xi^2+b^2\upsilon^2+c^2\zeta^2-1)+\mathrm{P}-2\mathrm{R}^2\right],$$

or

$$\Delta=2[\mathrm{QP+PQ+P-2R}]^{-2},\ \text{or}\ \Delta=\frac{2\mathrm{P}}{\mathrm{R}^2};$$

hence

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\xi.\Delta}=\frac{\xi\left[\frac{\mathrm{Q}}{a^2}+\mathrm{P}a^2-2\mathrm{R}\right]}{\mathrm{P}}=x,$$

or

$$x=\frac{\xi\left[\xi^2+\frac{b^2}{a^2}\upsilon^2+\frac{c^2}{a^2}\zeta^2-\frac{1}{a^2}+\xi^2+\frac{a^2}{b^2}\upsilon^2+\frac{a^2}{c^2}\zeta^2-2\xi^2-2\upsilon^2-2\zeta^2\right]}{\mathrm{P}};$$

and reducing,

$$\left.\begin{aligned} x&=\frac{\xi\left[\left(\frac{a}{b}-\frac{b}{a}\right)^2\upsilon^2+\left(\frac{a}{c}-\frac{c}{a}\right)^2\zeta^2-\frac{1}{a^2}\right]}{\frac{\xi^2}{a^2}+\frac{\upsilon^2}{b^2}+\frac{\zeta^2}{c^2}}\\ y&=\frac{\upsilon\left[\left(\frac{b}{c}-\frac{c}{b}\right)^2\zeta^2+\left(\frac{b}{a}-\frac{a}{b}\right)^2\xi^2-\frac{1}{b^2}\right]}{\frac{\xi^2}{a^2}+\frac{\upsilon^2}{b^2}+\frac{\zeta^2}{c^2}}\\ z&=\frac{\zeta\left[\left(\frac{c}{a}-\frac{a}{c}\right)^2\xi^2+\left(\frac{c}{b}-\frac{b}{c}\right)^2\upsilon^2-\frac{1}{c^2}\right]}{\frac{\xi^2}{a^2}+\frac{\upsilon^2}{b^2}+\frac{\zeta^2}{c^2}} \end{aligned}\right\}, \quad . \; . \; . \quad \text{(f)}$$

having found like expressions for y and z.

If we introduce the relations established in the second form of the equation of the surface of centres, see (c), sec. [120], we may easily show, though not at first sight apparent, that the preceding equations satisfy the *criterion of duality*,

$$x\xi+y\upsilon+z\zeta=1.$$

By the help of these three equations and the tangential equation of the surface of centres, we may eliminate ξ, υ, ζ, and express the projective equation of the surface of centres in terms of x, y, and z.

The *projective* equation of the surface of the centres of curvature has been given by Dr. Salmon, Professor of Divinity in the University of Dublin, and published in the Quarterly Journal of Pure and Applied Mathematics of Feb. 1858.

Although this surface has been familiarly known to the continental mathematicians since the time of Monge, none of them has ventured to grapple with the enormous difficulties which stand in the way of exhibiting its *projective* equation, or its equation in xyz. These difficulties have been surmounted by Dr. Salmon; and the resulting equation, which is of the twelfth degree, contains no fewer than eighty-three terms.

125.] To show the power and exemplify the reach of the combination of the methods of projective and tangential coordinates, it will be an apposite illustration to discuss the reciprocal polar of the "surface of centres." This investigation will afford a further instance of the great law which pervades all geometrical truths, that if one method of investigation be more easily applied to the discussion of the properties of curves or curved surfaces, their reciprocal polars will be best investigated by the other.

The reciprocal theorem to that of the surface of centres is the following:—

Let there be three concyclic *surfaces of the second order touched by a common tangent plane in three points; these points, two by two, will subtend right angles at the centre, and the locus of all the points of contact with the two variable hyperboloids will be a surface which may be called the "surface of contacts."*

Its projective equation may be found as follows.

Let the tangential equations of two of the surfaces, having a common tangent plane, be

$$a^2\xi^2+b^2\upsilon^2+c^2\zeta^2=1, \text{ and } a_,^2\xi^2+b_,^2\upsilon^2+c_,^2\zeta^2=1. \quad . \quad . \quad \text{(a)}$$

Subtracting one of these equations from the other, we shall have

$$(a^2-a_,^2)\xi^2+(b^2-b_,^2)\upsilon^2+(c^2-c_,^2)\zeta^2=0; \quad . \quad . \quad \text{(b)}$$

and as these surfaces are concyclic, we shall have

$$\frac{1}{a^2}-\frac{1}{a_,^2}=\frac{1}{b^2}-\frac{1}{b_,^2}=\frac{1}{c^2}-\frac{1}{c_,^2}=\frac{1}{k^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Making these substitutions in the preceding equation, there results

$$a^2a_,^2\xi^2+b^2b_,^2\upsilon^2+c^2c_,^2\zeta^2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Let the projective equation of one of the hyperboloids be

$$\frac{x_,^2}{a_,^2}+\frac{y_,^2}{b_,^2}+\frac{z_,^2}{c_,^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

and the equation of the tangent plane to this surface passing through the point $x_,, y_,, z_,$ be

$$\frac{x_,}{a_,^2}x+\frac{y_,}{b_,^2}y+\frac{z_,}{c_,^2}z=1; \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

and as this must coincide with the second of (a), we shall have

$$\left.\begin{aligned} &a_,^2\xi=x_,,\ b_,^2\upsilon=y_,,\ c_,^2\zeta=z_,, \\ \text{or} \quad &a_,^2\xi^2=\frac{x_,^2}{a_,^2},\ b_,^2\upsilon^2=\frac{y_,^2}{b_,^2},\ c_,^2\zeta^2=\frac{z_,^2}{c_,^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

Substituting these values in (d), we shall obtain

$$a^2a_,^2\xi^2+b^2b_,^2\upsilon^2+c^2c_,^2\zeta^2=\frac{a^2z_,^2}{a_,^2}+\frac{b^2y_,^2}{b_,^2}+\frac{c^2z_,^2}{c_,^2}=0, \quad . \quad . \quad \text{(h)}$$

or, reducing,

$$\left(\frac{1}{a^2}-\frac{1}{a_,^2}\right)(a^2x_,^2+b^2y_,^2+c^2z_,^2)=(x_,^2+y_,^2+z_,^2). \quad . \quad . \quad \text{(i)}$$

Equation (e) may be written in the form

$$\frac{x_,^2}{a_,^2}+y_,^2\left(\frac{1}{b^2}-\frac{1}{a^2}+\frac{1}{a_,^2}\right)+z_,^2\left(\frac{1}{c^2}-\frac{1}{a^2}-\frac{1}{a_,^2}\right)=1, \quad . \quad . \quad \text{(j)}$$

or

$$\frac{x_,^2}{a^2}+\frac{y_,^2}{b^2}+\frac{z_,^2}{c^2}-1=\left(\frac{1}{a^2}-\frac{1}{a_,^2}\right)(x_,^2+y_,^2+z_,^2). \quad . \quad . \quad . \quad \text{(k)}$$

Eliminating the quantity $\left(\frac{1}{a^2}-\frac{1}{a_{\prime}^2}\right)$ between this and the preceding expression, we find for the resulting equation

$$(x^2+y^2+z^2)^2=(a^2x^2+b^2y^2+c^2z^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right), \quad . \quad \text{(l)}$$

omitting the traits as no longer necessary.

This is the equation of the "*surface of contacts.*"

This equation may be written

$$\begin{aligned}&a^2b^2c^2(a^2x^2+b^2y^2+c^2z^2)\\&\quad=(b^2-c^2)^2a^2y^2z^2+(a^2-c^2)^2b^2x^2z^2+(a^2-b^2)^2c^2x^2y^2. \quad \text{(m)}\end{aligned}$$

If, instead of taking the concyclic surfaces with independent axes, and thus investigating the equation of the "surface of contacts" directly, we had derived the concyclic surfaces from the confocal surfaces of which they are the reciprocal polars, we should have obtained a projective equation for the surface of contacts more nearly in accord with the tangential equation of the surface of centres than the one above given.

To show this, let the radius of the *polarizing sphere* be r, the radius of the polarizing sphere being quite arbitrary; and let $a_{\prime}\, b_{\prime}\, c_{\prime}$ be the semiaxes of the concyclic ellipsoid; then, writing x, y, z for ξ, v, ζ,

$$a=\frac{r^2}{a_{\prime}}, \quad b=\frac{r^2}{b_{\prime}}, \quad c=\frac{r^2}{c_{\prime}};$$

making these substitutions in (n), sec. [119], and omitting the traits as no longer necessary, we shall find

$$(x^2+y^2+z^2)^2=\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right](a^2x^2+b^2y^2+c^2z^2-r^4), \quad . \quad \text{(n)}$$

which is identically the same in form as the tangential equation of the surface of centres.

This equation may also be reduced to the form

$$\left.\begin{aligned}&r^4[b^2c^2x^2+a^2c^2y^2+a^2b^2z^2]\\&\quad=(b^2-c^2)^2a^2y^2z^2+(a^2-c^2)^2b^2x^2z^2+(a^2-b^2)^2c^2x^2y^2.\end{aligned}\right\} \quad \text{(o)}$$

It should be observed that while the axes in the confocal surfaces are in the order of magnitude $a>b>c$, in the concyclic surfaces they are in the order of magnitude $c>b>a$.

We may show that this surface of contacts has four edges of contact perpendicular to the plane of xz.

To show this, let $x^2=\frac{r^4a^2}{(a^2-c^2)(a^2-b^2)}$, $z^2=\frac{r^4c^2}{(a^2-c^2)(b^2-c^2)}$.

Substitute these values of x and z in (o), and we shall have $y=\frac{0}{0}$, or any point taken on the axis of Y will be on the surface.

The sections of the surface of contacts in the plane of xz will be the curves

$$\left.\begin{aligned} & r^4(a^2z^2+c^2x^2)=(a^2-c^2)^2x^2z^2 \\ \text{and}\qquad & (a^2-b^2)^2c^2x^2+(b^2-c^2)^2a^2z^2=r^4a^2c^2, \\ \text{or}\qquad & (a^2-b^2)^2\frac{x^2}{a^2}+(b^2-c^2)^2\frac{z^2}{c^2}=r^4, \end{aligned}\right\} \quad .\ .\quad \text{(p)}$$

of which the former is the reciprocal polar of the evolute of an ellipse, while the latter is an ellipse.

We shall find that these two curves have four common points of contact. For if we make the variables x and z in each equal, the resulting value of z will be given by the expression

$$(a^2-c^2)^2(b^2-c^2)^2z^4-2(a^2-c^2)(b^2-c^2)c^2r^4z^2+c^4r^8=0;$$

but as this is a perfect square the two values of z^2 merge into one, and the resulting value becomes

$$\left.\begin{aligned} & z^2=\frac{r^4c^2}{(a^2-c^2)(b^2-c^2)}\,; \\ \text{in like manner}\qquad & x^2=\frac{r^4a^2}{(a^2-c^2)(a^2-b^2)}\,; \end{aligned}\right\} \quad .\ .\ .\ .\ .\quad \text{(q)}$$

and these are precisely the values of x^2 and z^2, which, substituted in the equation of the surface of contacts, give the value of y under the indefinite form $y=\frac{0}{0}$.

CHAPTER XIII.

ON THE APPLICATION OF THE METHOD OF TANGENTIAL COORDINATES TO THE INVESTIGATION OF THE PROPERTIES OF TRANSCENDENTAL AND OTHER CURVES OF A HIGHER ORDER THAN THE SECOND.

*On the tangential equation of the Caustic by reflection of the Circle**.

126.] Let the projective equation of the circle be

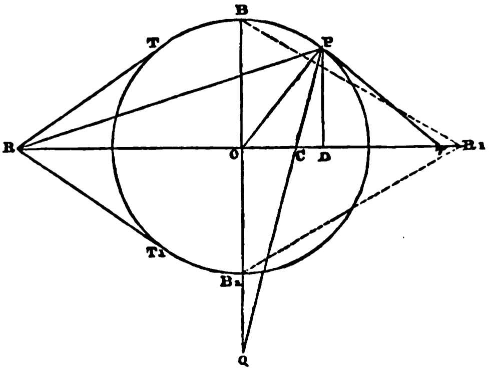
$$x^2+y^2=4a^2. \quad .\ .\ .\ .\ .\ .\ .\ . \quad \text{(a)}$$

---

* The general solution of this problem long baffled the skill of the most expert analysts; at length M. Gergonne announced, 'Annales de Mathématiques,' tom. xv. p. 346, "J'étais, depuis quelque temps, en possession de l'équation de la caustique par réflexion sur le cercle, qui n'avait encore été donnée par personne; mais je l'avais obtenu par des calculs trop prolixes, et sous une forme trop peu élégante pour songer à la publier," &c.

Some time after, the complete solution was given in the seventeenth volume of the same work by M. de St. Laurent, but in a most complicated and unmanageable form.—[Note to first edition.]

Fig. 21.



Let the axis of X pass through the radiating point, or "radiant," as for shortness it may be called. Let ρ be the reciprocal of RO, the distance of the radiant from the centre; then, as the lines RP, PC make equal angles with the radius PO, we shall have

$$\mathrm{RP} : \mathrm{PC} :: \mathrm{RO} : \mathrm{OC};$$

but $$\mathrm{OC}=\frac{1}{\xi},\ \mathrm{OD}=x,\ \mathrm{PD}=y,$$

hence $$y^2+\left(x+\frac{1}{\rho}\right)^2 : y^2+\left(x-\frac{1}{\xi}\right)^2 :: \frac{1}{\rho^2}:\frac{1}{\xi^2};$$

or reducing, and putting $4a^2$ for y^2+x^2, we get, dividing by the factor $(\xi-\rho)$,

$$\xi-\rho=\frac{x}{2a^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{b})$$

If we draw a tangent to the circle at P, meeting the axis X in τ, then $\mathrm{O}\tau=\frac{4a^2}{x}$ and $\frac{x}{2a^2}=\frac{2}{\mathrm{O}\tau}$; hence $\frac{2}{\mathrm{O}\tau}$, ξ, and $-\rho$ are in arithmetical progression, and therefore $\mathrm{O}\tau$, $\frac{1}{\xi}$, and $-\frac{1}{\rho}$ are in harmonical progression, which should be the case, seeing that PR, PO, PC, PT constitute an harmonic pencil.

Since the dual equation gives $y=\frac{1-x\xi}{v}$, combining this with (b) and the projective equation of the circle $x^2+y^2=4a^2$, we may eliminate y and x, so that the resulting equation becomes

$$4a^2(\xi^2+v^2)[1-a^2(\xi-\rho)^2]=1+4a^2\rho\xi, \quad . \quad . \quad . \quad (\mathrm{c})$$

the tangential equation of the caustic of the circle by reflection.

This equation may also be written

$$4a^2(\xi^2+v^2)[1-a(\xi-\rho)][1+a(\xi-\rho)]=1+4a^2\rho\xi. \quad . \quad \text{(d)}$$

When the radiant is at infinity, or $\rho=0$, the equation becomes

$$4a^2(\xi^2+v^2)(1-a^2\xi^2)=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

We shall show further on that this is the tangential equation of an epicycloid, the radius of the base circle being twice that of the rolling circle*.

If we solve (c) for v we obtain

$$v=\frac{1-2a^2\xi(\xi-\rho)}{2a\sqrt{1-a^2(\xi-\rho)^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

Hence, if we assign a series of values to ξ, we shall obtain corresponding values for v, so that the caustic may be defined or set out by the successive positions of the limiting tangent.

---

[*] The learned and accomplished editor of the Mathematical Papers in the 'Educational Times,' Mr. W. J. Miller of Huddersfield, derives the projective equations of the bicusped hypocycloid and cardioid from the general tangential equation of the caustic as follows.

"As an example of the method of finding the *projective* equation from the *tangential* equation, let us take the catacaustic of the circle for parallel rays; then, by putting $\rho=0$ in equation (c), the *tangential* equation is found to be

$$\Phi\equiv 4a^2(1-a^2\xi^2)(\xi^2+v^2)-1=0.$$

"Now assume $\sin^2\psi=2a^2(\xi^2+v^2)$; then from $\Phi=0$ we have

$$2a^2\xi^2=1-\cot^2\psi, \quad 2a^2v^2=\cos^2\psi\cot^2\psi;$$

$$\text{also } \frac{d\Phi}{d\xi}=8a^2\xi^2(1-2a^2\xi^2-a^2v^2)=4a^2\xi(\cos^2\psi+\cot^2\psi),$$

$$\text{and } \frac{d\Phi}{dv}=8a^2v(1-a^2\xi^2) \qquad =4a^2v\operatorname{cosec}^2\psi;$$

$$\therefore \frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{dv}v=4\cos^2\psi;$$

hence, applying the formulæ in Art. 2, we have

$$2(x^2-a^2)=3a^2\operatorname{cosec}^2\psi-\operatorname{cosec}^6\psi, \quad 2y^2=a^2\operatorname{cosec}^6\psi;$$

and, eliminating ψ, the *projective* equation is

$$4(x^2+y^2-a^2)^3=27a^4y^2,$$

which shows that the *catacaustic of the circle, for parallel rays, is a two-cusped epicycloid* whose base is concentric with the reflecting circle and has its radius (a) half the radius of that circle.

"We may further observe that the tangential equation of the *Cardioid*, or *one-cusped epicycloid*, referred to the centre of the base as origin, and the radius (a) through the cusp as positive axis of ξ, is

$$27a^2(1-a\xi)(\xi^2+v^2)=4.$$

This may be obtained from the equation in Question 1492 by putting $-\gamma=\rho=(3a)^{-1}$; for it can be easily shown (see Parkinson's 'Optics,' Art. 72) that, when the radiant point is in the *circumference* of the reflecting circle ($\gamma=-\rho$), the caustic is a *Cardioid* the radius (a) of whose base is one third of that of the reflecting circle."

In this expression, when $\rho=0$, or the radiant point is at infinity, we obtain

$$v=\frac{1-2a^2\xi^2}{2a\sqrt{1-a^2\xi^2}}. \qquad \text{(g)}$$

In the general equation (c) of the caustic, if we put $\xi=-\rho$, the resulting equation becomes

$$4a^2(\xi^2+v^2)=1, \qquad \text{(h)}$$

the tangential equation of the circle which generates the caustic. Hence, if the radiant be outside the circle, the tangent drawn from the radiant to the caustic is also a tangent to the circle.

If we assume, on the opposite side of the origin on the axis of X, a point $R_{,}$ which shall be equidistant from the origin, so that $OR=OR_{,}$, then $\xi=\rho$, and the general equation (c) becomes

$$4a^2v^2=1, \qquad \text{(i)}$$

or the tangent to the caustic drawn from the point $R_{,}$ will cut the vertical diameter of the circle in its circumference.

If in the equation (f) we put $v=0$, or make the limiting tangent vertical, then we shall have $1=2a^2\xi(\xi-\rho)$, or, solving this equation,

$$\xi=\rho\pm\frac{\sqrt{1+2a^2\rho^2}}{\sqrt{2}.a}. \qquad \text{(j)}$$

Hence, when the limiting tangent is vertical, it cuts the horizontal axis in two points at unequal distances from the vertical diameter of the circle; but when $\rho=0$, or the radiant is at infinity, these distances become equal.

To find when the axis of X is a tangent to the caustic.

When the limiting tangent coincides with the axis of X, then $v=\infty$; substituting this value of v in (f), we obtain

$$a(\xi-\rho)=1 \text{ or } \xi=\frac{1}{a}+\rho; \qquad \text{(k)}$$

when $\rho=0$, the cusp is at the middle point of the radius.

The tangent to the caustic parallel to the axis of X cuts the axis of Y at the distance

$$2a\sqrt{1-a^2\rho^2}$$

from the origin; for if in the general equation we put $\xi=0$, the resulting value of $\frac{1}{v}$ becomes

$$2a\sqrt{1-a^2\rho^2}. \qquad \text{(l)}$$

If the radiant be at infinity, $\rho=0$, and the horizontal tangent to the caustic is also a tangent to the circle.

127.] There are two cases of the general theorem which may repay discussion.

Let the radiant be at the extremity of the horizontal diameter of X; then $\rho=\frac{1}{2a}$, and $2a\rho=1$. Substituting this value in the general equation of the caustic, we find

$$a^2(\xi^2+v^2)(3-2a\xi)=1\text{. . . . .} \quad (a)$$

Now this, as we shall show further on, is the tangential of the *cardioid*, the radius of the rolling circle, as also that of the base, being $=\frac{2}{3}a$, or a third of the radius of the reflecting circle.

If we put $\xi=0$ in (a), the limiting tangent becomes horizontal, and cuts the vertical diameter of the circle at the distance $\sqrt{3}a$ from the origin.

If we put $v=\infty$, the limiting tangent must pass through the centre; and this condition gives $\xi=\frac{3}{2a}$, or the cusp is at the distance $\frac{2a}{3}$ from the centre.

If in this equation (a) we put $v=0$, or make the limiting tangent vertical, the points in which it cuts the axis of X will be given by the cubic equation

$$2a^3\xi^3-3a^2\xi^2+1=0;$$

or, putting the equation for the moment under the form

$$2x^3-3x^2+1=0, \text{ . . . .} \quad (b)$$

the three roots of this equation are $+1$, -1, $+\frac{1}{2}$. Hence the points in which the vertical limiting tangent cuts the axis of X are a, $-a$, and $2a$.

Since the vertical limiting tangent cuts the axis of X at the distance a from the centre, and the cusp at the distance $\frac{2}{3}a$ from the centre, it is clear that the distance between the cusp and the point where the vertical limiting tangent intersects the axis of X is $\frac{a}{3}$.

128.] If in the general equation, (d) sec. [126], we put $v=0$, or make the limiting tangent vertical, we shall have the following biquadratic equation to determine the points in which the limiting tangent, when vertical, cuts the axis of X—that is to say,

$$4a^4\xi^4-8a^4\rho\xi^3-4a^2(1-a^2\rho^2)\xi^2+4a^2\rho\xi+1=0. \text{ . .} \quad (a)$$

Now this expression is the square of the following,

$$2a^2\xi^2-2a^2\rho\xi-1=0;$$

or, solving this quadratic,

$$2a\xi=a\rho\pm\sqrt{2+a^2\rho^2}. \text{ . . . .} \quad (b)$$

These are the values of the segments of the axis of x cut off by the vertical limiting tangent.

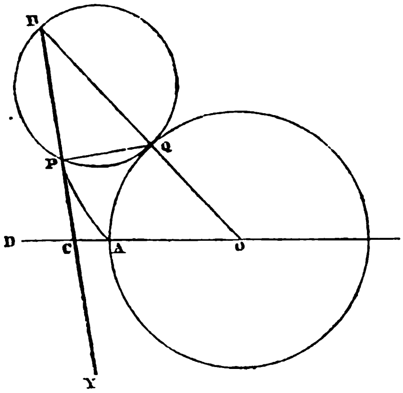
CHAPTER XIV.

ON EPICYCLOIDS AND HYPOCYCLOIDS.

129.] The theory of tangential coordinates may be applied with singular facility to the investigation of the properties of curves of this class, which comprises all those cases in which circles are supposed to roll, either on the outside or the inside of other circles assumed to be fixed, carrying along fixed points on their circumferences, which like tracing-points describe the curves in question.

Let r be the radius of the rolling circle, $R=2nr$ that of the fixed circle. At the beginning of the motion let the fixed or tracing-point coincide with the point of contact of the two circles. Through the centre of the fixed circle let the axes of coordinates be drawn, the axis of X passing through the first point of contact A. Let O be the centre of the fixed circle, Q the point of contact, B the extremity of the common diameter OQ, P the point on the rolling circle which has traced out the arc AP of the epicycloid, and which coincided with A at the beginning of the motion. As the curves are assumed not to slide, one on the other, the arc AQ=PQ.

Fig. 22.



Now the principle assumed in the *mise en équation* is this, that P is always on the circumference of an ever varying circle whose centre is the instantaneous point of contact Q of the two circles. Hence the magnitude and position of the instantaneous radius is always changing, and the limiting tangent to the epicycloid is always at right angles to this radius. QP is the changing radius; and BPC is the limiting tangent at right angles to PQ. Hence it follows that the limiting tangent CP and the common diameter OQB always meet in the same point B on the rolling circle.

Let the angle which the common diameter of the two circles makes with the axis of X be ϕ, then the arc described by the tracing-point of the moving circle is $2nr\phi$; for as the arcs of the circles which have been in contact are equal, putting for the moment this angle as ψ, $r\psi=2nr\phi$ or $\psi=2n\phi$. Hence the angle $OBC=n\phi$, $OC=\frac{1}{\xi}$, and the angle $BCD=(n+1)\phi$. Consequently in the triangle OBC, OB : OC :: sin OCB : sin OBC, or $2(n+1)r : \frac{1}{\xi} :: \sin(n+1)\phi : \sin n\phi$, or

$$\xi=\frac{\sin(n+1)\phi}{2r(n+1)\sin n\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

It is manifest that the limiting tangent meets the axis of Y at a distance $\frac{1}{v}$ from the origin, which will be equal to $OC \times \tan OCY$, or $\frac{1}{v}=\frac{1}{\xi}\tan(n+1)\phi$; or substituting the value of ξ found above,

$$v=\frac{\cos(n+1)\phi}{2r(n+1)\sin n\phi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Eliminating the trigonometrical functions of ϕ between these two equations, we shall obtain an equation between ξ and v, the tangential equation of the epicycloid.

When the circle rolls on the inside of the fixed circle, we must take its radius with the negative sign, so that the formulæ

$$\xi=\frac{\sin(n\pm1)\phi}{2r(n\pm1)\sin n\phi},\quad v=\frac{\cos(n\pm1)\phi}{2r(n\pm1)\sin n\phi} \quad . \quad . \quad . \quad \text{(c)}$$

will answer for hypocycloids as well as epicycloids.

Squaring these expressions and adding, we find

$$4r^2(n\pm1)^2(\xi^2+v^2)\sin^2 n\phi=1, \quad . \quad . \quad . \quad . \quad \text{(d)}$$

which enables us in all cases to find the value of $\sin n\phi$, and also of $\cos n\phi$ in terms of ξ and v.

130.] We shall now proceed to apply this theory to the following selected cases—namely, when $r=\frac{R}{2}$ or $n=1$, when $r=R$ or $n=\frac{1}{2}$, when $r=\frac{R}{4}$ or $n=2$; we shall also consider the case when the radius of the circle rolling inwardly is one half that of the base circle.

(*a*) In the general formulæ for epicycloids, let $n=1$, and the expressions (a) and (b) in the last section become $2r\xi=\frac{\sin 2\phi}{2\sin\phi}$, and $2rv=\frac{\cos 2\phi}{2\sin\phi}$, also $16r^2(\xi^2+v^2)\sin^2\phi=1$.

Now $2r\xi=\dfrac{\sin\phi\cos\phi}{\sin\phi}=\cos\phi$, or $\sin^2\phi=1-4r^2\xi^2$. Eliminating $\sin^2\phi$, the equation of the epicycloid becomes

$$16r^2(\xi^2+v^2)(1-4r^2\xi^2)=1. \quad . \quad . \quad . \quad . \quad \text{(a)}$$

But this is the equation that we found for the caustic by reflexion, see sec. [126] (e), when the radiant is at infinity. If we put $2r=a$, the equation becomes $4a^2(\xi^2+v^2)(1-a^2\xi^2)=1$, which is identical with (e) in sec. [126].

(β) To determine the epicycloid when $n=\frac{1}{2}$.

The general formulæ become

$$2r\xi=\frac{\sin(1+\frac{1}{2})\phi}{\frac{3}{2}\sin\frac{1}{2}\phi}, \quad 2rv=\frac{\cos(1+\frac{1}{2})\phi}{\frac{3}{2}\sin\frac{1}{2}\phi}. \quad . \quad . \quad . \quad \text{(b)}$$

Let $\phi=2\theta$. The equations now become

$$3r\xi=\frac{\sin 3\theta}{\sin\theta}, \quad 3rv=\frac{\cos 3\theta}{\sin\theta}.$$

Hence $9r^2(\xi^2+v^2)\sin^2\theta=1$.

Now $3r\xi\sin\theta=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta$. Dividing by $\sin\theta$, $3r\xi=2\cos^2\theta+(\cos^2\theta-\sin^2\theta)=3-4\sin^2\theta$, or $4\sin^2\theta=3(1-r\xi)$, hence $27r^2(\xi^2+v^2)(1-r\xi)=4$; or if we put $r=\frac{2}{3}a$, this equation becomes

$$a^2(\xi^2+v^2)(3-2a\xi)=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

This is the tangential equation of the *cardioid*, and is identical with the equation for the caustic by reflexion, when the radiant is at the extremity of the horizontal diameter of the reflecting circle, see (a), sec. [127].

(γ) Let $n=2$, or let the base circle receive four undulations of the epicycloid. The general equations become

$$\left.\begin{aligned} &6r\xi=\frac{\sin 3\phi}{\sin 2\phi}, \quad 6rv=\frac{\cos 3\phi}{\sin 2\phi} \\ \text{or} \quad &6^2r^2(\xi^2+v^2)\sin^2.2\phi=1. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Let $\sin 2\phi=c$, $\cos 2\phi=\sqrt{1-c^2}$. Then

$$\left.\begin{aligned} &\sin\phi+\cos\phi=\sqrt{1+c}, \ \sin\phi-\cos\phi=\sqrt{1-c}, \\ &2\sin\phi=\sqrt{1+c}+\sqrt{1-c}, \ 2\cos\phi=\sqrt{1+c}-\sqrt{1-c}, \\ \text{and } &\sin\phi\cos\phi=\frac{c}{2}. \end{aligned}\right\} \quad \text{(d)}$$

Hence $6r\xi.c=\sin 3\phi=\sin 2\phi\cos\phi+\cos 2\phi\sin\phi$,

and $6rv.c=\cos 3\phi=\cos 2\phi\cos\phi-\sin 2\phi\sin\phi$.

Multiplying these expressions together, we shall have

$$\begin{aligned} 6^2r^2\xi vc^2=&\sin 2\phi\cos 2\phi\cos^2\phi-\sin 2\phi\cos 2\phi\sin^2\phi \\ &+\cos^2 2\phi\sin\phi\cos\phi-\sin^2 2\phi\cos\phi\sin\phi. \end{aligned}$$

Hence $6^2r^2\xi\upsilon c^2 = \sin 2\phi \cos^2 2\phi + (\cos^2 2\phi - \sin^2 2\phi) \sin\phi \cos\phi$. Substituting the values for these sines and cosines as given in (d), we shall have $6^2r^4\xi\upsilon c^2 = 3c - 4c^3$; or dividing by c^3 and squaring,

$$\frac{6^4r^4\xi^2\upsilon^2}{c^2} = \frac{9}{c^4} - \frac{24}{c^2} + 16; \quad . \; . \; . \; . \; . \; . \; . \quad \text{(e)}$$

and if we substitute for $\frac{1}{c^2}$ its value as given in (c), namely $6^2r^2(\xi^2+\upsilon^2)$, on making the necessary reductions we shall find

$$27r^2[432r^4\xi^2\upsilon^2 - 27r^2(\xi^2+\upsilon^2) + 2](\xi^2+\upsilon^2) = 1. \quad . \; . \; . \quad \text{(f)}$$

This is the tangential equation of the *quadrantal epicycloid*.

Another solution of this question may be given.

131.] Since $R = 4r$, $n = 2$, and the general equations become $6r\xi \sin 2\phi = \sin 3\phi$ and $6r\upsilon \sin 2\phi = \cos 3\phi$, consequently

$$36r^2(\xi^2+\upsilon^2)\sin^2 2\phi = 1, \text{ and } \sin 3\phi = \frac{\xi}{\sqrt{\xi^2+\upsilon^2}}. \quad . \; . \; . \quad \text{(a)}$$

Let $$36r^2(\xi^2+\upsilon^2)\sin^2 2\phi = a^2\varpi^2 \sin^2.2\phi = 1. \quad . \; . \; . \; . \quad \text{(b)}$$

hence

$$\frac{2\xi^2}{\varpi^2} = 2\sin^2 2\phi \cos^2\phi + 2\cos^2 2\phi \sin^2\phi + 4\sin 2\phi \cos 2\phi \sin\phi \cos\phi. \quad \text{(c)}$$

Now $$\left.\begin{array}{c} \sin^2 2\phi = \dfrac{1}{a^2\varpi^2}, \; \cos^2 2\phi = \dfrac{a^2\varpi^2 - 1}{a^2\varpi^2} = \dfrac{M}{a^2\varpi^2}, \\ \text{writing M for } a^2\varpi^2 - 1; \\ 2\cos^2\phi = \dfrac{a\varpi + \sqrt{M}}{a\varpi}, \; 2\sin^2\phi = \dfrac{a\varpi - \sqrt{M}}{a\varpi}. \end{array}\right\} \; . \; . \quad \text{(d)}$$

Making these substitutions in the preceding value for $\sin 3\phi$, we shall have

$$2\xi^2a^3\varpi = a^3\varpi^3 + \sqrt{M}(4 - a^2\varpi^2).$$

Putting $a = \frac{3}{2}R$, we shall obtain for the tangential equation of the *quadrantal epicycloid*:—

$$(3R)^6(\xi^2+\upsilon^2)\xi^2\upsilon^2 = [16 - 27R^2(\xi^2+\upsilon^2)]^2. \quad . \; . \; . \quad \text{(e)}$$

On the Epicycloid whose base is a Semicircle.

132.] In this case $R = 2r$, and, as generally $R = 2nr$, $n = 1$, and the general formulæ in sec. [129] become

$$\xi = \frac{\sin 2\phi}{4r \sin\phi}, \quad \upsilon = \frac{\cos 2\phi}{4r \sin\phi}. \quad . \; . \; . \; . \; . \; . \quad \text{(a)}$$

Eliminating the trigonometrical functions, which presents no difficulty, we shall finally obtain as the tangential equation of the semicircular epicycloid:—

$$16r^2(1-r^2\xi^2)(\xi^2+v^2)=1. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

When the limiting tangent is parallel to the axis of X, $\xi=0$, and $v=\frac{1}{4r}$, as we might have anticipated.

When $r\xi=1$, the equation becomes $16r^2(\xi^2+v^2)\times 0=1$, or $v=\infty$, hence the limiting tangent at the cusp passes through the centre of the base circle.

On Hypocycloids.

The general formulæ, as given in sec. [129], which hitherto have been used to obtain the tangential equations of epicycloids may with the same facility be applied to the investigation of the properties of hypocycloids, by taking the radius of the rolling circle as negative.

133.] *To determine the equation of the hypocycloid, when the radius of the rolling circle is one half that of the base circle.*

In the general formulæ, sec. [129],

$$2r\xi=\frac{\sin(n-1)\phi}{(n-1)\sin n\phi}, \quad 2rv=\frac{\cos(n-1)\phi}{(n-1)\sin n\phi},$$

let $n=1$, and they become

$$2r\xi=\frac{\sin(1-1)\phi}{(1-1)\sin\phi}, \quad 2rv=\frac{\cos(1-1)\phi}{(1-1)\sin\phi}. \quad . \quad . \quad . \quad (f)$$

How are these expressions to be interpreted ?

Since $(1-1)\phi$ is an indefinitely small angle, we may write $(1-1)\phi$ instead of $\sin(1-1)\phi$, and then dividing by $(1-1)$, $\xi=\frac{\phi}{\sin\phi}$ a finite quantity; but since

$$\cos(1-1)\phi=1, \quad v=\frac{1}{(1-1)\sin\phi}=\frac{1}{0}=\infty; \quad . \quad . \quad . \quad (g)$$

hence, as ξ is finite and v is infinite, the limiting tangent always coincides with the axis of X, as we might have anticipated; for the locus of the tracing point on the hypocycloid is the diameter of the base circle.

134.] *On the hypocycloid whose radius is one fourth that of the base circle.* In this case $n=2$, and the general equations (c) in sec. [129] become

$$2r\xi=\frac{\sin\phi}{\sin 2\phi}, \quad 2rv=\frac{\cos\phi}{\sin 2\phi}, \quad \text{or} \quad \frac{1}{2r}\xi=2\cos\phi, \quad \frac{1}{2rv}=2\sin\phi;$$

hence

$$\frac{1}{16r^2}\left(\frac{1}{\xi^2}+\frac{1}{\upsilon^2}\right)=1. \quad \dots\dots \quad \text{(a)}$$

If we put R for the radius of the base circle, $R=4r$, and the equation of the hypocycloid becomes

$$\xi^2+\upsilon^2=R^2\xi^2\upsilon^2. \quad \dots\dots\dots \quad \text{(b)}$$

Since $\frac{1}{\xi}$ and $\frac{1}{\upsilon}$ denote the intercepts of the axes made by the limiting tangent, and since the sum of the squares of these intercepts is constant, we may infer that if a line of constant length revolves between the sides of a right angle, it will envelope a *quadrantal hypocycloid.*

135.] *To determine the involute of the quadrantal Hypocycloid.*

The projective equation of this curve is

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}. \quad \dots\dots\dots \quad \text{(a)}$$

Let l be the length of the string or elastic radius measured from the extremity A of the horizontal axis of the hypocycloid at the commencement of the motion at G; then at any point P on the hypocycloid, the varying radius PT of the involute at this point will be the line l, plus s the arc of the hypocycloid AP; and as this line is perpendicular to the limiting tangent QT, it is also a tangent to the hypocycloid. Now in sec. [5] we have shown that the length of a perpendicular let fall from a given point P on a limiting tangent is $=\frac{1-p\xi-q\upsilon}{\sqrt{\xi^2+\upsilon^2}}$; or as $l+s$ is the length of the perpendicular, and the projective coordinates of P are x and $-y$, we get

$$l+s=\frac{1-x\xi+y\upsilon}{\sqrt{\xi^2+\upsilon^2}}; \quad \dots\dots \quad \text{(b)}$$

but $\frac{\mathrm{d}y}{\mathrm{d}x}$ derived from (a) is the tangent of the angle that the varying radius (which is a tangent to the hypocycloid) makes with the axis of X, and this line is at right angles to the limiting tangent QT; hence

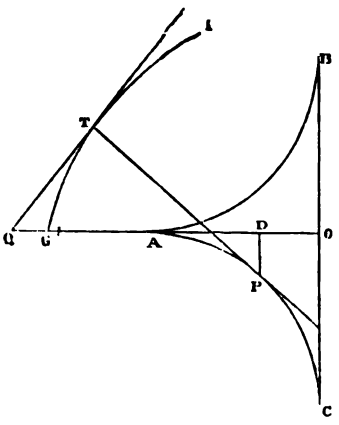
$$\frac{\upsilon}{\xi}=\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}},$$

or

consequently

$$\left.\begin{aligned} y^{\frac{2}{3}}&=\frac{a^{\frac{2}{3}}\upsilon^2}{\xi^2+\upsilon^2}, \quad x^{\frac{2}{3}}=\frac{a^{\frac{2}{3}}\xi^2}{\xi^2+\upsilon^2}; \\ y&=\frac{a\upsilon^3}{(\xi^2+\upsilon^2)^{\frac{3}{2}}}, \quad x=\frac{a\xi^3}{(\xi^2+\upsilon^2)^{\frac{3}{2}}}. \end{aligned}\right\} \quad \dots\dots \quad \text{(c)}$$

Fig. 23.



Now the arc s, measured from A to P, is $=\frac{3}{2}a^{\frac{1}{3}}y^{\frac{2}{3}}$, or $s=\frac{3}{2}a\frac{v^2}{\xi^2+v^2}$. Substituting these values of s, x, and y in the expression for the perpendicular (b), we obtain on reduction

$$\left[(l+a)\xi^2+\left(l+\frac{a}{2}\right)v^2\right]^2=\xi^2+v^2, \quad . \quad . \quad . \quad . \quad \text{(d)}$$

the tangential equation of the *involute* of the quadrantal hypocycloid. It is obvious that when l, the length of the string, the continuation of the elastic radius s, is very large as compared with a, the equation approximates to the tangential equation of a circle; and when $l=0$, or when the tracing point begins with A, the vertex of the hypocycloid, the equation becomes

$$\left(a\xi^2+\frac{av^2}{2}\right)^2=\xi^2+v^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

136.] *On the projective equation of the involute of the quadrantal Hypocycloid.*

The projective equation of the quadrantal hypocycloid is

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

The tangential equation of its involute, as shown in the preceding section, is

$$\xi^2+v^2=[(k+a)\xi^2+(k-a)v^2]^2, \quad . \quad . \quad . \quad \text{(b)}$$

hence

$$\frac{d\Phi}{d\xi}=[4\varpi(k+a)-2]\xi, \quad \frac{d\Phi}{dv}=[4\varpi(k-a)-2]v, \quad . \quad \text{(c)}$$

and

$$\Delta=\frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{d\upsilon}\upsilon=2\varpi^2, \text{ putting } \xi^2+\upsilon^2=\varpi^2;$$

consequently

$$\varpi^2x=[2\varpi(k+a)-1]\xi; \quad \varpi^2y=[2\varpi(k-a)-1]\upsilon. \quad . \quad \text{(d)}$$

The equation of the curve (b) may be written

$$\varpi=k\varpi^2+a(\xi^2-\upsilon^2).$$

It may also be put under the forms

$$\left.\begin{array}{l}\varpi=k\varpi^2-a\varpi^2+2a\xi^2, \text{ or } \varpi=k\varpi^2+a\varpi^2-2a\upsilon^2, \\ \text{or} \\ 2a\xi^2=\varpi-(k-a)\varpi^2, \; 2a\upsilon^2=(k+a)\varpi^2-\varpi.\end{array}\right\} \quad . \quad \text{(e)}$$

If we square the expressions in (d) we shall have

$$\left.\begin{array}{l}\varpi^4x^2=4\varpi^2(k+a)^2\xi^2-4\varpi(k+a)\xi^2+\xi^2, \\ \varpi^4y^2=4\varpi^2(k-a)^2\upsilon^2-4\varpi(k-a)\upsilon^2+\upsilon^2.\end{array}\right\} \quad . \quad . \quad . \quad \text{(f)}$$

Adding these expressions, bearing in mind that

$$\xi^2+\upsilon^2=\varpi^2, \text{ and that } \xi^2-\upsilon^2=\frac{\varpi(1-k\varpi)}{a}, \text{ from (e),}$$

dividing by ϖ^2, and putting

$$\text{U for } [x^2+y^2+4(k^2-a^2)],$$

we shall have

$$\text{U}\varpi^2=8k\varpi-3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

If we subtract the equations in (f) one from the other, we shall find

$$a(x^2-y^2)\varpi^3=4k(a^2-k^2)\varpi^3+8k^2\varpi^2-5k\varpi+1;$$

or putting

$$\text{V for } a(x^2-y^2)+4k(k^2-a^2),$$

the resulting equation becomes

$$\text{V}\varpi^3=8k^2\varpi^2-5k\varpi+1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(h)}$$

If we multiply this equation by 3, and add (g) to it, the resulting equation becomes

$$3\text{V}\varpi^2=(24k^2-\text{U})\varpi-7k. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

Eliminating ϖ from the two quadratic equations

$$\left.\begin{array}{l}\text{U}\varpi^2=8k\varpi-3, \\ 3\text{V}\varpi^2=(24k^2-\text{U})\varpi-7k,\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(j)}$$

we shall obtain the resulting expression in terms of x^2, y^2, k, and a, the projective equation of the involute of the quadrantal hypocycloid. The elimination of ϖ gives the final result,

$$\text{U}[\text{U}^2-13k^2\text{U}+128k^4]=\text{V}[18k\text{U}+128k^3-27\text{V}], \quad . \quad \text{(k)}$$

where

$$\text{U}=x^2+y^2+4(k^2-a^2), \text{ and } \text{V}=a(x^2-y^2)+4k(k^2-a^2).$$

137.] *On the tangential equation of the trigonal Hypocycloid.*

In this case $\mathrm{R}=3r$; hence, as $\mathrm{R}=2nr$, $n=\frac{3}{2}$; and if we substitute this value of n in the general equations,

$$\xi=\frac{\sin(n-1)\phi}{2r(n-1)\sin n\phi},\quad \upsilon=\frac{\cos(n-1)\phi}{2r(n-1)\sin n\phi},\quad . \quad . \quad . \quad \text{(a)}$$

putting $\phi=2\theta$, we shall have

$$\xi=\frac{\sin\theta}{r\sin 3\theta},\quad \upsilon=\frac{\cos\theta}{r\sin 3\theta}.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Hence

$$r^2(\xi^2+\upsilon^2)\sin^2 3\theta=1.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Dividing the expressions in (b) one by the other, we shall have

$$\sin\theta=\frac{\xi}{\sqrt{\xi^2+\upsilon^2}},\quad \cos\theta=\frac{\upsilon}{\sqrt{\xi^2+\upsilon^2}},\quad \sin 2\theta=\frac{2\xi\upsilon}{\xi^2+\upsilon^2},$$

$$\cos 2\theta=\frac{\upsilon^2-\xi^2}{\xi^2+\upsilon^2}.$$

But $$\sin 3\theta=\sin 2\theta\cos\theta+\cos 2\theta\sin\theta.\quad . \quad . \quad . \quad . \quad \text{(d)}$$

Substituting the preceding values of these expressions,

$$\sin 3\theta=\frac{2\xi\upsilon^2+(\upsilon^2-\xi^2)\xi}{(\xi^2+\upsilon^2)^{\frac{3}{2}}}.$$

But (c) gives $\sin 3\theta=\frac{1}{r(\xi^2+\upsilon^2)^{\frac{1}{2}}}$. Hence, equating these values of $\sin 3\theta$,

$$r\xi[3\upsilon^2-\xi^2]=\xi^2+\upsilon^2.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

We shall arrive at a more symmetrical equation of this curve if we choose the inclination of the axes at such an angle as will enable us to place two successive cusps of the curve on the axes of coordinates. Let the coordinate angle be 120°. Then, by the help of the formula given in sec. [4], which enables us to pass from a rectangular system to an oblique one, that is to say,

$$\upsilon=\frac{\upsilon_{,}-\xi\cos\omega}{\sin\omega},$$

as $\omega=120^{\circ}$, we shall have $\upsilon=\frac{2\upsilon_{,}+\xi}{\sqrt{3}}$.

Substituting this value of υ in the preceding equation, we shall have, putting R for the radius of the base circle,

$$\mathrm{R}\xi\upsilon(\xi+\upsilon)=\xi^2+\upsilon^2+\upsilon\xi,\quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

a symmetrical equation in ξ and υ.

This equation may be written in the form

$$\mathrm{R}\left(\frac{1}{\xi}+\frac{1}{v}\right)=\frac{1}{\xi^2}+\frac{1}{v^2}+\frac{1}{v\xi}. \quad \ldots\ldots \quad (\mathrm{g})$$

Now $\frac{1}{\xi}$ and $\frac{1}{v}$ are the segments of the axes of coordinates cut off by the tangent to the hypocycloid. Let l be the length of this tangent between the axes, and $s, s_{\prime}$ the segments of the axes cut off by it; then, as $l^2=\frac{1}{\xi^2}+\frac{1}{v^2}+\frac{1}{\xi v}$, we shall have

$$\mathrm{R}(s+s_{\prime})=l^2, \quad \ldots\ldots \quad (\mathrm{h})$$

or

$$\mathrm{R}=\frac{s^2+s_{\prime}^2}{s+s_{\prime}},$$

or the sum of the squares of the sides is to the sum of the sides in a constant ratio, while in the quadrantal hypocycloid the sum of the squares is constant.

Let P be the perpendicular from the centre on the limiting tangent to the trigonal hypocycloid. It may be shown that

$$2\mathrm{P}=\frac{1}{\{r(\xi+v)\xi v\}^{\frac{1}{3}}}$$[*].

138.] *On the tangential equation of the hexagonal Hypocycloid.*

Let the radius of the revolving circle be one sixth of the radius of the base circle, or $\mathrm{R}=2nr=6r$ or $n=3$.

Hence the general expressions now become

$$4r\xi=\frac{\sin 2\phi}{\sin 3\phi}, \quad 4rv=\frac{\cos 2\phi}{\sin 3\phi}.$$

Consequently

$$16r^2(\xi^2+v^2)\sin^2 3\phi=1. \quad \ldots\ldots \quad (\mathrm{a})$$

We shall also have

$$\left.\begin{array}{l} \sin 2\phi=\dfrac{\xi}{\sqrt{\xi^2+v^2}}, \quad \cos 2\phi=\dfrac{v}{\sqrt{\xi^2+v^2}}, \\[2ex] 2\cos^2\phi=\dfrac{\sqrt{\xi^2+v^2}+v}{\sqrt{\xi^2+v^2}}, \quad 2\sin^2\phi=\dfrac{\sqrt{\xi^2+v^2}-v}{\sqrt{\xi^2+v^2}}. \end{array}\right\} \quad \ldots \quad (\mathrm{b})$$

---

[*] It is beside the purpose of this work further to develope the numerous and beautiful properties of this curve, which in accordance with analogy I have named the trigonal hypocycloid. The object of this work is rather to develope new methods of investigation than to discuss at length properties of particular curves or surfaces. There is the less need to do so in the present case, as the properties of the tricusp hypocycloid have been made the subject of profound investigations by MM. Chasles, Cremona, and Steiner on the continent, and amongst ourselves by Messrs. Clifford, Laverty, Townsend, and others. The subject will be found treated with much elegance and research in the mathematical portion of the 'Educational Times' and other like publications.

Now
$$\left.\begin{aligned}2\sin^2 3\phi = 2\sin^2 2\phi\cos^2\phi + 2\cos^2 2\phi\sin^2\phi\\ +4\sin 2\phi\cos 2\phi\sin\phi\cos\phi.\end{aligned}\right\} \quad . \quad (c)$$

Substituting the preceding values of these expressions, and equating the two values of $\sin 3\phi$, we finally obtain

$$[1-8r^2(\xi^2+\upsilon^2)]^2(\xi^2+\upsilon^2)=64r^4\upsilon^2(3\xi^2-\upsilon^2)^2, \quad . \quad . \quad (d)$$

the tangential equation of the hexagonal hypocycloid.

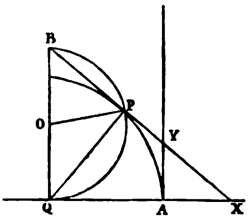
To exhibit the tangential equations of epicycloids and hypocycloids of more numerous convolutions than those above investigated, would require the solution of cubic and still higher equations.

139.] *On the tangential equation of the Cycloid.*

The cycloid may be considered as an epicycloid, the radius of the fixed circle being infinite.

Let A, the initial position of the moving point in contact with the horizontal straight line, be taken at the origin of coordinates, and let the circle be conceived as having rolled forward so as to bring the point A into the position P.

Fig. 24.



Then, as in the case of epicycloids, the limiting tangent is at right angles to the momentary radius PQ of the circle whose centre is the point of contact Q of the rolling circle with the horizontal line or axis of X, the other extremity being the tracing-point P.

Hence the limiting tangent *always* passes through B, the extremity of the vertical diameter of the rolling circle.

Let ξ and υ be the tangential coordinates of the limiting tangent, and let $r.2\phi$ be the arc which the circle has rolled over, ϕ being $=$ QBX; then manifestly, since QX $=$ arc QA, plus the tangential coordinate AX, we shall have

$$2r\tan\phi = 2r\phi + \frac{1}{\xi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Now $\tan\phi = \frac{\upsilon}{\xi}$, and $\phi = \tan^{-1}\left(\frac{\upsilon}{\xi}\right)$; consequently we shall have

$$2r\xi\left[\frac{\upsilon}{\xi}-\tan^{-1}\left(\frac{\upsilon}{\xi}\right)\right]=1, \text{ or } 2r\left[\upsilon-\xi\tan^{-1}\left(\frac{\upsilon}{\xi}\right)\right]=1, \quad (b)$$

the tangential equation of the cycloid.

If, instead of taking the origin at the extremity of the curve, we take it at the centre of the base, or at the distance $r\pi$ from the first origin, then using the formulæ given in sec. [3] for the transformation of coordinates, namely

$$\xi=\frac{\xi_{,}}{1-p\xi_{,}-qv_{,}},\ v=\frac{v_{,}}{1-p\xi_{,}-qv_{,}},$$

by which the translation of the origin and axes of coordinates in parallel directions may be effected. In this case we shall have $p=r\pi$, and $q=0$. Hence the formulæ now become

$$\xi=\frac{\xi_{,}}{1-r\pi\xi_{,}},\ v=\frac{v_{,}}{1-r\pi\xi_{,}}. \quad \ldots\ldots \quad \text{(c)}$$

Substituting these values in (b), we shall have

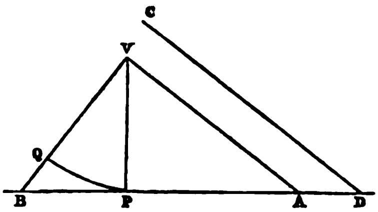
$$2rv_{,}+2r\xi_{,}\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{v_{,}}{\xi_{,}}\right)\right]=1.$$

Now it may easily be shown that the tangent of $\left[\frac{\pi}{2}-\tan^{-1}\left(\frac{v_{,}}{\xi_{,}}\right)\right]$ is $\frac{\xi_{,}}{v_{,}}$. Hence the tangential equation of the cycloid, when the origin is translated to the middle of the base, is

$$2r\left[v+\xi\tan^{-1}\left(\frac{\xi}{v}\right)\right]=1. \quad \ldots\ldots \quad \text{(d)}$$

140.] The vertex V and the hypotenuse AB of a right-angled triangle are given in position. The sides are VA and VB, while VP is the perpendicular on the hypotenuse. The line AD is taken always equal to the circular arc PQ. The line drawn through D parallel to VA will envelope a cycloid.

Fig. 25.



Let $VP=2r$, and the angle $PVQ=\phi$. Then we shall have $PQ=2r\phi=AD$, and $PA=2r\cot\phi$, and $\cot\phi=\frac{v}{\xi}$.

But $PD=PA+AD$, or

$$2r\left[v+\xi\tan^{-1}\left(\frac{\xi}{v}\right)\right]=1,$$

an equation which exactly coincides with (d), the tangential equation of a cycloid, the origin being at the middle point of the base.

141.] *On the tangential equation of the Logarithmic curve.*

Let the projective equation of the logarithmic curve be $y=ae^{\frac{x}{m}}$, and, without detracting from the generality of the expression, we may take a as the base of the system of logarithms whose modulus is m; then $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a}{m}e^{\frac{x}{m}}$; but $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\xi}{v}$; hence $\frac{m}{a}\frac{\xi}{v}=-e^{\frac{x}{m}}$, and $y=-m\frac{\xi}{v}$; substituting this value of y in the dual equation $yv=1-x\xi$, we shall find

$$\frac{x}{m}=\left(\frac{1}{m\xi}+1\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

and the tangential equation of the logarithmic curve becomes

$$\left(\frac{m}{a}\frac{\xi}{v}\right)^{m\xi}+e^{1+m\xi}=0. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

In any curve the subtangent $s=\left(x-\frac{1}{\xi}\right)$; but in the logarithmic $x=\frac{1}{\xi}+m$, as in (a); substituting this value of x,

$s=m$, or the subtangent is constant.

When $\xi=v$, or when the tangent is inclined at half a right angle to the axis of X, $y=-m$.

When $x=0$, $m\xi=-1$, and the general equation (b) becomes $\left(\frac{-1}{av}\right)^{-1}+1$, or $\frac{1}{v}=a$.

142.] *On the tangential equation of the Cissoid.*

Let the projective equation of the cissoid be

$$x(x^2+y^2)-ay^2=\mathrm{F}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a}$$

Then finding the values of $\frac{\mathrm{dF}}{\mathrm{d}x}, \frac{\mathrm{dF}}{\mathrm{d}y}$, and of $\frac{\mathrm{dF}}{\mathrm{d}x}x+\frac{\mathrm{dF}}{\mathrm{d}y}$, we shall have

$$\xi=\frac{3a-2x}{ax}, \quad v^2=\frac{4(a-x)^3}{a^2x^3}.$$

Eliminating x, we shall obtain for the tangential equation of the cissoid, the expression

$$27a^2v^2=4(a\xi-1)^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

143.] *On the tangential equation of the Lemniscate.*

Let the projective equation of the lemniscate be

$$(x^2+y^2)^2-4a^2(x^2-y^2)=0=\mathrm{F}. \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let $x^2+y^2=r^2$, then we shall have

$$r^4=4a^2(x^2-y^2), \; 8a^2x^2=r^2(4a^2+r^2), \; 8a^2y^2=r^2(4a^2-r^2). \quad \text{(b)}$$

Hence, using the formulæ of transition, and substituting in them the values of x and y in terms of r, we shall obtain

$$\left.\begin{aligned} 2a^2r^6\xi^2&=(4a^2+r^2)(r^2-2a^2)^2=16a^6-12a^4r^2+r^6 \\ 2a^2r^6v^2&=(4a^2-r^2)(r^2+2a^2)^2=16a^6+12a^4r^2-r^6 \end{aligned}\right\}. \quad . \quad \text{(c)}$$

Consequently,

$$r^6(\xi^2+\upsilon^2)=16a^4, \text{ and } a^2r^4(\xi^2-\upsilon^2)=r^4-12a^4. \quad . \quad . \quad \text{(d)}$$

Eliminating r from these equations, we obtain finally the tangential equation of the lemniscate

$$27a^4(\xi^2+\upsilon^2)^2=4[1-a^2(\xi^2-\upsilon^2)]^3. \quad . \quad . \quad . \quad . \quad \text{(e)}$$

144.] *On the tangential equation of the Cardioid.*

The projective equation of the cardioid is

$$(x^2+y^2-a^2)^2-4a^2\{(x-a)^2+y^2\}=0=\text{F}. \quad . \quad . \quad \text{(a)}$$

Hence

$$\left.\begin{aligned}&\frac{d\text{F}}{dx}=4x(x^2+y^2-3a^2)+8a^3,\ \frac{d\text{F}}{dy}=4y(x^2+y^2-3a^2),\\ \text{and}\quad &\frac{d\text{F}}{dx}x+\frac{d\text{F}}{dy}y=12a^2\{(x-a)^2-y^2\};\end{aligned}\right\} . \quad \text{(b)}$$

consequently

$$\xi=\frac{2a^3+x(x^2+y^2-3a^2)}{3a^2\{(x-a)^2-y^2\}},\ \upsilon=\frac{y(x^2+y^2-3a^2)}{3a^2\{(x-a)^2-y^2\}}. \quad . \quad \text{(c)}$$

Eliminating x and y between these equations and that of the curve (a), the resulting tangential equation of the cardioid is

$$27a^2(\xi^2+\upsilon^2)(1-a\xi)=4^*. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

---

* Mr. W. Spottiswoode, F.R.S., has given a very elegant solution of this problem in the mathematical papers published in the 'Educational Times' for 1865. It is as follows. "To eliminate x and y between the three equations

$$\frac{x(x^2+y^2-3a^2)+2a^3}{3\xi\{(x-a)^2+y^2\}}=\frac{y(x^2+y^2-3a^2)}{3\upsilon\{(x-a)^2+y^2\}}=\frac{(x^2+y^2-a^2)^2}{4\{(x-a)^2+y^2\}}=a^2. \quad . \quad . \quad \text{(a)}$$

"Let $x^2+y^2-a^2=r^2$, then the given equations become

$x(r^2-2a^2)+2a^3=3a^2\xi(r^2+2a^2-2ax)$, (1) $y(r^2-2a^2)=3a^2\upsilon(r^2+2a^2-2ax)$, (2)

$r^4=4a^2(r^2+2a^2-2ax)$. . . (3) Whence $(r^2-2a^2)x=\frac{3}{4}r^4\xi-2a^3$, . (4)

$$(r^2-2a^2)y=\tfrac{3}{4}r^4\upsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(5)}$$

"$\therefore\ (r^2-2a^2)^2(x^2+y^2)=(r^2-2a^2)^2(r^2+a^2)=\frac{9}{16}r^8(\xi^2+\upsilon^2)-3a^3r^4\xi+4a^6$; (6)

also, from (3) and (4),

$$x=\frac{3r^4\xi-8a^3}{4(r^2-2a^2)}=\frac{8a^4+4a^2r^2-r^4}{8a^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(7)}$$

"But multiplying out, and dividing throughout by r^4, (6) and (7) become

$$9(\xi^2+\upsilon^2)r^4-16r^2+48a^2(1-a\xi)=0, \quad . \quad . \quad . \quad . \quad . \quad \text{(8)}$$

$$r^6-6a^2(1-a\xi)=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(9)}$$

also

$$\{(8)+8(9)\}\div r^2 \text{ gives } 9(\xi^2+\upsilon^2)r^2-8=0,$$

whence finally

$$27a^2(\xi^2+\upsilon^2)(1-a\xi)=4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

the tangential equation of the cardioid."

On the projective equation of the curve whose tangential equation is

$$a^2\upsilon^2+b^2\xi^2=a^2b^2(\xi^2+\upsilon^2)^2, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

or the curve touched by one side of a right angle which moves along an ellipse, the other side passing through the centre.

145.] Let $\xi^2+\upsilon^2=\varpi^2$, then the equation of the curve may be written

$$\xi^2=\frac{a^2\varpi^2(1-b^2\varpi^2)}{a^2-b^2}, \text{ and } \upsilon^2=\frac{b^2\varpi^2(a^2\varpi^2-1)}{(a^2-b^2)}. \quad . \quad . \quad (b)$$

If we differentiate the equation $a^2b^2(\xi^2+\upsilon^2)^2-a^2\upsilon^2-b^2\xi^2=\Phi=0$, we shall have

$$\left.\begin{array}{l}\dfrac{d\Phi}{d\xi}=4a^2b^2(\xi^2+\upsilon^2)\xi-2b^2\xi, \text{ and } \dfrac{d\Phi}{d\upsilon}=4a^2b^2(\xi^2+\upsilon^2)\upsilon-2a^2\upsilon, \\ \text{and } \Delta=\dfrac{d\Phi}{d\xi}\,\xi+\dfrac{d\Phi}{d\upsilon}\,\upsilon=2a^2b^2(\xi^2+\upsilon^2).\end{array}\right\} \quad (c)$$

Hence
$$x=\frac{(2a^2\varpi^2-1)\xi}{a^2\varpi^4}, \text{ and } y=\frac{(2b^2\varpi^2-1)\upsilon}{b^2\varpi^4}. \quad . \quad . \quad (d)$$

If we square these equations separately, and introduce the values of ξ and υ given in (b), we shall have

$$a^2\varpi^6x^2(a^2-b^2)=4a^4\varpi^4-4a^2\varpi^2+1-4a^4b^2\varpi^6+4a^2b^2\varpi^4-b^2\varpi^2, \quad (e)$$

$$b^2\varpi^6y^2(a^2-b^2)=4b^4a^2\varpi^6-4a^2b^2\varpi^4+a^2\varpi^2-4b^4\varpi^4+4b^2\varpi^2-1\,; \quad (c_{\prime})$$

adding these expressions, and dividing by a^2-b^2, we shall have

$$[a^2x^2+b^2y^2+4a^2b^2]\varpi^4=4(a^2+b^2)\varpi^2-3\,; \quad . \quad . \quad . \quad (f)$$

or writing Q^4 for $a^2x^2+b^2y^2+4a^2b^2$, the equation becomes

$$Q^4\varpi^4=4(a^2+b^2)\varpi^2-3. \quad . \quad . \quad . \quad . \quad . \quad (g)$$

If we multiply (c) by b^2 and $(e_{\prime})$ by a^2, and then add the results, we shall have, dividing by (a^2-b^2),

$$a^2b^2(x^2+y^2)\varpi^6=(a^2+b^2)\varpi^2-1. \quad . \quad . \quad . \quad . \quad (h)$$

Let $x^2+y^2=R^2$, and this equation multiplied by 3 becomes

$$3a^2b^2R^2\varpi^6=3(a^2+b^2)\varpi^2-3.$$

If we subtract (g) from this expression, and divide by ϖ^2, we shall have

$$3a^2b^2R^2\varpi^4-Q^4\varpi^2=-(a^2+b^2). \quad . \quad . \quad . \quad . \quad (i)$$

We have now to eliminate ϖ^2 and ϖ^4 between the equations

$$3a^2b^2R^2\varpi^4-Q^4\varpi^2+(a^2+b^2)=0, \text{ and } Q^4\varpi^4-4(a^2+b^2)\varpi^2+3=0. \; (j)$$

Eliminating ϖ^4 and ϖ^2, we get finally for the projective equation of this curve the following expression,

$$\left.\begin{array}{l}Q^{12}-(a^2+b^2)^2Q^8-18a^2b^2(a^2+b^2)R^2Q^4 \\ \quad +16(a^2+b^2)^3a^2b^2R^2+27a^4b^4R^4=0.\end{array}\right\} \quad . \quad . \quad . \quad (k)$$

Substituting for Q^4 its value $[a^2x^2+b^2y^2+4a^2b^2]$, we obtain the projective equation of the curve, which is of the sixth degree.

When the curve is an equilateral hyperbola, $a^2+b^2=0$, and the general equation becomes

$$[(x^2-y^2)-4a^2]^3=27a^2(x^2+y^2)^2.$$

146.] *On the tangential equation of the curve inverse to the central ellipse.*

The projective equation of this curve is

$$a^2x^2+b^2y^2=(x^2+y^2)^2. \quad \ldots \quad \text{(a)}$$

Let

$$x^2+y^2=r^2, \quad \ldots \quad \text{(b)}$$

then the preceding equation becomes $a^2x^2+b^2y^2=r^4$. . . . (c)

Hence

$$x^2=\frac{r^2(r^2-b^2)}{a^2-b^2}, \quad y^2=\frac{r^2(a^2-r^2)}{a^2-b^2}; \quad \ldots \quad \text{(d)}$$

but

$$\left.\begin{array}{l} \dfrac{dF}{dx}=2(2r^2-a^2)x, \quad \dfrac{dF}{dy}=2(2r^2-b^2)y, \\ \text{and} \\ D=\dfrac{dF}{dx}x+\dfrac{dF}{dy}y=2r^4. \end{array}\right\} \quad \ldots \quad \text{(e)}$$

Consequently

$$\xi=\frac{(2r^2-a^2)x}{r^4}, \quad v=\frac{(2r^2-b^2)y}{r^4}; \quad \ldots \quad \text{(f)}$$

or, substituting the values of x and y given in (d),

$$(a^2-b^2)r^6\xi^2=4r^6-4a^2r^4+a^4r^2-4b^2r^4+4a^2b^2r^2-a^4b^2, \quad \text{(g)}$$

$$(a^2-b^2)r^6v^2=-4r^6+4a^2r^4-b^4r^2+4b^2r^4-4a^2b^2r^2+a^2b^4; \quad \text{(h)}$$

add together (g) and (h), divide by (a^2-b^2), and the resulting expression will become $r^6(\xi^2+v^2)=(a^2+b^2)r^2-a^2b^2$. . . . (i)

Multiply (g) by b^2 and (h) by a^2, and the resulting expression, dividing by (a^2-b^2), becomes

$$(b^2\xi^2+a^2v^2)r^6=-4r^6+4(a^2+b^2)r^4-3a^2b^2r^2; \quad \ldots \quad \text{(j)}$$

or if we put $U=b^2\xi^2+a^2v^2+4$, ... (k)

we shall have, dividing by r^2, $Ur^4=4(a^2+b^2)r^2-3a^2b^2$; . (l)

and if we write ϖ^2 for (ξ^2+v^2), (i) becomes

$$r^6\varpi^2=(a^2+b^2)r^2-a^2b^2. \quad \ldots \quad \text{(m)}$$

Multiply this expression by 3, subtract (l) from it, and divide by r^2, we get

$$3r^4\varpi^2-Ur^2+(a^2+b^2)=0. \quad \ldots \quad \text{(n)}$$

Multiply (l) by (a^2+b^2) and (n) by $3a^2b^2$, the resulting expressions will become

$$\left.\begin{array}{l} \mathrm{U}(a^2+b^2)r^4=4(a^2+b^2)^2r^2-3a^2b^2(a^2+b^2) \\ 9r^4a^2b^2\varpi^2=3\mathrm{U}a^2b^2r^2-3a^2b^2(a^2+b^2). \end{array}\right\} \quad . \quad . \quad . \quad (\mathrm{o})$$

Subtract one from the other and divide by r^2; consequently

$$\{\mathrm{U}(a^2+b^2)-9a^2b^2\varpi^2\}r^2=4(a^2+b^2)^2-3\mathrm{U}a^2b^2,$$

or

$$r^2=\frac{4(a^2+b^2)^2-3\mathrm{U}a^2b^2}{\mathrm{U}(a^2+b^2)-9a^2b^2\varpi^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{p})$$

(l) may be written

$$r^4=\frac{4(a^2+b^2)r^2-3a^2b^2}{\mathrm{U}},$$

and (n) may be written

$$r^4=\frac{\mathrm{U}r^2-(a^2+b^2)}{3\varpi^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{q})$$

Equating these values of r^4, we get another expression for r^2—that is to say,

$$r^2=\frac{\mathrm{U}(a^2+b^2)-9a^2b^2\varpi^2}{\mathrm{U}^2-12(a^2+b^2)\varpi^2}. \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{r})$$

If we now eliminate r^2 from this and the preceding equation (p), we shall have the equation

$$a^2b^2\mathrm{U}^3-(a^2+b^2)^2\mathrm{U}^2-18a^2b^2(a^2+b^2)\varpi^2\mathrm{U}+27a^4b^4\varpi^4$$
$$+16(a^2+b^2)^3\varpi^2=0\,;$$

and if we substitute for U and ϖ their values, the tangential equation of the inverse curve of the central ellipse becomes

$$\left.\begin{array}{l} a^2b^2[b^2\xi^2+a^2v^2+4]^3-(a^2+b^2)^2[b^2\xi^2+a^2v^2+4]^2 \\ -18a^2b^2(a^2+b^2)(\xi^2+v^2)[b^2\xi^2+a^2v^2+4]+16(a^2+b^2)^3(\xi^2+v^2) \\ +27a^4b^4(\xi^2+v^2)^2=0. \end{array}\right\} \quad (\mathrm{s})$$

If in this equation we make $b^2=-a^2$, the original curve becomes an equilateral hyperbola, and the preceding equation is reduced to

$$27a^4(\xi^2+v^2)^2=4[1-a^2(\xi^2-v^2)]^3,$$

which is identical with (c) in sec. [143], writing $2a$ for a, and is therefore the tangential equation of the lemniscate.

147.] *On the reciprocal polar of the evolute of an ellipse.*

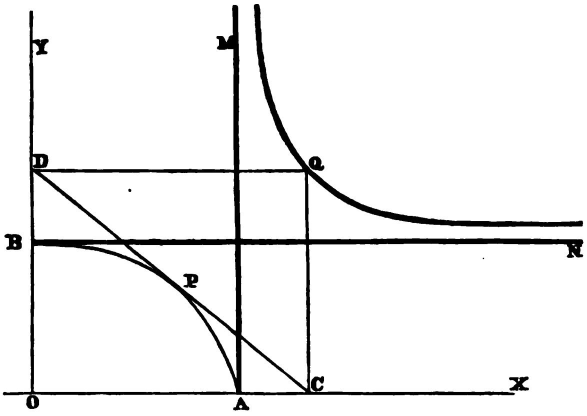
The equation of this curve is

$$a^2y^2+b^2x^2=x^2y^2,$$

and it may be generated as follows.

Let a tangent to an ellipse be drawn meeting the major and minor axes produced in the points C and D. If ordinates to the axes be erected at these points, the locus of their points of intersection will be the curve whose equation is given above.

Fig. 26.



The tangents to the ellipse at A and B will be asymptotes to the curve.

Let $$F=x^2y^2-a^2y^2-b^2x^2=0\,; \quad \ldots\ldots \quad (a)$$

then $$\frac{dF}{dx}=2xy^2-2b^2x,\quad \frac{dF}{dy}=2x^2y-2a^2y,$$

and $$\left.\begin{aligned}&\\ &\frac{dF}{dx}x+\frac{dF}{dy}y=2x^2y^2.\end{aligned}\right\} \quad \ldots\ldots \quad (b)$$

Hence

$$\xi=\frac{y^2-b^2}{xy^2},\quad \upsilon=\frac{x^2-a^2}{x^2y},\quad \text{or}\quad x^2\xi=\frac{y^2x^2-b^2x^2}{xy^2}=\frac{a^2y^2}{xy^2}\,;$$

consequently $\xi=\frac{a^2}{x^3}$. In like manner $\upsilon=\frac{b^2}{y^3}$.

Let $$\alpha a=1,\quad b\beta=1,\quad x^2=\frac{a^{\frac{4}{3}}}{\xi^{\frac{2}{3}}},\quad y^2=\frac{b^{\frac{4}{3}}}{\upsilon^{\frac{2}{3}}}.$$

Substituting these values in (a),

$$\frac{b^2a^{\frac{4}{3}}}{\xi^{\frac{2}{3}}}+\frac{a^2b^{\frac{4}{3}}}{\upsilon^{\frac{2}{3}}}=\frac{a^{\frac{4}{3}}b^{\frac{4}{3}}}{\xi^{\frac{2}{3}}\upsilon^{\frac{2}{3}}},\quad \text{or}\quad \frac{b^{\frac{2}{3}}}{\xi^{\frac{2}{3}}}+\frac{a^{\frac{2}{3}}}{\upsilon^{\frac{2}{3}}}=\frac{1}{\xi^{\frac{2}{3}}\upsilon^{\frac{2}{3}}}\,;$$

or, as

$$a=\frac{1}{\alpha},\quad b=\frac{1}{\beta},$$

the tangential equation of the reciprocal polar of the evolute becomes

$$\left(\frac{\xi}{a}\right)^{\frac{2}{3}}+\left(\frac{v}{\beta}\right)^{\frac{2}{3}}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

precisely the form of the projective equation of the evolute of the ellipse.

On the tangential equation of the semicubical parabola.

148.] The projective equation of the semicubical parabola may be written in the form

$$27ay^2-4x^3=0=\mathrm{F}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Taking the partial differentials,

$$\frac{\mathrm{dF}}{\mathrm{d}y}=54ay,\ \frac{\mathrm{dF}}{\mathrm{d}x}=-12x^2, \text{ and } \mathrm{D}=\frac{\mathrm{dF}}{\mathrm{d}y}y+\frac{\mathrm{dF}}{\mathrm{d}x}x=54ay^2-12x^3.$$

Consequently $x=\frac{3}{\xi}$, $y=\frac{-2}{v}$. Making these substitutious in (a), we shall have

$$a\xi^3-v^2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

the tangential equation of the semicubical parabola.

On the tangential equation of the Cubical parabola.

149.] The projective equation of the cubical parabola may be written under the form

$$3a^2y-x^3=0=\mathrm{F}.$$

Hence $$\frac{\mathrm{dF}}{\mathrm{d}x}=-3x^2,\ \frac{\mathrm{dF}}{\mathrm{d}y}=3a^2, \text{ and } \frac{\mathrm{dF}}{\mathrm{d}y}y+\frac{\mathrm{dF}}{\mathrm{d}x}x=\mathrm{D}=-2x^3.$$

Consequently $x=\frac{3}{2\xi}$, $y=-\frac{1}{2v}$.

Substituting these values of x and y in the projective equation, we obtain for the tangential equation of the cubical parabola

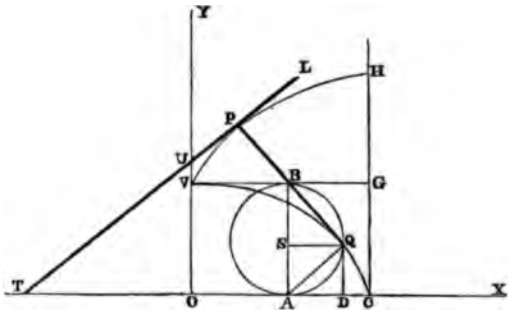
$$9v+4a^2\xi^3=0.$$

On the Involute of the Cycloid.

150.] Let the cycloid be conceived as placed in the position O V C (fig. 27), O being the centre, V the vertex, and C the cusp; and let the unwinding of the curve be conceived to commence at the point V, and let P Q be the position of the elastic radius. Now P Q is a perpendicular to the limiting tangent LT at P, and

$$\mathrm{P\,Q}=\text{the arc }\mathrm{Q\,V}=2\mathrm{Q\,B}=4r\cos\phi.$$

Fig. 27.



But we have shown in sec. [5] that the expression for a perpendicular on a limiting tangent let fall from a point Q, whose projective coordinates are x and y, is

$$P=\frac{1-x\xi-yv}{\sqrt{\xi^2+v^2}}.$$

The angle ABQ=LTO$=\phi$, $OT=-\frac{1}{\xi}$, and $OU=\frac{1}{v}$. Hence

$$\cos\phi=\frac{v}{\sqrt{\xi^2+v^2}}.$$

But in this case as $P=4r\cos\phi$,

$$4rv=1+x\xi-yv, \text{ since } \xi \text{ is negative.} \quad . \quad . \quad . \quad \text{(a)}$$

Now $\quad y=QD=2r\sin^2\phi=\frac{2r\xi^2}{\xi^2+v^2}$, or $yv=\frac{2r\xi^2 v}{\xi^2+v^2}$,

and $x=OC-AC+AD$, or $x=2r\left\{\left(\frac{\pi}{2}-\phi\right)+\sin\phi\cos\phi\right\}$, since the angle which the limiting tangent at P makes with the vertical is the complement of the angle which the elastic radius makes with the same vertical; hence

$$x\xi=2r\left(\frac{\pi}{2}-\phi\right)\xi+\frac{2r\xi^2 v}{\xi^2+v^2}.$$

Consequently

$$x\xi-yv=2r\xi\left(\frac{\pi}{2}-\phi\right),$$

and therefore

$$4rv=1+2r\xi\left(\frac{\pi}{2}-\phi\right). \quad \text{Now } \tan\left(\frac{\pi}{2}-\phi\right)=\cot\phi=\frac{v}{\xi};$$

consequently

$$4rv-2r\xi\tan^{-1}\left(\frac{v}{\xi}\right)=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

If we now translate in parallel directions the axes of coordinates to the vertex V of the curve, which we may effect by the help of the formulæ in sec. [3],

$$\xi=\frac{\xi_{\prime}}{1+2rv_{\prime}}, \quad v=\frac{v_{\prime}}{1+2rv_{\prime}},$$

the preceding expression becomes

$$2r\left\{v-\xi\tan^{-1}\left(\frac{v}{\xi}\right)\right\}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

But this is exactly the expression we have found for the tangential equation of a cycloid when the origin is at a cusp; see (b), sec. [139]. Hence the involute of a cycloid is a cycloid having its axis parallel to the axis of the former, at the distance $2r$ from it, and having its cusp at the vertex V of the original cycloid.

Since $PQ=4r\cos\phi$, and $BQ=2r\cos\phi$, it is evident that the flexible radius is always bisected by the horizontal tangent to the original cycloid at V.

When the axes of coordinates are translated in parallel directions to the vertex V of the cycloid, the tangential equation of the cycloid assumes the very simple form

$$2r\xi\tan^{-1}\left(\frac{\xi}{v}\right)=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

CHAPTER XV.

ON PARALLEL CURVES.

151.] Two straight lines in a plane are said to be parallel when a perpendicular to the one is also perpendicular to the other; whence it follows that these perpendicular distances, wherever taken, are equal.

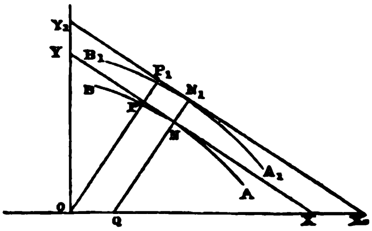
So also two concentric circles may be said to be parallel, since the differences of their radii are equal, and they are at right angles to their respective circles.

We may widen this definition and say that two or more curves are parallel when the differences of their coincident normals are constant.

In general the parallel curves will be of different orders. Thus while the tangential equation of the ellipse is of the second degree, the tangential equation of its parallel curve will be of the fourth degree.

Let N Q, $N_{,}Q$ be two coincident normals to the curves AB and $A_{,}B_{,}$. Let the constant difference of these normals be h. Let $OP=P$, and $OP_{,}=P_{,}$.

Fig. 28.



Now as $OX : OX_{,} :: P : P_{,}$ or $\frac{1}{\xi} : \frac{1}{\xi_{,}} :: P : P_{,}$

$$\xi = \frac{P\xi_{,}}{P} = \frac{P\xi_{,}}{P_{,}-h} = \frac{\xi_{,}}{1-\frac{h}{P_{,}}}; \text{ but } \frac{1}{P_{,}} = \sqrt{\xi_{,}^2+\upsilon_{,}^2} = \varpi_{,}.$$

Hence

$$\xi = \frac{\xi_{,}}{1-h\varpi_{,}}, \quad \upsilon = \frac{\upsilon_{,}}{1-h\varpi_{,}}, \text{ and also } \varpi = \frac{\varpi_{,}}{1-h\varpi_{,}}. \quad . \quad . \quad (a)$$

Making these substitutions for ξ and υ in the original tangential equation of the curve, we shall obtain the tangential equation of the parallel curve.

We shall now proceed to apply this theory to a few examples.

152.] *To determine the tangential equation of the curve parallel to the ellipse.*

The tangential equation of the ellipse referred to its centre and axis is $a^2\xi^2+b^2\upsilon^2=1$.

Substituting the values above given for ξ and υ, we shall have

$$[(a^2-h^2)\xi^2+(b^2-h^2)\upsilon^2-1]^2=4h^2(\xi^2+\upsilon^2). \quad . \quad . \quad (a)$$

Hence it follows that if a series of elastic radii or strings be applied to the evolute of an ellipse, of all the curves that may be thus traced out by a tracing point at the extremity of the radius, there is only one that will be an ellipse; all the rest will be curves of a higher order than the second.

The length of a quadrant of the evolute of an ellipse is manifestly the difference between the normals to the ellipse at the extremities of the minor and major axes. Let s be the arc of the quadrant of the evolute, then

$$s=\frac{a^2}{b}-\frac{b^2}{a}=\frac{a^3-b^3}{ab}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Let α and β be the semiaxes of the evolute. These semiaxes are evidently the differences between the semiaxes of the ellipse and the coincident normals, or

$$\alpha = a - \frac{b^2}{a} = \frac{a^2-b^2}{a} = ae^2, \text{ and } \beta = \frac{a^2}{b} - b = \frac{a^2-b^2}{b} = \frac{a^2e^2}{b};$$

consequently

$$\frac{\alpha}{\beta} = \frac{b}{a}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(c)}$$

Now the length of the elastic radius or string that may describe the ellipse by a tracing point at its extremity is the quadrant of the evolute, plus the difference between the coincident semiaxes of the ellipse and evolute; hence, the length of this radius being l,

$$l = s + (a - \alpha), \text{ or } l = \frac{a^3-b^3}{ab} + a - ae^2 = \frac{a^2}{b}.$$

This is the length of elastic radius or string that will describe the conjugate ellipse.

To express the length of this elastic radius in terms of the semiaxes of the evolute.

Now $l = \frac{a^2}{b}$, and $\alpha = ae^2$, $\beta = \frac{a^2e^2}{b}$; eliminating a and b, we find

$$l = \frac{\beta^3}{\beta^2 - \alpha^2}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(d)}$$

Hence, if the semiaxes of the evolute of an ellipse be α and β, all other lengths of flexible radii or strings greater or less than $\frac{\beta^3}{\beta^2-\alpha^2}$ will describe curves whose tangential equations are of the fourth order.

Hence also, if we assume the surface of the earth to be an oblate spheroid, and to be covered by a sea or atmosphere of uniform depth, the outer surface of this enveloping shell will not be a surface of the second order, but one whose tangential equation is of the fourth order; and if we imagined successive shells or strata of uniform thickness to be applied, the equations of the outer surfaces would all be of the same order.

This is a curious illustration of a breach of the law of continuity. However small the thickness of the shell may be, while the interior surface is that of an ellipsoid the tangential equation of the outer surface will be of the fourth degree.

153.] *The normals of a parabola are increased by the constant line h ; to find the tangential equation of the curve.*

The tangential equation of the parabola referred to its focus as origin, the axis of the parabola being the axis of X, is, as shown in sec. [49] (b),

$$k\xi^2 + kv^2 + \xi = 0.$$

If we now substitute for ξ and v their values given in the preceding section, we shall have

$$[k\xi^2+kv^2+\xi]^2=h^2\xi^2(\xi^2+v^2), \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

the tangential of the curve which is always at the distance h from the parabola, or the curve whose normals differ from the coincident normals of the parabola by the constant quantity h.

The parallel curve of the evolute of the ellipse whose tangential equation is

$$a^2\xi^2+b^2v^2=(a^2-b^2)^2\xi^2v^2, \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

will have for its tangential equation the following expression :—

$$\left.\begin{aligned}&[a^2\xi^2(1+h^2\xi^2)+b^2v^2(1+h^2v^2)+\{h^2(a^2+b^2)-(a^2-b^2)^2\}\xi^2v^2]^2\\ &\qquad =4h^2(a^2\xi^2+b^2v^2)^2(\xi^2+v^2).\end{aligned}\right\}\text{(c)}$$

To find the parallel curve of the quadrantal hypocycloid.

Its equation is $\xi^2+v^2=a^2\xi^2v^2$.

Hence the equation of its parallel curve is

$$[1-2ah\xi v](\xi^2+v^2)=h^2(\xi^2+v^2)^2+a^2\xi^2v^2. \quad . \quad . \quad . \quad \text{(d)}$$

To find the equations of the parallel curves of the involute of the quadrantal hypocycloid.

154.] The tangential equation of the involute of the quadrantal hypocycloid is, as in sec. [136],

$$[(k+a)\xi^2+(k-a)v^2]^2=\xi^2+v^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

To obtain the equation of the curve parallel to this curve, we have only to substitute in this equation the values of ξ and v indicated in [151], and the equation of the parallel curve, or of one whose normals differ from those of the former by the constant h, is

$$[(k+h+a)\xi^2+(k+h-a)v^2]^2=\xi^2+v^2, \quad . \quad . \quad . \quad \text{(b)}$$

a form exactly the same as the preceding, only having $k+h$ instead of k. Whence it follows that whatever the lengths may be of the successive elastic radii, the difference between the axes of any one of the involutes is constant, and independent of the length of the elastic radius. Hence these successive curves constitute bands of equal width, like concentric circles.

We may easily find the tangential equation of the curve that is parallel to the semicubical parabola. We have only to substitute for ξ and v the values $\dfrac{\xi_{\prime}}{1-h\varpi_{\prime}}$ and $\dfrac{v_{\prime}}{1-h\varpi_{\prime}}$ in the equation of the curve $a\xi^3=v^2$, as given in sec. [148]. The equation will become on reduction

$$a^2\xi^6-h^2v^6+(h^2v^2-2a\xi)\xi^2v^2+v^4=4. \quad . \quad . \quad . \quad \text{(c)}$$

In like manner we may find the equation of the curve parallel to the cycloid, whose tangential equation is

$$2rv\left[1+\frac{\xi}{v}\tan^{-1}\left(\frac{\xi}{v}\right)\right]=1,$$

to be

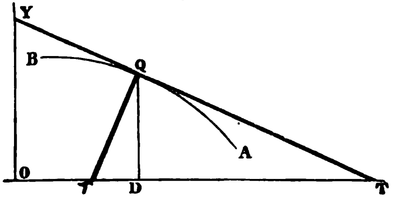
$$\left[2rv+2r\xi\tan^{-1}\left(\frac{\xi}{v}\right)-1\right]^2=h^2(\xi^2+v^2). \quad . \quad . \quad . \quad \text{(d)}$$

CHAPTER XVI.

ON THE TANGENTIAL EQUATIONS OF EVOLUTES.

155.] The system of tangential coordinates supplies a general method of determining the tangential equation of the evolute of any curve whose projective equation is given. To determine the equation of the evolute of a curve by the ordinary methods often requires eliminations that are wholly impracticable.

Fig. 29.



Let A B be an arc of a given curve, whose equation is

$$F(x, y)=0.$$

Let Q τ be a normal and Q T a tangent to the curve A B. Let O T$=\frac{1}{\xi}$, O $\tau=\frac{1}{\xi_{\prime}}$, O D$=x$, Q D$=y$; and as T Q τ is a right angle, $\overline{\text{Q D}}^2=\text{D T}\times\text{D}\,\tau$,

$$\text{or} \quad y^2=\left(\frac{1}{\xi}-x\right)\left(x-\frac{1}{\xi_{\prime}}\right), \quad \text{or } [x-(y^2+x^2)\xi]\xi_{\prime}=1-x\xi. \quad . \quad \text{(a)}$$

But, as we have shown in sec. [22], $\xi=\dfrac{\frac{dF}{dx}}{\frac{dF}{dx}x+\frac{dF}{dy}y}$.

If we substitute this value of ξ in the preceding equation, we shall have

$$\xi_{\prime}=\frac{\frac{dF}{dy}}{\frac{dF}{dy}x-\frac{dF}{dx}y}, \text{ and also } v_{\prime}=\frac{-\frac{dF}{dx}}{\frac{dF}{dy}x-\frac{dF}{dx}y}. \quad . \quad . \quad . \quad \text{(b)}$$

156.] We may apply this method to a few examples.

To determine the tangential equation of the evolute of the ellipse.

The equation of the ellipse is $a^2y^2+b^2x^2-a^2b^2=\mathrm{F}=0$; hence

$$\frac{\mathrm{dF}}{\mathrm{d}y}=2a^2y,\quad \frac{\mathrm{dF}}{\mathrm{d}x}=2b^2x.$$

Substituting these values in the preceding formulæ, we find

$$x=\frac{a^2}{(a^2-b^2)\xi_{,}},\quad y=\frac{b^2}{(a^2-b^2)v}.$$

Introducing these values of x and y in the equation of the ellipse, we obtain, omitting the accents,

$$a^2\xi^2+b^2v^2=(a^2-b^2)^2\xi^2v^2, \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

the tangential equation of the evolute of the ellipse.

157.] To find the tangential equation of the evolute of the parabola. Let the projective equation of the parabola be $y^2=4k(2k+x)$, the origin being taken on the axis at the distance $2k$, or the semiparameter, from the vertex. Hence $\frac{\mathrm{dF}}{\mathrm{d}y}=2y$, $\frac{\mathrm{dF}}{\mathrm{d}x}=4k$; making these substitutions in the general formulæ, we find for the tangential equation of the evolute $k\xi^3=v^2$, the tangential equation of the semicubical parabola.

158.] *On the evolute of the semicubical parabola.*

Let the projective equation of the semicubical parabola be written in the form

$$3ay^2-2x^3=0=\mathrm{F}.$$

Then

$$\frac{\mathrm{dF}}{\mathrm{d}y}=6ay,\quad \frac{\mathrm{dF}}{\mathrm{d}x}=-6x^2;$$

consequently

$$\xi=\frac{a}{x(a+x)},\quad v=\frac{x}{y(a+x)}.$$

Hence

$$\frac{\xi}{v}=\frac{ay}{x^2},\quad \text{or}\quad x=\tfrac{2}{3}\frac{av^2}{\xi^2},$$

and

$$y=\frac{\xi}{v}\frac{x^2}{a}=\tfrac{4}{9}\frac{a^2v^3}{\xi^3}.$$

Substituting these values of x and y in the dual equation, we shall have for the tangential equation of the evolute of the semicubical parabola the expression

$$4av^4=3\xi^2(3\xi-2av^2).$$

159.] *On the evolute of the cubical parabola.*

The projective equation of the cubical parabola is

$$3a^2y-x^3=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Hence $\frac{dF}{dx}=-3x^2$, $\frac{dF}{dy}=3a^2$, $\frac{dF}{dy}x-\frac{dF}{dx}y=3a^2x+3x^2y$.

Consequently, ξ and v being the tangential coordinates of the evolute,

$$\xi=\frac{3a^2}{3x(a^2+xy)},\quad v=\frac{3x^2}{3x(a^2+xy)}; \quad . \; . \; . \; . \quad \text{(b)}$$

dividing one by the other, $x=a\sqrt{\frac{v}{\xi}}$. . . . . (c)

Substituting this value of x in the equation of the curve (a), we find

$$y=\frac{a}{3}\left(\frac{v}{\xi}\right)^{\frac{3}{2}};$$

and if we substitute these values of x and y in the dual equation $x\xi+yv=1$, we shall obtain as a final result the following equation for the evolute of the cubical parabola,

$$9\xi^3=a^2v(3\xi^2+v^2)^2. \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(d)}$$

On the evolute of the quadrantal Hypocycloid.

160.] The tangential and projective equations of the quadrantal hypocycloid are

$$\xi^2+v^2=a^2\xi^2v^2,\ \text{and}\ x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}; \quad . \; . \; . \quad \text{(a)}$$

differentiating this latter, and applying the usual formulæ of transition, we obtain

$$\frac{dF}{dy}=\tfrac{2}{3}\,y^{-\frac{1}{3}},\quad \frac{dF}{dx}=\tfrac{2}{3}\,x^{-\frac{1}{3}},$$

and

$$\frac{dF}{dy}x-\frac{dF}{dx}y=\tfrac{2}{3}\,(xy^{-\frac{1}{3}}-yx^{-\frac{1}{3}}).$$

Consequently

$$\xi=\frac{x^{\frac{1}{3}}}{x^{\frac{4}{3}}-y^{\frac{4}{3}}}=\frac{x^{\frac{1}{3}}}{a^{\frac{2}{3}}(x^{\frac{2}{3}}-y^{\frac{2}{3}})},\ \text{and}\ v=\frac{-y^{\frac{1}{3}}}{a^{\frac{2}{3}}(x^{\frac{2}{3}}-y^{\frac{2}{3}})};$$

hence $\frac{\xi^3}{v^3}=\frac{-x}{y}$, or $y\xi^3+xv^3=0$; combining this expression with the dual equation $x\xi+yv=1$, we get

$$x=\frac{\xi^3}{\xi^4-v^4},\quad y=\frac{-v^3}{\xi^4-v^4}.$$

Substituting these values of x and y in the projective equation of the curve, we obtain

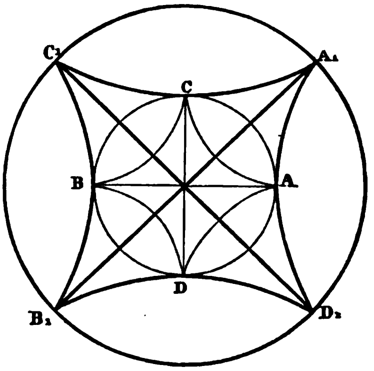
$$\xi^2+v^2=a^2(\xi^2-v^2)^2, \quad . \; . \; . \; . \; . \; . \; . \quad \text{(b)}$$

the tangential equation of a quadrantal hypocycloid, which we may reduce to the usual form by turning the axes of coordinates through an angle of 45°. In sec. [2] it has been shown that this transformation may be effected by putting $\xi=\xi_{,}\cos\theta+v_{,}\sin\theta$, $v=\xi_{,}\sin\theta-v_{,}\cos\theta$, or in this case, as $\theta=\frac{\pi}{4}$, $\sqrt{2}\xi=\xi_{,}+v_{,}$, $\sqrt{2}v=\xi_{,}-v_{,}$; hence $\xi^2-v^2=2\xi_{,}v_{,}$; consequently equation (b) may now be written

$$\xi^2+v^2=(2a)^2\xi^2v^2. \quad . \quad \text{(c)}$$

Hence it follows that the evolute of the quadrantal hypocycloid is also a quadrantal hypocycloid whose parameter is twice that of the original hypocycloid, and whose axes make angles of 45° with the original axes. If we take the evolute of this hypocycloid, its parameter will be four times that of the original hypocycloid, and its cusps will lie in the same direction.

Fig. 30.



On the tangential equation of the evolute of the Lemniscate.

161.] Let the projective equation of the lemniscate be

$$(x^2+y^2)^2=4a^2(x^2-y^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Putting r^2 for x^2+y^2, we may write this equation under the forms

$$x^2=\frac{r^4+4a^2r^2}{8a^2},\quad y^2=\frac{4a^2r^2-r^4}{8a^2}. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Hence

$$\frac{\mathrm{dF}}{\mathrm{d}x}=4r^2x-8a^2x,\quad \frac{\mathrm{dF}}{\mathrm{d}y}=4r^2y+8a^2y.$$

Substituting these values in the general formulæ for determining the tangential equations of evolutes, we shall have

$$\xi=\frac{2a^2+r^2}{4a^2x},\quad v=\frac{2a^2-r^2}{4a^2y},\quad \text{or } x=\frac{2a^2+r^2}{4a^2\xi},\quad y=\frac{2a^2-r^2}{4a^2v}. \quad . \quad \text{(c)}$$

Between (b) and (c) we may eliminate x and y.

The elimination of x gives

$$[1-2a^2\xi^2]r^4+4a^2[1-2a^2\xi^2]r^2+4a^4=0.$$

The elimination of y gives

$$[1+2a^2v^2]r^4-4a^2[1+2a^2v^2]r^2+4a^4=0.$$

If we subtract these equations, one from the other, we shall have

$$(\xi^2+v^2)r^2=4-4a^2(\xi^2-v^2). \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Adding these equations together, we shall have

$$[1-a^2(\xi^2-\upsilon^2)]r^4-4a^4(\xi^2+\upsilon^2)r^2+4a^4=0. \quad . \quad . \quad (e)$$

Eliminating r between these equations, we get

$$4[1-a^2(\xi^2-\upsilon^2)]^3-4a^4[1-a^2(\xi^2-\upsilon^2)](\xi^2+\upsilon^2)^2+a^4(\xi^2+\upsilon^2)^2=0, \ (f)$$

the tangential equation of the evolute of the lemniscate.

The preceding equation may be written also under the form

$$8[1-a^2(\xi^2-\upsilon^2)]^2-4[1-a^2(\xi^2-\upsilon^2)](1+4a^4\xi^2\upsilon^2)+a^4(\xi^2+\upsilon^2)^2=0. (g)$$

We shall now apply this method to a more difficult example.

162.] *To find the evolute of the curve*

$$a^2x^2+b^2y^2=(x^2+y^2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

This is the equation of the locus of the foot of a perpendicular from the centre to a tangent to an ellipse. Let

$$x^2+y^2=r^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and the equation of the curve may now be written in either of the forms

$$x^2=\frac{r^4-b^2r^2}{a^2-b^2} \text{ or } y^2=\frac{a^2r^2-r^4}{a^2-b^2}. \quad . \quad . \quad . \quad . \quad . \quad (c)$$

If we differentiate (a), and substitute in the general formulæ for determining the tangential coordinates of the evolute the values thus found for $\frac{dF}{dy}$ and $\frac{dF}{dx}$, we shall find, after some simple reductions,

$$x=\frac{2r^2-b^2}{(a^2-b^2)\xi} \text{ and } y=\frac{a^2-2r^2}{(a^2-b^2)\upsilon}. \quad . \quad . \quad . \quad (d)$$

If we now eliminate x between the first of (c) and the first of (d), and y between the second equations of the same groups, we shall have two equations by which we may eliminate r. Hence

$$\left.\begin{array}{l}[a^2\xi^2-b^2\xi^2-4]r^4+[b^4\xi^2-a^2b^2\xi^2+4b^2]r^2-b^4=0, \\ [b^2\upsilon^2-a^2\upsilon^2-4]r^4+[a^4\upsilon^2-a^2b^2\upsilon^2+4a^2]r^2-a^4=0.\end{array}\right\} \quad . \quad (e)$$

If we multiply the first of (e) by a^4 and the second by b^4, we shall have

$$\left.\begin{array}{l}[a^6\xi^2-a^4b^2\xi^2-4a^4]r^4+[a^4b^4\xi^2-a^6b^2\xi^2+4a^4b^2]r^2-a^4b^4=0, \\ [b^6\upsilon^2-a^2b^4\upsilon^2-4b^4]r^4+[a^4b^4\upsilon^2-a^2b^6\upsilon^2+4a^2b^4]r^2-a^4b^4=0.\end{array}\right\} \ (f)$$

Subtract one from the other, divide by $(a^2-b^2)r^2$, and the resulting equation will become

$$[a^4\xi^2+b^4\upsilon^2-4(a^2+b^2)]r^2+[4a^2b^2-a^4b^2\xi^2-a^2b^4\upsilon^2]=0. \quad (g)$$

Hence

$$r^2=\frac{a^2b^2[a^2\xi^2+b^2\upsilon^2-4]}{[a^4\xi^2+b^4\upsilon^2-4(a^2+b^2)]}. \quad . \quad . \quad . \quad . \quad (h)$$

If we divide equations (f) by the coefficients of r^4, and then subtract them one from the other, we shall obtain another expression for r^2—that is to say,

$$r^2 = \frac{[a^4\xi^2 + b^4\upsilon^2 - 4(a^2+b^2)]}{[(a^2-b^2)\xi^2 - 4]\,[(a^2-b^2)\upsilon^2+4]}. \quad . \quad . \quad . \quad . \qquad \text{(i)}$$

Between (h) and (i) we may eliminate r; so that the resulting tangential equation of the evolute of the inverse curve of the central ellipse is as follows :

$$\left.\begin{array}{l} a^2b^2[a^2\xi^2 + b^2\upsilon^2 - 4]\,[(a^2-b^2)\xi^2 - 4]\,[(a^2-b^2)\upsilon^2+4] \\ \quad = [a^4\xi^2 + b^4\upsilon^2 - 4(a^2+b^2)]^2. \end{array}\right\} . \qquad \text{(j)*}$$

If in this equation we make $-b^2 = a^2$, it will coincide with (g), sec. [161], the tangential equation of the evolute of the lemniscate.

CHAPTER XVII.

ON REVOLVING ANGLES, PROJECTIVE AND TANGENTIAL.

163.] These constitute a large class of problems. It is an obvious question to ask what may be the locus of the point in which a perpendicular from a fixed point meets a tangent to a given curve; and it is equally pertinent to inquire if a right angle move along a given curve, one side passing through a fixed point, what will be the curve enveloped by the other side of the angle.

To these questions answers have hitherto been obtained by independent methods, each adapted to the particular case under inquiry. But they may all be solved by one uniform method, as simple as it is universal, and one that requires neither skill nor ingenuity in its application.

It is proper to give distinct definitions of these curves. When a right angle moves along a curve whose projective equation is given, one side of the right angle passing through the origin, the other side will envelope a curve which may be called the *tangential pedal* of the given curve.

When a right angle moves so that one side shall touch a curve whose tangential equation is given, while the other side passes through the origin, the vertex of the right angle will describe a curve which may be called the *projective pedal* of the given curve.

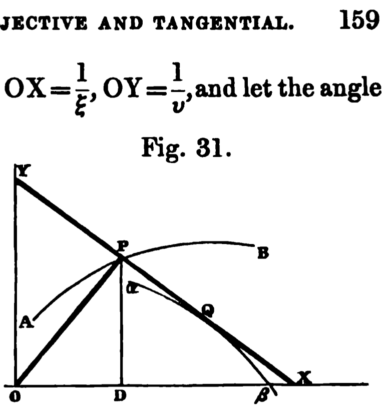
Nothing can be more simple than the proof of these important propositions.

Let XY be a tangent to a curve $\alpha\beta$, and OP a perpendicular on

---

* Professor Price, in his able treatise on the Infinitesimal Calculus, vol. i. p. 383, well observes, "Theoretically, the equations to the evolutes of all curves may be found by means of the given equations; but the difficulty of elimination is in all cases, save in two or three besides the above, so great as to be beyond the present powers of analysis."

Fig. 31.



this tangent. Let PD $=y$, OD $=x$, OX $=\frac{1}{\xi}$, OY $=\frac{1}{v}$, and let the angle which O P makes with the axis of X be ϕ. Now $\tan\phi=\frac{y}{x}$, and $\cot\phi=\frac{\xi}{v}$. Hence $\frac{y\xi}{xv}=1$, or $y\xi-xv=0$. Combining this relation with the dual equation $x\xi+yv=1$, eliminating one variable at a time, we shall have

$$x=\frac{\xi}{\xi^2+v^2},\ y=\frac{v}{\xi^2+v^2},\ \xi=\frac{x}{x^2+y^2},\ v=\frac{y}{x^2+y^2}.\quad .\quad .\quad \text{(a)}$$

(α) Let a perpendicular be let fall from the origin of coordinates on a tangent to the curve $\alpha\beta$, whose tangential equation is $\phi(\xi,v)=0$. The projective equation of the locus of the point in which the perpendicular from the centre meets the tangent will be obtained by the mere substitution of $\frac{x}{x^2+y^2}$ and $\frac{y}{x^2+y^2}$ for ξ and v in the preceding equation; so that the projective equation of the projective pedal will be $\phi\left[\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right]=0$.

(β) Let AB be a curve (see preceding figure) along which a right angle moves, one side passing through the origin. The tangential equation of the curve $\alpha\beta$ which the other side envelopes, will be found by substituting $\frac{\xi}{\xi^2+v^2}$ and $\frac{v}{\xi^2+v^2}$ for x and y in $f(x,y)=0$, the projective equation of the given curve AB; so that the tangential equation of the envelope will be $f\left[\frac{\xi}{\xi^2+v^2},\frac{v}{\xi^2+v^2}\right]=0$.

These are very remarkable as well as general theorems. They admit of the widest application; and there are no difficulties of elimination to contend with. The solutions are obtained by simple substitution, and they afford another striking illustration of that duality which pervades all mathematical truth.

164.] We may illustrate this theory by its application to a few examples.

(α) The projective equation of a straight line is $ay+bx=ab$; for y and x substitute their values $\frac{v}{\xi^2+v^2}$ and $\frac{\xi}{\xi^2+v^2}$, hence $av+b\xi=ab(\xi^2+v^2)$, the tangential equation of a parabola. Hence the *tangential pedal* of a straight line is a parabola.

(β) The tangential equation of a point is $a\xi+bv=1$. For ξ and v

substitute their values $\frac{x}{x^2+y^2}$ and $\frac{y}{x^2+y^2}$; the resulting equation becomes $ax+by=x^2+y^2$, the projective equation of a circle passing through the origin. Hence the projective pedal of a point is a circle.

(γ) Let the general tangential equation of a conic section be

$$a\xi^2+a_{\prime}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{\prime}v=1.$$

Substituting for ξ and v their values as given above, we shall have

$$ax^2+a_{\prime}y^2+2\beta xy+2(\gamma x+\gamma_{\prime}y)(x^2+y^2)=(x^2+y^2)^2;$$

this is the projective equation of the locus of the foot of a perpendicular let fall from any point in its plane on a tangent to the curve. It is manifestly a curve of the fourth degree.

(δ) The tangential equation of a circle when the origin is at the extremity of a diameter is $r^2v^2+2r\xi=1$; and if we substitute for v and ξ their values $\frac{y}{x^2+y^2}$ and $\frac{x}{x^2+y^2}$, we shall have

$$(x^2+y^2)^2-2rx(x^2+y^2)=r^2y^2,$$

the projective equation of a cardioid, the origin being at the cusp of the curve.

165.] The projective equation of a circle, the origin being anywhere in its plane, is

$$x^2+y^2-2px-2qy=r^2-p^2-q^2. \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Hence, if a right angle move along this circle, one side passing through the origin, the other side will envelope the curve whose tangential equation is

$$(r^2-p^2-q^2)(\xi^2+v^2)+2p\xi+2qv=1. \quad . \quad . \quad . \quad \text{(b)}$$

Now, as the coefficients of the squares of the variables are equal, and as the rectangle vanishes, this is the tangential equation of a conic section whose focus is at the origin and whose centre coincides with the centre of the circle, seeing that p and q are the projective coordinates both of the circle and the conic section.

When the point is within the circle the locus is an ellipse, when on the circle it is an infinitesimal parabola degenerating to a straight line passing through the centre; and when the point is outside, the locus is an hyperbola.

Let the tangential equation of the parabola be, see (d) sec. [49],

$$f\xi^2+f_{\prime}v^2+g\xi v+h\xi+h_{\prime}v=0;$$

and if we substitute in this equation the values of ξ and v, we shall have

$$fx^2+f_{\prime}y^2+gxy+(hx+h_{\prime}y)(x^2+y^2)=0,$$

a curve of the third degree which passes through the origin.

Let the tangential equation of the ellipse referred to its centre and axes be $a^2\xi^2+b^2v^2=1$, then the equation of the projective pedal of this curve will be $a^2x^2+b^2y^2=(x^2+y^2)^2$, a well-known curve.

166.] Let the projective and tangential equations of the semicubical parabola be $ay^2=x^3$, and $v^2-a\xi^3=0$. If a right angle move along this curve while one side passes through a cusp, the other side will envelope the curve whose equation is $av^2(\xi^2+v^2)=\xi^3$; and the locus of the foot of the perpendicular on a tangent to this curve will be $y^2(y^2+x^2)=ax^3$.

167.] The tangential equation of a cycloid, the middle of the base being the origin, is $2r\left\{v+\xi\tan^{-1}\left(\frac{\xi}{v}\right)\right\}=1$. . . . . (a)

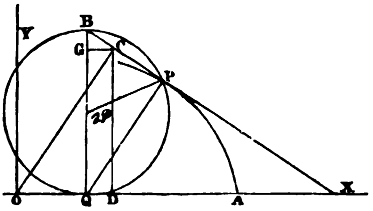
Consequently the locus of the foot of the perpendicular on a tangent to the curve is

$$2r\left\{y+x\tan^{-1}\left(\frac{x}{y}\right)\right\}=x^2+y^2. \quad . \quad . \quad . \quad . \quad . \qquad \text{(b)}$$

An independent geometrical proof of this theorem may be given.

Let O C be a perpendicular on the tangent P B to the cycloid at P. Let O D $=x$, C D $=y$, and the angle P B Q $=\phi$.

Fig. 32.



In the adjoining figure, since CD and OD are the coordinates of C, the foot of the perpendicular OC on the tangent PB to the cycloid, we shall have

$$\text{OQ}=\text{OA}-\text{QA}=2r\left\{\frac{\pi}{2}-\phi\right\}$$

and $$\text{OQ}=\text{OD}-\text{QD}=x-(2r-y)\tan\phi.$$

Equating these two values of OQ, we shall have the above expression for the locus.

168.] *A right angle moves along a cycloid; one side passes through the centre of the base, the other side will envelope a curve whose tangential equation may be thus found.*

When the cycloid is referred to projective coordinates, the origin being at the centre of the base, we know that its equation is

$$r\cos^{-1}\left(\frac{y-r}{r}\right)=x-\sqrt{2ry-y^2}.$$

If we make the necessary substitutions, the equation of the enveloped curve will become

$$r\cos^{-1}\left\{\frac{v}{2r(\xi^2+v^2)}\right\}^{\frac{1}{2}}=\frac{\xi-\sqrt{2rv(\xi^2+v^2)-v^2}}{\xi^2+v^2}.$$

169.] Let the tangential equation of the cardioid be

$$a^2(\xi^2+v^2)(3-2a\xi)=1;$$

and if we put $a=3c$, the equation will become

$$27c^2(\xi^2+v^2)(1-2c\xi)=1.$$

Substituting for ξ and v their values, we get

$$\frac{27c^2}{(x^2+y^2)}\left(\frac{x^2+y^2-2cx}{x^2+y^2}\right)=1,$$

or

$$27c^2(x^2+y^2-2cx)=(x^2+y^2)^2 \quad . \quad . \quad . \quad . \quad \text{(a)}$$

is the projective pedal of the cardioid.

The projective pedal of the evolute of the cubical parabola whose tangential equation is $9\xi^3=a^2v(3\xi^2+v^2)^2$, see sec. [159], will become, on making the indicated substitutions,

$$9x^3(x^2+y^2)^2=a^2y(3x^2+y^2)^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

170.] Let the curve be the quadrantal hypocycloid whose tangential equation may be written

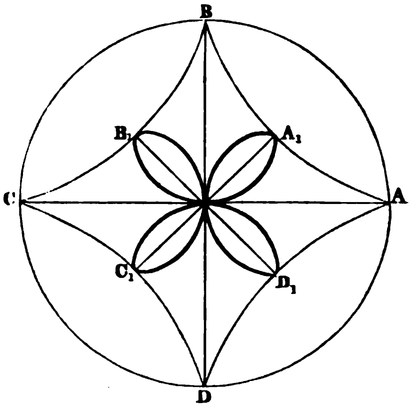
$$\xi^2+v^2=l^2\xi^2v^2.$$

For ξ and v substitute their values $\frac{x}{x^2+y^2}$ and $\frac{y}{x^2+y^2}$; the resulting equation becomes

$$(x^2+y^2)^3=l^2x^2y^2.$$

And the polar equation of this curve, putting $x=r\cos\theta$, $y=r\sin\theta$, becomes $2r=l\sin 2\theta$.

Fig. 33.



Hence the locus of the foot of the perpendicular on a tangent to the quadrantal hypocycloid becomes the looped curve, as shown in figure 33.

171.] The equation of the logocyclic curve is

$$y^2(2a-x)=x(a-x)^2,$$

as we shall show further on.

The equation of its tangential pedal is therefore

$$2a(\xi^2+v^2)=\xi[1+a^2(\xi^2+v^2)]\,;$$

or if a right angle move along the logocyclic curve so that one side shall pass through the cusp, the other side will envelope the curve of which the preceding is the tangential equation.

169.] The projective equation of the lemniscate is

$$(x^2+y^2)^2=a^2(x^2-y^2).$$

The tangential equation of its tangential pedal is therefore $a^2(\xi^2-v^2)=1$, the tangential equation of an equilateral hyperbola.

Hence, if a right angle moves along a lemniscate, one side passing through the centre, the other side will envelope an equilateral hyperbola.

172.] The projective equation of the evolute of an ellipse is

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

the equation of the tangential pedal is

$$\left(\frac{\xi}{a}\right)^{\frac{2}{3}}+\left(\frac{v}{b}\right)^{\frac{2}{3}}=(\xi^2+v^2)^{\frac{2}{3}}. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

The tangential equation of the evolute of an ellipse is

$$b^2\xi^2+a^2v^2=(a^2-b^2)^2\xi^2v^2, \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

the projective equation of the projective pedal is

$$(b^2x^2+a^2y^2)(x^2+y^2)^2=(a^2-b^2)^2x^2y^2. \quad . \quad . \quad . \quad \text{(d)}$$

The projective equation of the cissoid is

$$y^2(2r-x)=x^2\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

the tangential equation of its tangential pedal is $2rv^2-\xi=0$, the equation of a parabola with its axis in a reverse position.

Hence, if a right angle move along a cissoid so that one side shall always pass through the cusp, the other side will envelope a parabola.

The tangential equation of the cissoid is

$$3r^{\frac{1}{3}}v^{\frac{2}{3}}=1-2r\xi\,;$$

the projective equation of the projective pedal is therefore

$$3r^{\frac{1}{3}}(y^2+x^2)^{\frac{1}{3}}y^{\frac{2}{3}}=x^2+y^2-2rx. \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

173.] The tangential equation of the curve generated by the extremity of a line of constant length h added to the normals of an ellipse, as shown in sec. [152], (a), is

$$[(a^2-h^2)\xi^2+(b^2-h^2)\nu^2-1]^2=4h^2(\xi^2+\nu^2). \quad . \quad . \quad (a)$$

If now in this equation we substitute for ξ and ν the values

$$\frac{x}{x^2+y^2}, \text{ and } \frac{y}{x^2+y^2},$$

we shall have for the resulting equation of the locus

$$[(a^2-h^2)x^2+(b^2-h^2)y^2-(x^2+y^2)^2]^2=4h^2(x^2+y^2)^3, \quad . \quad (b)$$

a curve of the eighth degree. When $h=0$, we get the equation of the foot of the perpendicular,

$$a^2x^2+b^2y^2=(x^2+y^2)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

as may be otherwise investigated.

Hence, if a right angle so move that one of its sides shall pass through the centre of an ellipse while the other side envelopes the curve parallel to the ellipse, the vertex of the right angle will move along the curve whose projective equation is (b), a curve of the eighth order.

174.] When we take the parallel curve to the parabola whose equation is given in sec. [153], (a), that is to say,

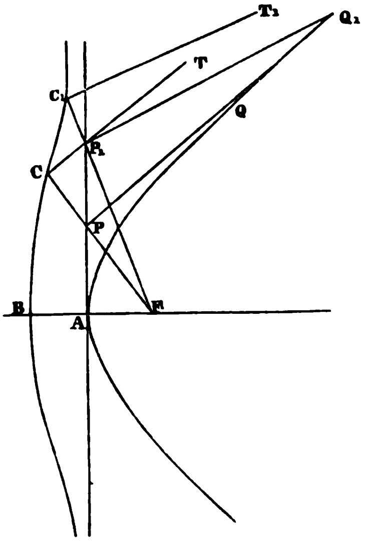
$$[k(\xi^2+\nu^2)+\xi]^2=h^2\xi^2(\xi^2+\nu^2), \quad . \quad . \quad . \quad . \quad . \quad (a)$$

it becomes

$$(x^2+y^2)(k+x)^2=h^2x^2, \quad . \quad . \quad , \quad . \quad . \quad . \quad . \quad (b)$$

the projective equation of a conchoid.

Fig. 34.



Indeed we might have anticipated this—because, if we let fall a focal perpendicular to a tangent to a parabola, the locus of the foot of the perpendicular is the straight line perpendicular to the axis touching the parabola at its vertex; and if we produce all these perpendiculars by the constant line h, their extremities will describe a conchoid of which the *modulus* is h, the *rule* the vertical tangent to the parabola, and the *pole* the focus of the parabola.

Let $F\,A\,Q\,Q_{,}$ be the parabola, F the focus, A the vertex, and $P\,Q$, $P_{,}\,Q_{,}$ tangents to it. P moves along the line $A\,P\,P_{,}$. Let $P\,C=P_{,}C_{,}=h$, then $C\,T$, $C_{,}T_{,}$ parallel to $P\,Q$, $P_{,}\,Q_{,}$ are tangents to the parallel curve, and $CC_{\prime}$ are evidently points in the conchoid $BCC_{\prime}$.

Hence, if a right angle move along a conchoid, one side passing through the pole, the other side will envelope the curve parallel to the parabola, which has its focus at the pole of the conchoid. When $h=0$, the conchoid degenerates into the vertical tangent to the parabola, and the curve parallel to the parabola becomes the parabola itself.

CHAPTER XVIII.

ON RIGHT ANGLES REVOLVING ROUND FIXED POINTS IN THE PLANE OF A CURVE.

175.] Let the tangential equation of the given curve be $\phi(\xi,v)=0$, the fixed point being taken as origin. Let a right angle revolve round this point, one side meeting a tangent to the curve in its point of contact; to determine the projective equation of the locus of the point in which this tangent meets the other side of the right angle.

The equations of the locus are

$$x=\frac{\frac{d\Phi}{dv}}{\frac{d\Phi}{dv}\xi-\frac{d\Phi}{d\xi}v}, \text{ and } y=\frac{-\frac{d\Phi}{d\xi}}{\frac{d\Phi}{dv}\xi-\frac{d\Phi}{d\xi}v}.$$

This theorem may be established as follows:—

Let $(x_{\prime}, y_{\prime})$ be the projective coordinates of the point of contact, and (x, y) the projective coordinates of the required locus. As the tangent to the given curve passes through the points $(x_{\prime}, y_{\prime})$ and (x, y), we shall have the equations

$$x_{\prime}\xi+y_{\prime}v=1 \quad \text{(a), and} \quad x\xi+yv=1; \quad . \quad . \quad (\text{a}_{\prime})$$

and as the sides of the revolving angle are at right angles, we shall also have

$$y_{\prime}y+x_{\prime}x=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

We have also by the formulæ of transition, see sec. [22],

$$x_{\prime}=\frac{\frac{d\Phi}{d\xi}}{\frac{d\Phi}{d\xi}\xi+\frac{d\Phi}{dv}v}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Eliminating $x_{\prime}$, $y_{\prime}$, and y from the four equations (a), $(\text{a}_{\prime})$, (b), and (c), we shall have

$$x=\frac{\frac{d\Phi}{dv}}{\frac{d\Phi}{dv}\xi-\frac{d\Phi}{d\xi}v}, \text{ as also } y=\frac{-\frac{d\Phi}{d\xi}}{\frac{d\Phi}{dv}\xi-\frac{d\Phi}{d\xi}v}. \quad . \quad . \quad . \quad \text{(d)}$$

176.] To apply these general formulæ to some particular examples.

The tangential equation of an ellipse, referred to its axes, is

$$a^2\xi^2+b^2v^2-1=0=\Phi.$$

Hence $\frac{d\Phi}{dv}=2b^2v$, $\frac{d\Phi}{d\xi}=2a^2\xi$; substituting these values in the general formula (d), the resulting equation becomes

$$a^2b^2(a^2x^2+b^2y^2)=(a^2-b^2)^2x^2y^2. \quad . \quad . \quad . \quad . \quad (a)$$

Comparing this equation with the one given in sec. [147], we shall see that the locus is the reciprocal polar of the evolute of a certain ellipse.

177.] *A right angle revolves round the point of inflexion of a cubical parabola; one side meets a tangent at its point of contact with the curve, the other side will meet the same tangent in a point, of which the projective equation is required.*

Let the projective equation of the cubical parabola, asshown in sec. [149], be

$$3a^2y-x^3=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The tangential equation of the same curve is

$$4a^2\xi^3=9v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$\left.\begin{aligned} &\text{Hence} && \frac{d\Phi}{dv}=9,\ \frac{d\Phi}{d\xi}=-12a^2\xi^2,\\ &\text{and} && \Delta_{\prime}=\frac{d\Phi}{dv}\,\xi-\frac{d\Phi}{d\xi}\,v=3\xi(3+4a^2\xi v).\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (c)$$

Consequently $x=\frac{3}{\xi(3+4a^2\xi v)}$ and $y=\frac{4a^2\xi}{(3+4a^2\xi v)}$.

Therefore $\frac{y}{x}=\frac{4a^2\xi^2}{3}$, or $\xi=\sqrt{\frac{3y}{4a^2x}}$.

Introducing this value of ξ into the equation of the curve (b), we find for the value of v:—$\frac{y}{6ax}\sqrt{\frac{3y}{x}}=v$.

Substituting these values of ξ and v in the dual equation, the resulting equation becomes

$$[y^2+3x^2]^2y=12a^2x^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

the projective equation of the required locus.

178.] *A right angle revolves round the cusp of a semicubical parabola; one of its sides meets a tangent to the curve at its point of contact; the other side will meet the same tangent in a point, of which the locus is required.*

Let $$3v^2-a\xi^3=0=\Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$
be the tangential equation of the semicubical parabola, then

$$\frac{d\Phi}{dv}=6v,\ \frac{d\Phi}{d\xi}=-3a\xi^2,\ \text{and}\ \Delta_{,}=\frac{d\Phi}{dv}\xi-\frac{d\Phi}{d\xi}v=3\xi v(a\xi+2).$$

Hence

$$x=\frac{2}{\xi(a\xi+2)},\ y=\frac{a\xi}{v(a\xi+2)},\ \text{and}\ \frac{y}{x}=\frac{a\xi^2}{2v}. \quad . \quad . \quad . \quad \text{(b)}$$

Substituting in the equation (a) of the curve this value of v as derived from (b),

$$\xi=\frac{4y^2}{3ax^2}\ \text{(c)};\ \text{but (b) gives}\ v=\frac{a\xi^2x}{2y}. \quad . \quad . \quad . \quad \text{(c}'\text{)}$$

Substituting these values of ξ and v in the dual equation $x\xi+yv=1$, and reducing, we obtain the projective equation of the locus,

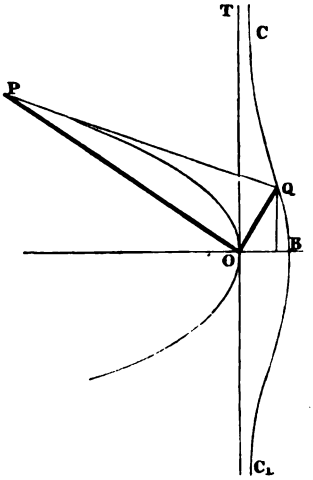
$$12x^2y^2+8y^4=9ax^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

179.] *A right angle revolves round the vertex of a parabola; one side of the angle meets a tangent to the curve at the point of contact; the other side of the angle will meet the tangent in a point whose locus is a curtate cissoid.*

Let $kv^2-\xi=0=\Phi$ be the tangential equation of a parabola, the axis being taken in a reverse position.

Let P O Q be the right angle revolving round the vertex O, one

Fig. 35.



side O P meeting the tangent in P, the other side O Q meeting the tangent in Q.

Now, taking the partial differentials, we shall have

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\upsilon}=2k\upsilon,\ \frac{\mathrm{d}\Phi}{\mathrm{d}\xi}=-1,\ \text{and}\ \Delta_{\prime}=\frac{\mathrm{d}\Phi}{\mathrm{d}\upsilon}\xi-\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\upsilon=2k\upsilon\xi+\upsilon.$$

Hence $x=\dfrac{\frac{\mathrm{d}\Phi}{\mathrm{d}\upsilon}}{\Delta_{\prime}}=\dfrac{2k\upsilon}{2k\xi\upsilon+\upsilon}$, $y=\dfrac{1}{2k\xi\upsilon+\upsilon}$.

Consequently $\dfrac{x}{y}=2k\upsilon$ or $\upsilon=\dfrac{x}{2ky}$, and $\xi=\dfrac{2k-x}{2kx}$.

Substituting these values of ξ and υ in the original equation, we shall find on reduction the equation of the locus

$$2y^2(2k-x)=x^3. \quad \dots\dots\dots \quad \text{(a)}$$

If now we substitute $\bar{y}^2$ for $2y^2$, we shall have $\bar{y}^2(2k-x)=x^3$, the projective equation of the cissoid. Hence, the abscissæ remaining the same, if we shorten the ordinates of the cissoid Q B in the ratio of 1 : $\sqrt{2}$, we shall obtain the curve C B $C_{\prime}$, the required locus; or if the ordinate of the cissoid be represented by the diagonal of a square, the ordinate of the required locus will be its side.

180.] If the revolving angle, instead of being a right angle, should be one whose tangent is τ, we must transform (b), sec. [175] into

$$\tau=\frac{yx_{\prime}-y_{\prime}x}{xx_{\prime}+yy_{\prime}}; \quad \dots\dots\dots \quad \text{(a)}$$

and eliminating $x_{\prime}$, $y_{\prime}$, and y between this expression and (a), ($a_{\prime}$), with (c) in the same section, we shall have

$$\left.\begin{aligned} x&=\frac{\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}-\frac{\mathrm{d}\Phi}{d\upsilon}\tau}{\left\{\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\xi+\frac{\mathrm{d}\Phi}{\mathrm{d}\upsilon}\right\}+\left\{\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\upsilon-\frac{\mathrm{d}\Phi}{\mathrm{d}\upsilon}\xi\right\}\tau},\\ \text{and}\qquad y&=\frac{\frac{\mathrm{d}\Phi}{\mathrm{d}\upsilon}+\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\tau}{\left\{\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\xi+\frac{\mathrm{d}\Phi}{\mathrm{d}\upsilon}\upsilon\right\}+\left\{\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\upsilon-\frac{\mathrm{d}\Phi}{\mathrm{d}\upsilon}\xi\right\}\tau}. \end{aligned}\right\} \quad \dots \quad \text{(b)}$$

We may apply these formulæ to one or two examples. An angle whose tangent is τ, revolves round the centre of an ellipse, one side passes through the point of contact of a tangent, while the other side of this angle meets the tangent to the ellipse in a point of which the locus is required.

Let $a^2\xi^2+b^2\upsilon^2=1$ be the tangential equation of the ellipse; then $\frac{d\Phi}{d\xi}=2a^2\xi$, $\frac{d\Phi}{d\upsilon}=2b^2\upsilon$.

Substituting these values in the preceding formulæ, we shall have

$$x=\frac{a^2\xi-b^2\upsilon\tau}{\{a^2\xi^2+b^2\upsilon^2\}+(a^2-b^2)\xi\upsilon\tau},\ y=\frac{b^2\upsilon+a^2\xi\tau}{\{a^2\xi^2+b^2\upsilon^2\}+(a^2-b^2)\xi\upsilon\tau};\quad \text{(c)}$$

dividing one by the other, we shall have

$$\frac{y}{x}=\frac{b^2\upsilon+a^2\xi\tau}{a^2\xi-b^2\upsilon\tau}\ \text{or}\ \frac{\upsilon}{\xi}=\frac{a^2(y-\tau x)}{b^2(x+\tau y)}.\quad \dots\dots \quad \text{(d)}$$

But the equation of the curve gives $a^2+b^2\frac{\upsilon^2}{\xi^2}=\frac{1}{\xi^2}$, and the dual equation gives $x+y\frac{\upsilon}{\xi}=\frac{1}{\xi}$. From these equations, eliminating $\frac{1}{\xi}$, we shall have

$$a^2+b^2\frac{\upsilon^2}{\xi^2}=x^2+2xy\frac{\upsilon}{\xi}+y^2\frac{\upsilon^2}{\xi^2},\quad \dots\dots \quad \text{(e)}$$

or

$$a^2-x^2+(b^2-y^2)\frac{\upsilon^2}{\xi^2}=2xy\frac{\upsilon}{\xi}.$$

Introducing into this equation the value of $\frac{\upsilon}{\xi}$ as given in (d), the resulting equation will be

$$(a^2-x^2)b^4(x+\tau y)^2+(b^2-y^2)a^4(y-\tau x)^2=2xya^2b^2(x+\tau y)(y-\tau x).\ \text{(e)}$$

If we assume the revolving angle to be half a right angle, $\tau=1$, and the preceding equation becomes

$$\left[\frac{a^2y^2}{b^2}+\frac{b^2x^2}{a^4}-(a^2+b^2)\right](x^2+y^2)=2(a^2-b^2)xy\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right].\quad \text{(f)}$$

Analogous expressions may be found for the parabola and other curves. The application of the method presents but little difficulty.

CHAPTER XIX.

TO INVESTIGATE THE LOCI WHEN THE REVOLVING ANGLES IN THE TWO PRECEDING CHAPTERS ARE OTHER THAN RIGHT ANGLES.

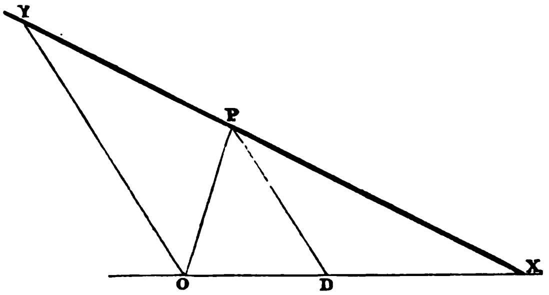
181.] We may, with very slight modifications, apply this method to determine the equations of curves generated by the motion of an angle ω, whether the angle ω move along a given curve, or be the intersection of a tangent with a line drawn from the centre at the angle ω.

We can show that the lines x, y, ξ, v may be transformed into

$$\left.\begin{aligned} x&=\frac{\xi}{\xi^2+v^2+\kappa\xi v}, \quad y=\frac{v}{\xi^2+v^2+\kappa\xi v}, \\ \xi&=\frac{x}{x^2+y^2+\kappa xy}, \quad v=\frac{y}{x^2+y^2+\kappa xy}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (a)$$

Let O X and O Y be the axes of coordinates meeting at the angle

Fig. 36.



ω. Let X Y be the limiting tangent. Let the revolving angle be O P X. Let P D be drawn parallel to the axis O Y. Let P D$=y$, O D$=x$. Then, as the supplement of the angle YOX$=$PDO$=$OPX, the triangles X O P and P O D are similar.

Hence $x=r^2\xi$; but $r^2=\dfrac{1}{\xi^2+v^2+\kappa\xi v}$, putting κ for $2\cos\omega$.

Consequently $x=\dfrac{\xi}{\xi^2+v^2+\kappa\xi v}$.

In like manner we may show that $\xi=\dfrac{x}{x^2+y^2+\kappa xy}$.

When we shall have obtained the equation of the locus of the moving angle ω by the help of oblique coordinates, we may pass back again to rectangular coordinates and exhibit the equation of the required locus referred to rectangular coordinates.

182.] To apply this method to an example or two may suffice.

To find the projective equation of the point in which a tangent to an ellipse is met by a diameter which makes with it an angle of 60°.

The tangential equation of an ellipse, referred to its centre and axes, is

$$a^2\xi^2+b^2v^2=1 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and if we alter the angle of ordination from a right angle to 60° by the help of the formula in sec. [4], we shall have

$$(3a^2+b^2)\xi^2+4b^2v^2-4b^2\xi v=3 ; \quad . \quad . \quad . \quad . \quad (b)$$

and if we now substitute for ξ and v their values as given in (a), since $2\cos\omega=\kappa=1$, we shall have

$$(3a^2+b^2)x^2+4b^2y^2-4b^2xy=3(x^2+y^2+xy)^2, \quad . \quad . \quad \text{(c)}$$

the projective equation of the curve along which an angle of 60° moves, one side touching the ellipse while the other passes through the centre.

183.] The tangential equation of the trigonal hypocycloid, the axes of coordinates being inclined at an angle of 60°, is

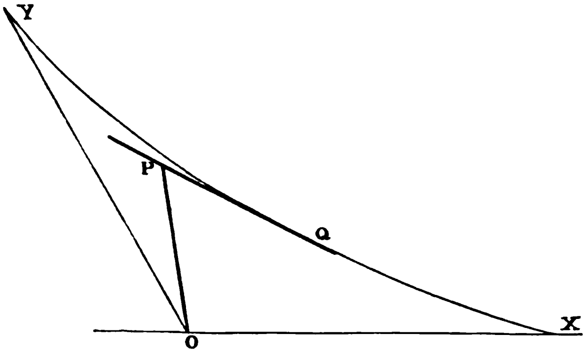
$$\text{R}(\xi+v)\xi v=\xi^2+v^2+\xi v, \text{ see sec. [137].}$$

If we now draw a tangent to this curve and a radius meeting this tangent at an angle of 60°, the projective equation of the locus of this point will be

$$\text{R}(x+y)xy=(x^2+y^2+xy)^2. \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let O X, O Y be the axes of coordinates, inclined at an angle of

Fig. 37.



120°, so that Y O X may be equal to 120°. Let the angle $\text{OPQ}=60°$; then the above equation will be the locus of the point P, the intersection at an angle of 60° of the line O P with P Q, a tangent to the trigonal hypocycloid, having two out of three of its cusps at the points X and Y.

CHAPTER XX.

ON THE LOCI OF RIGHT ANGLES REVOLVING AROUND GIVEN CURVES.

184.] *A right angle revolves so as to touch with its sides a given curve, to determine the projective equation of the vertex of the right angle.*

Let P and $\text{P}_{\prime}$ be the perpendiculars let fall from the origin on the sides of the revolving right angle; and let x and y be the projective coordinates of the vertex of the right angle. Let the tangential equation of the given curve $\phi(\xi, v)=0$ be transformed into $\phi[\mathrm{P}, \sin\lambda, \cos\lambda]=0$, by writing $\sin\lambda$ for $\mathrm{P}\xi$ and $\cos\lambda$ for $\mathrm{P}v$. It is manifest that $x^2+y^2=\mathrm{P}^2+\mathrm{P}_{,}^2$.

Since the limiting tangent passes through the point (xy), we shall have $x\xi+yv=1$ or $y\cos\lambda+x\sin\lambda=\mathrm{P}$; and for the perpendicular $\mathrm{P}_{,}$ at right angles to the former, $y\sin\lambda-x\cos\lambda=\mathrm{P}_{,}$.

Making these substitutions in the given equation of the curve, we shall have

$$\phi[(y\cos\lambda+x\sin\lambda)\sin\lambda, \cos\lambda]=0; \quad . \quad . \quad . \quad (a)$$

and for the perpendicular $\mathrm{P}_{,}$ at right angles to the former,

$$\phi[(y\sin\lambda-x\cos\lambda)\sin\lambda, \cos\lambda]=0. \quad . \quad . \quad . \quad (b)$$

Between these equations, eliminating the trigonometrical quantities, we shall obtain the projective equation of the locus in terms of x, y, and the constants of the given tangential equation of the curve.

To apply this method of investigation to some examples.

185.] Let the tangential equation of the ellipse referred to its centre and axes be $a^2\xi^2+b^2v^2=1$. Multiply by P^2, and the resulting expression becomes

$$a^2\cos^2\lambda+b^2\sin^2\lambda=\mathrm{P}^2=(y\cos\lambda+x\sin\lambda)^2,$$

or

$$a^2\cos^2\lambda+b^2\sin^2\lambda=y^2\cos^2\lambda+2xy\sin\lambda\cos\lambda+x^2\sin^2\lambda;$$

for the perpendicular $\mathrm{P}_{,}$, at right angles to P, we shall have

$$a^2\sin^2\lambda+b^2\cos^2\lambda=y^2\sin^2\lambda-2xy\sin\lambda\cos\lambda+x^2\cos^2\lambda;$$

adding these equations,

$$a^2+b^2=x^2+y^2,$$

or the locus is a concentric circle whose radius $=\sqrt{a^2+b^2}$, a well-known theorem.

Let the curve be the parabola.

186.] The tangential equation of the parabola, see sec. [49], (i), the origin being at the vertex, is

$$kv^2+\xi=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Multiply by P^2, and we shall have $k\cos^2\lambda+\mathrm{P}\sin\lambda=0$, or, substituting for P its value, we shall have

$$k\cos^2\lambda+x\sin^2\lambda+y\sin\lambda\cos\lambda; \quad . \quad . \quad . \quad . \quad (b)$$

for the other side at right angles to the former, we shall have

$$k\sin^2\lambda+x\cos^2\lambda-y\sin\lambda\cos\lambda; \quad . \quad . \quad . \quad . \quad (c)$$

adding these expressions, $k+x=0$, or the locus is the directrix.

To investigate the equation of the vertex of the right angle which envelopes the involute of the quadrantal hypocycloid.

187.] The equation of this involute, see sec. [136], is

$$(l+a)\xi^2+(l-a)\upsilon^2=\sqrt{\xi^2+\upsilon^2}, \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

or, multiplying by P^2,

$$(l+a)\sin^2\lambda+(l-a)\cos^2\lambda=P=y\cos\lambda+x\sin\lambda. \quad . \quad \text{(b)}$$

For the other side, at right angles to the former, we shall have

$$(l+a)\cos^2\lambda+(l-a)\sin^2\lambda=y\sin\lambda-x\cos\lambda; \quad . \quad . \quad \text{(c)}$$

writing $\cos 2\lambda$ for $\cos^2\lambda-\sin^2\lambda$, squaring these expressions, and adding,

$$2l^2+2a^2\cos^2 2\lambda=x^2+y^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

or

$$\cos 2\lambda=\left[\frac{x^2+y^2-2l^2}{2a^2}\right]^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

and

$$\sin 2\lambda=\left[\frac{2a^2+2l^2-(x^2+y^2)}{2a^2}\right]^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

If we multiply (b) and (c) together, the resulting expression becomes

$$2l^2-2a^2\cos^2 2\lambda=(y^2-x^2)\sin 2\lambda-2xy\cos 2\lambda; \quad . \quad . \quad \text{(g)}$$

or if we write for brevity M and N for the values of $\cos 2\lambda$ and $\sin 2\lambda$ given in (e) and (f), we shall have

$$4l^2-(x^2+y^2)=(y^2-x^2)\,N-2xyM, \quad . \quad . \quad . \quad \text{(h)}$$

the projective equation of the locus of the vertex of a right angle rolling round the involute of the quadrantal hypocycloid.

A right angle revolves, touching with its sides the curve parallel to the parabola; to determine the locus of the vertex.

188.] The tangential equation of the curve parallel to the parabola is

$$[k(\xi^2+\upsilon^2)+\xi]^2=h^2\xi^2(\xi^2+\upsilon^2), \quad . \quad . \quad . \quad . \quad \text{(a)}$$

the focus being the origin, see sec. [153].

Multiply by P^2, first taking the square root, and we shall have

$$k+P\sin\lambda=h\sin\lambda,$$

or

$$k+x\sin^2\lambda+y\sin\lambda\cos\lambda-h\sin\lambda=0; \quad . \quad . \quad . \quad \text{(b)}$$

for the perpendicular at right angles to the former we shall have

$$k+x\cos^2\lambda-y\sin\lambda\cos\lambda+h\cos\lambda=0; \quad . \quad . \quad . \quad \text{(c)}$$

adding these expressions,

$$2k+x=h(\sin\lambda-\cos\lambda). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Consequently

$$\sin 2\lambda=\frac{h^2-(2k+x)^2}{h^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

Subtracting (c) from (b),

$$h(\cos\lambda+\sin\lambda)=y\sin 2\lambda-x\cos 2\lambda\,; \quad . \quad . \quad . \quad \text{(f)}$$

and

$$h(\sin\lambda-\cos\lambda)=2k+x, \text{ from (d)}.$$

Consequently

$$h^2\cos 2\lambda=(2k+x)x\cos 2\lambda-(2k+x)y\sin 2\lambda,$$

or

$$(2k+x)y\tan 2\lambda=(2k+x)x-h^2\,; \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

and substituting for tan 2λ its value derived from (e),

$$[h^2-(2k+x)]^2y^2=\{(2k+x)x-h^2\}^2[2h^2-(2k+x)^2], \quad . \quad \text{(h)}$$

the projective equation of the required curve.

When $h=0$, the equation is satisfied by putting $2k+x=0$ or $x=-2k$, and y indefinite, or the locus is the directrix when the curve is a parabola.

A right angle revolves, touching with its sides a cycloid; to determine the locus of the vertex.

189.] Let the tangential equation of the cycloid be

$$2r\left[v+\xi\tan^{-1}\left(\frac{\xi}{v}\right)\right]=1, \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

the origin being taken at the centre of the base.

Multiply this expression by P the perpendicular from the origin on the limiting tangent, the resulting expression is

$$2r[\cos\lambda+\lambda\sin\lambda]=\mathrm{P}\,;$$

and as

$$\mathrm{P}=y\cos\lambda+x\sin\lambda,$$

we shall have

$$[y-2r]\cos\lambda+[x-2r\lambda]\sin\lambda=0. \quad . \quad . \quad . \quad . \quad \text{(b)}$$

If we take the perpendicular at right angles to the former, we shall have

$$[y-2r]\sin\lambda-\left[x-2r\left(\lambda+\frac{\pi}{2}\right)\right]\cos\lambda=0. \quad . \quad . \quad \text{(c)}$$

Eliminating sin λ, cos λ from these expressions, we obtain

$$\left[2r\lambda-\left(x-r\frac{\pi}{2}\right)\right]^2=\left(r\frac{\pi}{2}\right)^2-(2r-y)^2. \quad . \quad . \quad . \quad \text{(d)}$$

But if we eliminate λ between (b) and (c), we shall have

$$\sin 2\lambda=\frac{2r-y}{r\frac{\pi}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

or $2\lambda=\sin^{-1}\left(\dfrac{2r-y}{r\frac{\pi}{2}}\right)$; substituting this value of 2λ in the preceding equation, the resulting expression becomes

$$\left[r\sin^{-1}\left(\frac{2r-y}{r\frac{\pi}{2}}\right)-\left(x-r\frac{\pi}{2}\right)\right]^2=\left(r\frac{\pi}{2}\right)^2-(2r-y)^2, \quad . \quad \text{(f)}$$

the projective equation of the vertex of a right angle revolving round a cycloid.

A right angle revolves round a fixed point, whose sides meet a curve in certain points, to determine the tangential equation of the curve which is enveloped by the line joining these points.

190.] Let $$f(x, y)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$
be the equation of the curve, the origin being taken at the fixed point. Let
$$y=mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$
be the equation of one of the vectors, and let
$$x\xi+yv=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$
be the equation of the limiting tangent. Now (a) and (c) may be written $f(x, m)$ and $x=\frac{1}{\xi+mv}$; eliminating x between these two equations, the resulting expression will become
$$f(\xi, v, m)=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

The other side of the right angle will give
$$f(\xi, v, m_{\prime})=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$
and as the angle is assumed to be a right angle, we shall have the additional equation
$$mm_{\prime}+1=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$
Eliminating m and $m_{\prime}$ between these three equations, the resulting equation in ξ and v will be the tangential equation of the sought curve.

An illustration of this method may suffice.

A right angle revolves meeting two given straight lines in the points A *and* B; *the line which joins the points* A *and* B *will envelope a conic section whose focus is at the origin, and will also touch the two given lines.*

191.] Let $\frac{x}{a}+\frac{y}{b}=1$, (a), and $\frac{x}{a_{\prime}}+\frac{y}{b_{\prime}}=1$ . . . . . . ($a_{\prime}$)
be the projective equations of the two given straight lines; let
$$y=nx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$
be the projective equation of one of the sides of the right angle, and let
$$x\xi+yv=1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

be the dual equation; eliminating x and y between these three equations, we shall have

$$n=\frac{b}{a}\left(\frac{1-a\xi}{bv-1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

For the other straight line we shall have

$$n_{\prime}=\frac{b_{\prime}}{a_{\prime}}\left(\frac{1-a_{\prime}\xi}{b_{\prime}v-1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d}_{\prime}\text{)}$$

But as the revolving angle is a right angle, $nn_{\prime}+1=0$; consequently, substituting for n and $n_{\prime}$ their values, we shall have as the resulting expression

$$\xi^2+v^2-\left(\frac{1}{a}+\frac{1}{a_{\prime}}\right)\xi-\left(\frac{1}{b}+\frac{1}{b_{\prime}}\right)v+\frac{1}{aa_{\prime}}+\frac{1}{bb_{\prime}}=0, \quad . \quad . \quad \text{(e)}$$

the tangential equation of a conic section.

Since the coefficients of the squares of the variables are equal, and the rectangle ξv does not appear, the focus of the conic section is at the origin. The given straight lines are tangents to the curve; for the equation (e) is satisfied by putting $\xi=\frac{1}{a}$, and $v=\frac{1}{b}$, or $\xi=\frac{1}{a_{\prime}}$, and $v=\frac{1}{b_{\prime}}$.

The curve is a parabola when $aa_{\prime}+bb_{\prime}=0$; for then the absolute term vanishes. But when $aa_{\prime}+bb_{\prime}=0$, the given lines are at right angles.

192.] *A right angle revolves round a point in the plane of a conic section meeting the curve in two points; the line joining these points will envelope a conic section, one of whose foci is at the given point.*

Let the origin of coordinates be taken at the given point, and let the equation of the curve be

$$Ax^2+A_{\prime}y^2+2Bxy+2Cx+2C_{\prime}y=1. \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Let $y=nx$ (b) and $x\xi+yv=1$ (c); eliminating x and y between (a), (b), and (c), the resulting equation may be written

$$n^2+\frac{2[B+C_{\prime}\xi+Cv-\xi v]}{A_{\prime}+2C_{\prime}v-v^2}\,n+\frac{A+2C\xi-\xi^2}{A_{\prime}+2C_{\prime}v-v^2}=0. \quad . \quad . \quad \text{(d)}$$

If now we take the other side of the right angle, bearing in mind that $mm_{\prime}+1=0$, we shall have

$$n^2-\frac{2[B+C_{\prime}\xi+Cv-\xi v]}{A+2C\xi-\xi^2}\,n+\frac{A_{\prime}+2C_{\prime}v-v^2}{A+2C\xi-\xi^2}=0. \quad . \quad . \quad \text{(e)}$$

Now, as n must have the same roots in each equation, the coefficients must be identical. Equating the absolute term in each, we shall have $(A+2C\xi-\xi^2)^2=(A_{\prime}+2C_{\prime}v-v^2)^2$.

And taking the square root on each side,

$$(A+2C\xi-\xi^2)=\pm(A_{\prime}+2C_{\prime}v-v^2).$$

We cannot, *à priori*, say which of these signs we should select; but on comparing the second terms, and dividing out the common numerator $(B+C_{,}\xi+C\upsilon-\xi\upsilon)$, we find

$$(A_{,}+2C_{,}\upsilon-\upsilon^2)=-(A+2C\xi-\xi^2).$$

Consequently the resulting equation becomes

$$\xi^2+\upsilon^2-2C\xi-2C_{,}\upsilon=A+A_{,}. \quad . \quad . \quad . \quad . \quad . \quad (f)$$

Now, as the squares of the variables have the same coefficients, and the term in $\xi\upsilon$ does not appear, this conic section has its focus at the origin.

RECAPITULATION.

193.] It may be well to recapitulate the propositions that have been established in the preceding Chapters, on the description of curves by the motion of straight lines and angles constrained to move under certain conditions.

In Chapter XV. it has been shown that a curve may be drawn parallel to another curve at the distance h from it, by substituting

$$\frac{\xi_{,}}{1-h\sqrt{\xi_{,}^2+\upsilon_{,}^2}} \text{ and } \frac{\upsilon_{,}}{1-h\sqrt{\xi_{,}^2+\upsilon_{,}^2}} \text{ for } \xi \text{ and } \upsilon,$$

in the tangential equation of the curve $\phi(\xi,\upsilon)=0$.

In Chapter XVI. it has been established that if a right angle revolves touching at its vertex a curve whose projective equation is $f(x,y)=0=F$, the other side of the right angle will envelope a curve (the evolute) whose tangential equation may be found by eliminating x and y between the expressions

$$\xi=\frac{\frac{dF}{dy}}{\frac{dF}{dy}x-\frac{dF}{dx}y}, \quad \upsilon=\frac{-\frac{dF}{dx}}{\frac{dF}{dy}x-\frac{dF}{dx}y}, \text{ and } f(x,y)=0.$$

In Chapter XVII. it has been proved that if a right angle revolves so that its vertex shall move along a curve whose equation is $f(x,y)=0$, while one side of the angle passes through the origin, the other side will envelope a curve whose tangential equation may be found by substituting, for x and y in the preceding equation, $\frac{\xi}{\xi^2+\upsilon^2}$ and $\frac{\upsilon}{\xi^2+\upsilon^2}$.

It has been moreover established in the same Chapter that if a right angle revolves, one side passing through a fixed point, while the other side envelopes a curve whose tangential equation is $\phi(\xi,\upsilon)=0$, the projective equation of the vertex of the right angle will be found by substituting $\frac{x}{x^2+y^2}$ and $\frac{y}{x^2+y^2}$ for ξ and υ in the given tangential equation of the curve.

In Chapter XVIII. it has been shown that if round a fixed point in the plane of a curve a right angle revolves so that one side shall pass through the point of contact of a tangent drawn to the curve, the locus of the point in which the other side of the right angle meets the same tangent will be found by eliminating ξ and υ between $\phi(\xi, \upsilon)=0$, the tangential equation of the curve, and the equations

$$x=\frac{\dfrac{d\Phi}{d\upsilon}}{\dfrac{d\Phi}{d\upsilon}\xi-\dfrac{d\Phi}{d\xi}\upsilon}, \text{ and } y=\frac{-\dfrac{d\Phi}{d\xi}}{\dfrac{d\Phi}{d\upsilon}\xi-\dfrac{d\Phi}{d\xi}\upsilon}.$$

In Chapter XIX. it is shown how the preceding methods of describing curves by the motion of right angles constrained to move under certain conditions may be extended to the description of curves by the motion of any angle whatever.

In Chapter XX. it has been established that if a right angle revolves touching with its sides a curve whose tangential equation, $\phi(\xi, \upsilon)=0$, is given, the coordinates of the vertex x and y are found by eliminating ξ and υ between the equations $\phi(\xi, \upsilon)=0$, $x\xi+y\upsilon=1$, $x\xi_{\prime}+y\upsilon_{\prime}=1$, and, as the lines are supposed to be at right angles, $\xi\xi_{\prime}+\upsilon\upsilon_{\prime}=0$.

Finally, if a right angle revolves round a point, meeting in points a curve whose equation is $f(x, y)$, the line joining these points envelopes a curve whose tangential equation may be found by eliminating x, y, and n between the equations

$$\left.\begin{array}{c} f(x, y)=0, \quad x\xi+y\upsilon=1, \quad x_{\prime}\xi+y_{\prime}\upsilon=1, \\ y=nx, \quad y_{\prime}=n_{\prime}x, \text{ and } nn_{\prime}+1=0. \end{array}\right\}$$

CHAPTER XXI.

ON PEDAL TANGENTIAL COORDINATES.

194.] Besides the system of tangential coordinates which has had its principles developed in the foregoing pages, there is another, which for distinction may be called pedal tangential coordinates. The object of the former method is to develope the duality of the properties of space in an analytical form; while the scope of the latter is to derive new curves from others already known, by the mechanical action, as it were, of a very simple law.

Let $\mathrm{F}(x, y)=0$ be the projective equation of any plane curve or other locus. In this equation for x and y let their reciprocals ξ and υ be substituted, $x\xi$ and $y\upsilon$ being each $=1$, and the new equation, $\mathrm{F}\left(\frac{1}{\xi}, \frac{1}{\upsilon}\right)=0$, becomes the tangential equation of a curve, in general of a different species, but derived from the projective equation $F(x, y)=0$ by a very simple law.

A few illustrations will render this method of pedal tangential coordinates very obvious and plain.

Let the projective equation of a straight line be $\frac{x}{a}+\frac{y}{b}=1$. For x and y substitute $\frac{1}{\xi}$ and $\frac{1}{v}$, and the equation becomes

$$\frac{1}{a\xi}+\frac{1}{bv}=1, \text{ or } ab\xi v-a\xi-bv=0,$$

the tangential equation of a parabola, see sec. [49].

195.] Let us take the common projective equation of the circle $x^2+y^2=a^2$. Substitute for x and y their reciprocals, and the equation becomes

$$\xi^2+v^2=a^2\xi^2v^2, \quad \ldots\ldots\ldots \quad \text{(a)}$$

the tangential equation of the quadrantal hypocycloid.

It is obvious that as $\frac{1}{\xi^2}+\frac{1}{v^2}=a^2$, the portion of the limiting tangent which revolves between the axes is of constant length and equal to a.

Let us assume the central projective equation of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; for x and y put their equivalents $\frac{1}{\xi}$ and $\frac{1}{v}$. The resulting equation becomes

$$a^2\xi^2+b^2v^2=a^2b^2\xi^2v^2, \quad \ldots\ldots \quad \text{(b)}$$

the tangential equation of the evolute of an ellipse.

Hence the line joining the feet of the projective coordinates x, y of an ellipse referred to its axes envelopes the evolute of another ellipse.

Now the semiaxes of this evolute along the axes of X and Y are b and a. We may satisfy ourselves that this is so, by putting $\frac{1}{v}$ and $\frac{1}{\xi}$ successively equal to 0.

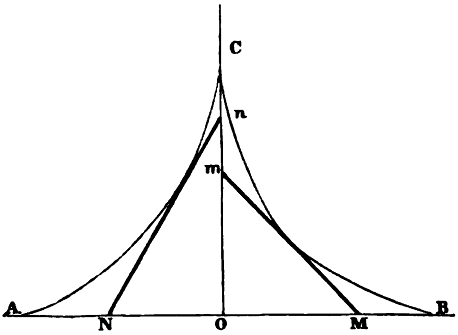
Now if A and B are the semiaxes of the ellipse of whose evolute b and a are the semiaxes, we shall have $a=\frac{A^2e^2}{B}$ and $b=Ae^2$, hence

$$\frac{a}{b}=\frac{A}{B} \text{ and } A=\frac{a^2b}{a^2-b^2}, \; B=\frac{b^2a}{a^2-b^2}. \quad \ldots\ldots \quad \text{(c)}$$

It is evident that the portion of the limiting tangent which revolves between the axes of coordinates is equal to the semidiameter of the ellipse which makes an equal angle with the axis of X; and as the sum of the reciprocals of the squares of any two semidiameters of the ellipse at right angles to each other is constant, so must the reciprocals of the squares of any two tangents drawn to the evolute at right angles to each other be constant.

196.] In the base of the evolute AB, let two points M and N be taken so that $\overline{\mathrm{AN}} \times \overline{\mathrm{NB}} = \overline{\mathrm{OM}}^2$; and if from M and N two tangents be drawn to the curve, of which the intercepted parts are l and l', we shall have $l^2 + l_{\prime}^2 = a^2 + b^2$.

Fig. 38.



It is evident that $l^2 + l_{\prime}^2 = \overline{\mathrm{OM}}^2 + \overline{\mathrm{ON}}^2 + \overline{\mathrm{O}m}^2 + \overline{\mathrm{O}n}^2$.

But $\overline{\mathrm{OM}}^2 + \overline{\mathrm{ON}}^2 = a^2$.

We also have $\frac{\overline{\mathrm{ON}}^2}{a^2} + \frac{\overline{\mathrm{O}n}^2}{b^2} = 1$, or $\overline{\mathrm{O}n}^2 = b^2 - \frac{b^2}{a^2}\overline{\mathrm{ON}}^2$; and in like manner $\overline{\mathrm{O}m}^2 = b^2 - \frac{b^2}{a^2}\overline{\mathrm{OM}}^2$; hence

$$l^2 + l_{\prime}^2 = a^2 + 2b^2 - \frac{b^2}{a^2}\left(\overline{\mathrm{OM}}^2 + \overline{\mathrm{ON}}^2\right) = a^2 + 2b^2 - \frac{b^2}{a^2}(a)^2,$$

or

$$l^2 + l_{\prime}^2 = a^2 + b^2.$$

197.] Let two tangents at right angles, one to the other, be drawn to two concentric and coaxial evolutes whose axes are connected by the relations

$$\frac{1}{b^2} - \frac{1}{a^2} = \frac{1}{k^2}, \quad \frac{1}{b_{\prime}^2} - \frac{1}{a_{\prime}^2} = \frac{1}{k^2}.$$

Let the segments of the tangents be l and $l_{\prime}$, we shall have

$$\frac{1}{l^2} + \frac{1}{l_{\prime}^2} = \frac{1}{a^2} + \frac{1}{b_{\prime}^2}.$$

Let $\frac{1}{a^2\xi^2} + \frac{1}{b^2\upsilon} = 1$, or putting $\frac{1}{a} = \mathrm{A}$, $\frac{1}{b} = \mathrm{B}$, the result becomes

$$\frac{\mathrm{A}^2}{\xi^2} + \frac{\mathrm{B}^2}{\upsilon^2} = 1, \text{ or } \frac{\mathrm{A}^2}{l^2\xi^2} + \frac{\mathrm{B}^2}{l^2\upsilon^2} = \frac{1}{l^2}; \quad . \quad . \quad . \quad . \quad \text{(a)}$$

but it may be shown that if ω be the angle which l makes with the axis of X, $l\xi=\frac{1}{\cos\omega}$, and $lv=\frac{1}{\sin\omega}$; consequently

$$A^2\cos^2\omega+B_{,}^2\sin^2\omega=\frac{1}{l^2}, \text{ or, as } B^2-A^2=\frac{1}{k^2},$$

substituting, $A^2+\frac{\sin^2\omega}{k^2}=\frac{1}{l^2}$. For $l_{,}$, we shall have $A_{,}^2+\frac{\cos^2\omega}{k^2}=\frac{1}{l_{,}^2}$, consequently

$$A^2+A_{,}^2+\frac{1}{k^2}=\frac{1}{l^2}+\frac{1}{l_{,}^2}; \text{ or, as } A_{,}^2+\frac{1}{k^2}=B_{,}^2,$$

we shall have

$$A^2+B_{,}^2=\frac{1}{l^2}+\frac{1}{l_{,}^2}, \text{ or } \frac{1}{a^2}+\frac{1}{b_{,}^2}=\frac{1}{l^2}+\frac{1}{l_{,}^2}. \quad . \quad . \quad . \quad \text{(b)}$$

On the tangential pedal of the Semicubical Parabola.

198.] The equation of the semicubical parabola is $ay^2=x^3$, its tangential equation is $v^2=a\xi^3$, and the tangential equation of its pedal is also $v^2=a\xi^3$. Hence the pedal curve of the semicubical parabola is the semicubical parabola itself. Consequently if we join the feet of the ordinates of one branch of the curve by a straight line, this straight line will be the envelope or limiting tangent of the other branch. Thus while one of the branches may be traced out by a point, the other may be enveloped by a straight line.

199.] Let us assume the equation of the fourth degree,

$$a^2y^2+b^2x^2=x^2y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Substituting for x and y their values $\frac{1}{\xi}$ and $\frac{1}{v}$, we get

$$a^2\xi^2+b^2v^2=1,$$

the tangential equation of an ellipse referred to its centre and axes.

Hence the line which joins the feet of the ordinates x and y in the equation (a) envelopes the ellipse whose semiaxes are a and b; and if we erect perpendiculars to the axes of coordinates from the points in which they are met by the limiting tangent, these lines will meet on the curve (a), see sec. [147].

CHAPTER XXII.

ON THE RADIUS OF CURVATURE AND THE RECTIFICATION OF PLANE CURVES.

200.] The method of tangential coordinates is peculiarly applicable to the rectification of plane curves.

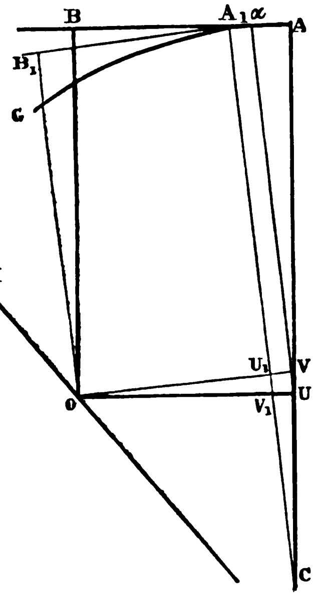
In the common formulæ for rectification derived from the methods of projective or polar coordinates, the element ds of the arc is assumed to be the hypotenuse of an infinitesimal right-angled triangle, of which the sides are the infinitesimals dx and dy; while in polar coordinates, the element ds is the hypotenuse of the right-angled triangle of which the sides are dr and $rd\theta$. We must then take the square roots of these expressions to adapt them for integration. Thus

$$ds = \sqrt{dx^2+dy^2}, \text{ and } ds = \sqrt{dr^2+r^2d\theta^2}.$$

Now a little consideration will show that these expressions are arbitrary. They have nothing to do directly with the curvature of the curve itself. They depend on the previous establishment of a particular system of coordinates. Had the system not been invented, such a method of rectification would have been any thing but obvious.

In the following method, the primary element on which the length and curvature of a given curve depend, is the radius of curvature at the extremity of the element; and it is easy to show that the arc of any given curve may be exhibited as made up of two distinct elements, the one depending on the curvature of a circle whose radius varies from point to point, while the other element, which is arbitrary, is the differential of a straight line.

Fig. 39.



As the centres of curvature move along the evolute of the given curve, it would seem that the coordinates of the centre of curvature which is found on the evolute, must be necessary elements in the rectification of a curve; but we may elude the difficulty in the following way.

Let $C\,A$, $C\,A_{\prime}$ be the radii of curvature at the extremities of the element of the arc of the curve $A\,G$. From an arbitrarily assumed, but fixed point O, let perpendiculars $O\,B$, $O\,B_{\prime}$ be let fall on the tangents to the curve at the points A and $A_{\prime}$.

$A\,B$ and $A_{\prime}\,B_{\prime}$ may be called *protangents* of the curve, seeing that they are the *projections* of the radius vector OA on the *tangents* of the curve.

Let $A\,B = t$, and $A_{\prime}B_{\prime} = t_{\prime}$. From the point O let fall the perpendiculars $O\,U$, $O\,U_{\prime}$ on the radii of curvature $A\,C$, $A_{\prime}C$. These are equal to $A\,B$

and $A_{\prime}B_{\prime\prime}$, and therefore equal to t and $t_{\prime}$ respectively. Through U draw the line U a parallel to O $B_{\prime}$ or to $V_{\prime}A_{\prime}$.

Now, though we cannot determine the increment of t, or the value of $t_{\prime}-t$ from a consideration of the lines A B and $A_{\prime}B_{\prime\prime}$, seeing that they have no point in common, yet by comparing their equals O U and O $U_{\prime\prime}$, the opposite sides of the rectangles A B O U and $A_{\prime}B_{\prime}O\,U_{\prime\prime}$, we may see that $t-t_{\prime}=U\,V_{\prime}$ ultimately. Let λ be the angle which the perpendicular O B or P makes with the axis O X; then, putting $AC=R$ the radius of curvature, let $AU=OB=P$. Also $d\lambda$ being the element of the angle between A C and $A_{\prime}C$ the radii of curvature, and also between the perpendiculars P and $P_{\prime\prime}$, which are parallel to them, we shall evidently have, when the radii of curvature are indefinitely near,

$$(R-P)d\lambda=UV_{\prime}=t-t_{\prime}=dt. \quad . \quad . \quad . \quad . \quad \text{(a)}$$

As $V_{\prime}U_{\prime}$ is ultimately $=P-P_{\prime}=dP$, and $V_{\prime}U_{\prime}=t d\lambda$, we shall have ultimately

$$\frac{dP}{d\lambda}=t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

or

$$\frac{d^2P}{d\lambda^2}=\frac{dt}{d\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Introducing this value of $\frac{dt}{d\lambda}$ in the preceding equation, we shall have

$$R=P+\frac{d^2P}{d\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

an elegant and simple expression for the radius of curvature.

[I find this formula entered in an old note-book of mine as having been discovered on the 17th August, 1837.]

201.] Since the elementary arc of the curve ds is manifestly equal to $Aa+aB$ and $Aa=Pd\lambda$, while $aA_{\prime}$ is equal to $UV_{\prime\prime}$, the opposite side of the rectangle $aA_{\prime}V_{\prime}U$, and $UV_{\prime}=dt$, ultimately, we shall consequently have

$$ds=Pd\lambda+dt. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Since $dt=(R-P)d\lambda$, the sign of dt will depend on the sign of $(R-P_{\prime})$.

We have shown in (b), sec. [200], that $\frac{dP}{d\lambda}=t$; and if we integrate the expression (a), we shall have $s=\int Pd\lambda+t$; or putting for t its value $\frac{dP}{d\lambda}$, we shall finally obtain the following simple formula for the rectification of a plane curve,

$$s=\int P d\lambda+\frac{dP}{d\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

From the preceding demonstration it follows that dt will be positive so long as $R>P$, that it will be $=0$ when $R=P$, and that it will be negative when $R<P$.

Hence it follows that when at any point of a curve the radius of curvature shall be equal to the perpendicular from the origin on the tangent, we shall have

$$\frac{d^2P}{d\lambda^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (c)$$

Consequently in the general formula for rectification,

$$s=\int P d\lambda\pm\frac{dP}{d\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (d)$$

the upper sign must be taken when $R>P$, the lower when $R<P$.

When the point O, and C the centre of curvature, are on opposite sides of the tangent to the curve, the formula becomes

$$s=\frac{dP}{d\lambda}-\int P d\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (e)$$

Hence, having obtained the value of P, we may obtain the value of t, the *protangent*, by simple differentiation; for

$$t=\frac{dP}{d\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (f)$$

202.] These observations may throw some light on a well-known theorem in the rectification of the elliptic quadrant. In Fagnani's theorem there is a critical point at which the quadrant of the ellipse is divided into two sections, whose difference is equal to the difference of the semiaxes a and b; at this point $\frac{dP}{d\lambda}=a-b$, and therefore $\frac{d^2P}{d\lambda^2}=0$, or $R=P$. That is to say, the radius of curvature is equal to the perpendicular from the centre on the tangent at the critical point, in Fagnani's theorem. It is easy to show this.

In the ellipse the radius of curvature at any point is equal to the product of the squares of the semiaxes divided by the cube of the perpendicular from the centre on the tangent at that point, or $R=\frac{a^2b^2}{P^3}$. But $R=P$ in this case. Hence $P^2=ab$.

In the ellipse as $\frac{1}{P^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{ab}$, and the equation of the curve gives $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, eliminating y and x successively, we shall have

$$x=\frac{a^{\frac{3}{2}}}{\sqrt{a+b}},\ y=\frac{b^{\frac{3}{2}}}{\sqrt{a+b}}.$$

These are the coordinates of the critical point in Fagnani's theorem. For the angles which this perpendicular makes with the axes, we get

$$\cos^2\lambda=\frac{b}{a+b},\ \sin^2\lambda=\frac{a}{a+b}.$$

The tangent of the angle between the perpendicular on the tangent and the radius vector of the critical point is

$$=\frac{a-b}{\sqrt{ab}},\text{ since } t=a-b,\text{ and } \mathrm{P}=\sqrt{ab}.$$

203.] Let the tangential equation of a plane curve be $\phi(\xi, v)=0$, and, assuming the value found for t in sec. [27], we may transform the expression for the radius of curvature, namely $\mathrm{R}=\mathrm{P}\pm\frac{\mathrm{d}t}{\mathrm{d}\lambda}$, into

$$\mathrm{R}=\frac{1}{\sqrt{\xi^2+v^2}}\pm\frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}v-\frac{\mathrm{d}\Phi}{\mathrm{d}v}\xi}{\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\xi+\frac{\mathrm{d}\Phi}{\mathrm{d}v}v}\right]\frac{1}{\sqrt{\xi^2+v}}. \quad . \quad . \quad \text{(a)}$$

While the quadrature of curves may be most easily effected by the use of the integral $\int y\mathrm{d}x$ derived from the projective equation of the curves, so their rectification by the aid of formulæ derived from their tangential equations may with the greatest facility be obtained.

Since $s=\int \mathrm{P}\mathrm{d}\lambda+\frac{\mathrm{dP}}{\mathrm{d}\lambda}$, we shall have

$$s=\int \mathrm{P}\mathrm{d}\lambda+\left[\frac{\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}v-\frac{\mathrm{d}\Phi}{\mathrm{d}v}\xi}{\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\xi+\frac{\mathrm{d}\Phi}{\mathrm{d}v}v}\right]\frac{1}{\sqrt{\xi^2+v^2}}. \quad . \quad . \quad . \quad \text{(b)}$$

Thus the rectification of a curve, whose tangential equation is given, may in general be very simply effected by the use of these simple and general formulæ.

204.] *On the rectification of the circle by the method of tangential coordinates, the origin being taken any where in the plane of the circle.*

Let the general equation of the circle be

$$a\xi^2+a_{,}v^2+2\beta\xi v+2\gamma\xi+2\gamma_{,}v=1\ ; \quad . \quad . \quad . \quad . \quad \text{(a)}$$

this equation becomes, writing $\cos\lambda$ for $P\xi$, and $\sin\lambda$ for Pv,

$$P^2-2(\gamma\cos\lambda+\gamma_{\prime}\sin\lambda)P=a\cos^2\lambda+2\beta\cos\lambda\sin\lambda+a_{\prime}\sin^2\lambda.$$

Solving this equation for P, we obtain the result,

$$P=\gamma\cos\lambda+\gamma_{\prime}\sin\lambda$$
$$\pm\sqrt{(a+\gamma^2)\cos^2\lambda+(a_{\prime}+\gamma_{\prime}^2)\sin^2\lambda+2(\beta+\gamma\gamma_{\prime})\cos\lambda\sin\lambda}.$$

Now in sec. [14] it has been shown that, when the section is a circle, $a+\gamma^2=a_{\prime}+\gamma_{\prime}^2$, and $\beta+\gamma\gamma_{\prime}=0$.

Hence $$P=\gamma\cos\lambda+\gamma_{\prime}\sin\lambda\pm\sqrt{(a+\gamma^2)},$$

and $$\int P d\lambda=\gamma\sin\lambda-\gamma_{\prime}\cos\lambda\pm\sqrt{a+\gamma^2}.\lambda,$$

and $$\frac{dP}{d\lambda}=+\gamma_{\prime}\cos\lambda-\gamma\sin\lambda.$$

Hence $$\int P d\lambda+\frac{dP}{d\lambda}=\sqrt{a+\gamma^2}.\lambda.$$

In sec. [12], (19), it is shown that the radius of the circle is $=\sqrt{a+\gamma^2}$.

205.] *On the radius of curvature of the ellipse.*

We have shown in sec. [200] that the radius of curvature of any curve may be expressed in terms of the perpendicular on the tangent from the origin, and the angle which this perpendicular makes with one of the axes of coordinates; or

$$R=P+\frac{d^2P}{d\lambda^2}.$$

In the ellipse $P^2=a^2\cos^2\lambda+b^2\sin^2\lambda$, consequently

$$P\frac{d^2P}{d\lambda^2}=\frac{a^2b^2\sin^4\lambda-a^4\cos^4\lambda-b^4\sin^4\lambda+a^2b^2\cos^4\lambda}{P^2}.$$

Hence $$P^2+P\frac{d^2P}{d\lambda^2}=\frac{a^2b^2}{P^2} \text{ or } P+\frac{d^2P}{d\lambda^2}=\frac{a^2b^2}{P^3}.$$

206.] *On the radius of curvature of the parabola.*

Since $P=\frac{k}{\cos\lambda}$, the origin being at the focus,

$$\frac{d^2P}{d\lambda^2}=k\frac{(1+\sin^2\lambda)}{\cos^3\lambda}.$$

Hence

$$P+\frac{d^2P}{d\lambda^2}=\frac{2k}{\cos^3\lambda}; \text{ or as } N\cos\lambda=2k,$$

eliminating $\cos\lambda$, we obtain $R=\frac{N^3}{(2k)^2}$.

On the rectification of the ellipse.

207.] The tangential equation of the ellipse referred to its centre and axes is $a^2\xi^2+b^2v^2=1$. Multiply by P^2, for $P\xi$ put $\cos\lambda$, and for Pv put $\sin\lambda$, and then the equation will become $P=a\sqrt{1-e^2\sin^2\lambda}$; and as $t=\frac{dP}{d\lambda}$, $t=-\frac{ae^2\sin\lambda\cos\lambda}{\sqrt{1-e^2\sin^2\lambda}}$.

Hence

$$s=a\int d\lambda\,\sqrt{1-e^2\sin^2\lambda}-\frac{ae^2\sin\lambda\cos\lambda}{\sqrt{1-e^2\sin^2\lambda}}, \quad . \quad . \quad . \quad \text{(a)}$$

an elliptic function of the second order.

t is taken with a negative sign as the perpendicular continues to diminish.

At the critical point, substituting for $\sin^2\lambda$ and $\cos^2\lambda$ their values $\frac{a}{a+b}$ and $\frac{b}{a+b}$, the expression for the curve as far as this point from the extremity of the major axis becomes

$$s=a\int d\lambda\,\sqrt{1-e^2\sin^2\lambda}-(a-b),$$

the limits of integration being $\lambda=0$, and $\lambda=\sin^{-1}\sqrt{\frac{a}{a+b}}$.

On the rectification of the parabola.

208.] The tangential equation of the parabola is

$$k(\xi^2+v^2)=\xi,$$

where k is one fourth of the parameter.

Multiply by P^2, and the equation becomes $P=\frac{k}{\cos\lambda}$ and $t=\frac{dP}{d\lambda}=k\tan\theta\sec\theta$; the positive sign of t to be taken, since the radius of curvature is always greater than the focal perpendicular.

Hence $$s=k\int\frac{d\lambda}{\cos\lambda}+k\tan\lambda\sec\lambda. \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

This expression for the arc of a parabola is the foundation of parabolic trigonometry.

On the radius of curvature and the rectification of the cycloid.

209.] It has been shown in sec. [139], (d), that if the origin of coordinates be taken at the centre of the cycloid, its tangential equation will be

$$1=2r\left[v+\xi\tan^{-1}\left(\frac{\xi}{v}\right)\right]. \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

If we multiply this expression by P, it becomes

$$P=2r[\cos\lambda+\lambda.\sin\lambda]; \quad \ldots\ldots \quad (b)$$

taking its first differential,

$$\frac{dP}{d\lambda}=2r\lambda\cos\lambda; \quad \ldots\ldots\ldots\ldots \quad (c)$$

taking the second differential of P,

$$\frac{d^2P}{d\lambda^2}=2r\cos\lambda-2r\lambda\sin\lambda. \quad \ldots\ldots \quad (d)$$

Consequently

$$R=P+\frac{d^2P}{d\lambda^2}=4r\cos\lambda. \quad \ldots\ldots\ldots \quad (e)$$

We have also

$$\int P d\lambda=4r\sin\lambda-2r\lambda.\cos\lambda; \quad \ldots\ldots \quad (f)$$

and adding the value of $\frac{dP}{d\lambda}=2r\lambda\cos\lambda$, as given in (c), we shall have finally

$$s=\int P d\lambda+\frac{dP}{d\lambda}=4r\sin\lambda. \quad \ldots\ldots \quad (g)$$

If we square the expressions in (e) and (g), we shall have

$$R^2+s^2=4r^2; \quad \ldots\ldots\ldots\ldots \quad (h)$$

or *the square of the radius of curvature at any point of a cycloid, together with the square of the arc measured from that point to the vertex, are equal to the square of the diameter of the generating circle.*

This property holds also in the case of epicycloids and hypocycloids, with some slight modification. Simple as the theorem is, we do not recollect to have met with it.

The perpendicular from the centre on a tangent to the cycloid is equal to the radius of curvature at the same point, when $\lambda\tan\lambda=1$.

For $P=2r\{\cos\lambda+\lambda\sin\lambda\}=4r\cos\lambda=R$;

consequently $\qquad \lambda\tan\lambda=1$, or $\lambda=\cot\lambda$. . . . . . (i)

On the rectification of the evolute of the ellipse.

210.] Let the tangential equation of the evolute of the ellipse be

$$a^2v^2+b^2\xi^2=(a^2-b^2)^2\xi^2v^2. \quad \ldots\ldots\ldots \quad (a)$$

Then the perpendicular from the origin on the centre will be

$$P=\frac{(a^2-b^2)\sin\lambda\cos\lambda}{\sqrt{a^2\sin^2\lambda+b^2\cos^2\lambda}}, \quad \ldots\ldots\ldots \quad (b)$$

and

$$\int P d\lambda=\sqrt{a^2\sin^2\lambda+b^2\cos^2\lambda}. \quad \ldots\ldots \quad (c)$$

Now

$$\frac{dP}{d\lambda}=\frac{(a^2-b^2)(b^2\cos^4\lambda-a^2\sin^4\lambda)}{(a^2\sin^2\lambda+b^2\cos^2\lambda)^{\frac{3}{2}}}; \quad . \quad . \quad . \quad . \qquad (d)$$

and as

$$s=\int P d\lambda+\frac{dP}{d\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (e)$$

we shall have, making these substitutions,

$$s=\frac{a^2b^2}{\{a^2\sin^2\lambda+b^2\cos^2\lambda\}^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (f)$$

When $\lambda=0$, $s_{\prime}=\frac{a^2}{b}$; and when $\lambda=90^{\circ}$, $s_{\prime\prime}=\frac{b^2}{a}$; hence $s_{\prime}-s_{\prime\prime}=\frac{a^2}{b}-\frac{b^2}{a}$ or $=\frac{a^3-b^3}{ab}$, a result already obtained in a different way.

On the rectification of the semicubical parabola, and its radius of curvature.

211.] The tangential equation of the semicubical parabola is

$v^2=k\xi^3$, see sec. [148].

Multiplying this equation by P, the perpendicular from the origin, and referring the angle λ to the normal passing through the cusp of the curve as the axis of X, we shall have

$$P=k\frac{\sin^3\lambda}{\cos^2\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

Hence

$$\int P d\lambda=k\cos\lambda+\frac{k}{\cos\lambda}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

and

$$\frac{dP}{d\lambda}=\frac{2k}{\cos^3\lambda}-k\cos\lambda-\frac{k}{\cos\lambda}; \quad . \quad . \quad . \quad . \qquad (c)$$

consequently

$$s=\int P d\lambda+\frac{dP}{d\lambda}=\frac{2k}{\cos^3\lambda}+C. \quad . \quad . \quad . \quad . \quad . \qquad (d)$$

But at the cusp $s=0$ and $\lambda=0$, hence

$$s=2k(\sec^3\lambda-1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (e)$$

This is a very simple and elegant expression for the arc of the semicubical parabola as compared with that usually given in projective coordinates—that is to say,

$$s=\frac{(9x+4a)^{\frac{3}{2}}-(4a)^{\frac{3}{2}}}{27a^{\frac{1}{2}}}.$$

See Gregory's examples, p. 416.

If we differentiate the expression (c), we shall have

$$\frac{d^2P}{d\lambda^2}=\frac{6k\sin\lambda}{\cos^4\lambda}-\frac{k\sin^3\lambda}{\cos^2\lambda}; \quad \ldots\ldots \quad (f)$$

but $P=k\frac{\sin^3\lambda}{\cos^2\lambda}$.

Hence
$$R=P+\frac{d^2P}{d\lambda^2}=\frac{6k\sin\lambda}{\cos^4\lambda}, \quad \ldots\ldots \quad (g)$$

a simple expression for the radius of curvature of the semicubical parabola.

It may easily be shown that the following relation exists between the arc, radius of curvature, and the perpendicular, on the tangent, from the cusp of the semicubical parabola

$$\frac{6P}{R}=\frac{2k}{s+2k}-\left(\frac{2k}{s+2k}\right)^{\frac{4}{3}}. \quad \ldots\ldots \quad (h)$$

On the rectification of parallel curves.

212]. Let $s_{\prime}$ be the arc of the parallel curve, and s that of the primitive.

Applying formula (b), sec. [201], we get

$$s_{\prime}=\int P_{\prime}d\lambda+\frac{dP_{\prime}}{d\lambda}; \text{ but } P_{\prime}=P+h, \text{ and } \frac{dP_{\prime}}{d\lambda}=\frac{dP}{d\lambda};$$

hence

$$s_{\prime}=\int Pd\lambda+h\lambda+\frac{dP}{d\lambda}. \quad \ldots\ldots \quad (a)$$

But the arc of the primitive curve is

$$s=\int Pd\lambda+\frac{dP}{d\lambda}; \quad \ldots\ldots \quad (b)$$

consequently

$$s_{\prime}-s=h\lambda; \quad \ldots\ldots \quad (c)$$

or the difference of the corresponding arcs of any two parallel curves is equal to a circular arc whose radius is h, the constant difference of the normals of the two curves.

*On the radius of curvature, and the rectification of epicycloids and hypocycloids**.

213.] Assuming the general expressions given in sec. [129] for the tangential coordinates of these curves, namely

$$\xi=\frac{\sin(n+1)\phi}{2r(n+1)\sin n\phi}, \quad v=\frac{\cos(n+1)\phi}{2r(n+1)\sin n\phi},$$

---

* In the 49th proposition of the first book of the 'Principia,' Newton shows, by a purely geometrical method, that all epicycloids and hypocycloids are rectifiable. He does not, however, discuss the relations which exist between the arcs and radii of curvature of these curves.

squaring these expressions, adding them, multiplying by P^2, and taking the square root, we shall have

$$P=2r(n+1)\sin n\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Let λ be the angle which P makes with the axis of X; then an inspection of figure 40 will show that $(n+1)\phi=\frac{\pi}{2}+\lambda$, $\frac{d\phi}{d\lambda}=\frac{1}{n+1}$. For

$$BXC=OZX+ZOX \text{ or } (n+1)\phi=\frac{\pi}{2}+\lambda. \quad . \quad . \quad (b)$$

Hence

$$\int Pd\lambda=-2r\frac{(n+1)^2}{n}\cos n\phi+C,$$

and

$$\frac{dP}{d\lambda}=\frac{dP}{d\phi}\frac{d\phi}{d\lambda}=2rn\cos n\phi;$$

consequently

$$s=\int Pd\lambda+\frac{dP}{d\lambda}=-2r\frac{(2n+1)}{n}\cos n\phi+C.$$

At the cusp, the arc is $=0$, consequently $0=-2r\frac{(2n+1)}{n}+C$.

Hence, by subtraction, $s=2r\frac{(2n+1)}{n}(1-\cos n\phi)$; or, as this latter factor is $=2\sin^2\left(\frac{n\phi}{2}\right)$, we shall have finally

$$s=4r\frac{(2n+1)}{n}\sin^2\left(\frac{n\phi}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

This is a general formula for all epicycloids, and also for hypocycloids when we write $2n-1$ instead of $2n+1$.

Since R, the radius of curvature, is $=P+\frac{d^2P}{d\lambda^2}$, and since $\frac{dP}{d\lambda}=2rn\cos n\phi$, we shall have

$$\frac{d}{d\lambda}\cdot\left(\frac{dP}{d\lambda}\right)=\frac{d}{d\phi}\cdot\left(\frac{dP}{d\lambda}\right)\frac{d\phi}{d\lambda}=n\phi,$$

or

$$\frac{d^2P}{d\lambda^2}=-2rn\sin n\phi\,.\,\frac{d\phi}{d\lambda}=-\frac{2rn^2}{n+1}\sin n\phi;$$

and as $P=2r(n+1)\sin n\phi$,

$$R=P+\frac{d^2P}{d\lambda^2}=\frac{2r(2n+1)}{n+1}\sin n\phi. \quad . \quad . \quad . \quad . \quad . \quad (d)$$

This is a general form for the radius of curvature of all epicycloids.

Since, when $n\phi=0$, $R=0$, we may infer that in this class of curves the radius of curvature is $=0$ at the cusp.

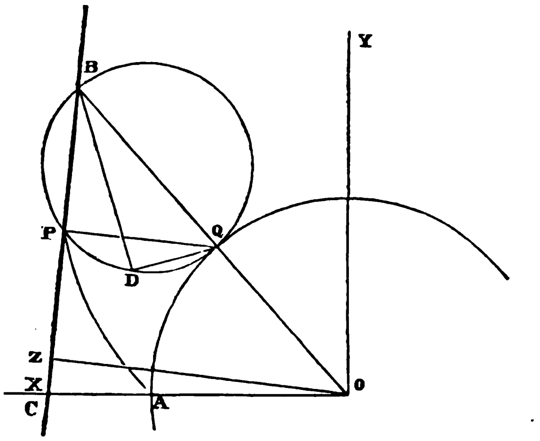
The radius of curvature will be a maximum when $\sin n\phi$ is a maximum, or when $\phi=\frac{\pi}{2n}$.

Let us apply these two general formulæ—

$$\left.\begin{aligned} s&=4r\left(\frac{2n\pm1}{n}\right)\sin^2\left(\frac{n\phi}{2}\right),\\ R&=2r\left(\frac{2n\pm1}{n\pm1}\right)\sin . n\phi. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

We may first give a simple geometrical illustration of these formulæ. Let B P Q be the rolling circle, and A Q the fixed circle;

Fig. 40.



bisect the angle P B Q by the line B D, and join Q D, then evidently

$$PQ=2r\sin\phi, \text{ and } \overline{DQ}^2=4r^2\sin^2\left(\frac{\phi}{2}\right);$$

consequently

$$R=\left(\frac{2n\pm1}{n\pm1}\right)\overline{PQ}, \text{ and } s=\frac{(2n\pm1)}{nr}\overline{DQ}^2. \quad . \quad . \quad . \quad (f)$$

Hence, while the radius of curvature of an epicycloid or hypocycloid, at any point P of the curve, is proportional to the line $\overline{PQ}$ (the instantaneous radius, the chord of the arc QP which has rolled along the fixed arc AQ), so the arc of the curve up to that point from the cusp is proportional to the square of the chord of half the arc QDP.

214.] We may now apply these formulæ to determine the lengths of the arcs and the radii of curvature of several of these curves.

(α) In the cardioid whose equation is given in sec. [130], we have $R=2nr=r$ or $2n=1$; substituting this value of n in the general formulæ, we shall have

$$s=16r\sin^2\left(\frac{\phi}{4}\right), \text{ (a)}, \quad \text{and} \quad R=\frac{8r}{3}\sin\left(\frac{\phi}{4}\right), \quad . \quad . \quad \text{(b)}$$

when $\phi=\pi$,

$$s=8r, \quad R=\frac{8r}{3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

(β) In the semicircular epicycloid whose equation is given in sec. [130], *a*, we have $R=2nr=2r$, or $n=1$; hence

$$s=12r\sin^2\frac{\phi}{2}, \quad R=3r\sin\phi. \quad . \quad . \quad . \quad . \quad \text{(d)}$$

(γ) In the quadrantal epicycloid as $R=2nr=4r$, $n=2$, and

$$s=10r\sin^2\phi, \quad R=\frac{10}{3}r\sin 2\phi. \quad . \quad . \quad . \quad . \quad \text{(e)}$$

(δ) In the trigonal epicycloid, as the base circle is three times the rolling circle, $2n=3$,

$$s=\frac{16}{3}r\sin^2\left(\frac{3\phi}{4}\right), \quad R=\frac{16}{5}r\sin\left(\frac{3\phi}{2}\right). \quad . \quad \text{(f)}$$

215.] *On the curvature and rectification of hypocycloids.*

The formulæ in this case are, $s=4r\left(\frac{2n-1}{n}\right)\sin^2\left(\frac{n\phi}{2}\right)$ and $R=2r\left(\frac{2n-1}{n-1}\right)\sin n\phi$.

(α) Let the rolling circle be one half the base circle, then $2n=2$, or $n=1$. Hence

$$R=\frac{2r}{0}, \quad s=4r\sin^2\left(\frac{\phi}{2}\right);$$

as the hypocycloid is in this case the diameter of the base curve, we must evidently have $R=\infty$, and $s=4r$.

(β) Let the curve be the quadrantal hypocycloid whose equation is given in sec. [134].

As, in this case, the base circle is four times the rolling circle, we must have $n=2$, and the resulting expressions become

$$s=6r\sin^2\phi, \quad R=6r\sin 2\phi.$$

(γ) In the trigonal hypocycloid, as $2n=3$, $n=\frac{3}{2}$, we shall have

$$s=\frac{16r}{3}\sin^2\left(\frac{3\phi}{4}\right),\quad R=8r\sin\frac{3\phi}{2};$$

$\sin\frac{3\phi}{4}$ is a maximum when $\phi=120°$.

Hence the length of one of the arcs of the trigonal hypocycloid is $=\frac{16r}{3}$, and the sum of lengths of the three arcs is $=16r$.

It is needless to pursue these illustrations further; what is very remarkable is this, that though we may not be able to give the equation of an epicycloid or hypocycloid either in tangential or projective coordinates, we may notwithstanding find finite and exact expressions for their arcs and radii of curvature.

Thus, if the base circle be 100 times the rolling circle, we shall have for s, $s=(8\frac{2}{25})r\sin^2(50\phi)$,

$$R=2r\,.\,\frac{199}{99}\sin(100\phi).$$

216.] If we eliminate the angle ϕ between the two equations, sec. [213], (e), we shall have the following relation between any arc of an epicycloid and the radius of curvature at its extremity,

$$(n+1)^2R^2+n^2s^2=4rn(2n+1)s. \quad . \quad . \quad . \quad . \quad \text{(a)}$$

There are several important consequences which may be drawn from this equation.

(a) When $s=0$, $R=0$, or at a cusp the radius of curvature $=0$. When $R=0$, $s=0$, or $ns=4r(2n+1)$.

We may write the preceding equation in the form

$$(n+1)^2R^2+n^2(4r-s)^2=16n^2r^2+4nrs. \quad . \quad . \quad \text{(b)}$$

If we now assume $n=\infty$, or the base circle a straight line, dividing by $n^2=(n+1)^2$, we shall have

$$R^2+(4r-s)^2=16r^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

a property of the cycloid established in sec. [209], (h).

On the radius of curvature of the cubical parabola.

217.] If we measure the angle λ from the normal to the curve passing through the point of inflexion at the origin, the equation of the curve $\xi=a^2v^3$, when multiplied by P^2, will become

$$P^2\cos\lambda=a^2\sin^3\lambda; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

differentiating this expression,

$$\frac{2P}{a^2}\left(\frac{dP}{d\lambda}\right)=\frac{\sin^2\lambda}{\cos^2\lambda}(3\cos^2\lambda+\sin^2\lambda), \quad . \quad . \quad . \quad . \quad \text{(b)}$$

or

$$\frac{2P}{a^2}\left(\frac{dP}{d\lambda}\right)=1-2\cos^2\lambda+\frac{1}{\cos^2\lambda}; \quad . \quad . \quad . \quad . \quad . \quad (c)$$

differentiating again,

$$\frac{4}{a^2}\left(\frac{dP}{d\lambda}\right)^2+\frac{4P}{a^2}\left(\frac{d^2P}{d\lambda^2}\right)=8\sin\lambda\cos\lambda+\frac{4\sin\lambda}{\cos^3\lambda}. \quad . \quad (d)$$

If now we square (b) and divide by P^2, we shall have

$$\frac{4}{a^2}\left(\frac{dP}{d\lambda}\right)^2=9\sin\lambda\cos\lambda+\frac{6\sin^3\lambda}{\cos\lambda}+\frac{\sin^5\lambda}{\cos^3\lambda}; \quad . \quad . \quad (e)$$

and if we subtract this expression from (d), the result will be

$$\frac{4P}{a^2}\left(\frac{d^2P}{d\lambda^2}\right)=\frac{\sin\lambda}{\cos^3\lambda}\{3-4\sin^2\lambda\cos^2\lambda\}. \quad . \quad . \quad . \quad (f)$$

The equation of the curve gives

$$\frac{4P^2}{a^2}=\frac{4\sin^3\lambda\cos^2\lambda}{\cos^3\lambda};$$

adding these expressions,

$$\frac{4P}{a^2}\left\{P+\frac{d^2P}{d\lambda^2}\right\}=\frac{3\sin\lambda}{\cos^3\lambda}; \quad \text{and as} \quad R=P+\frac{d^2P}{d\lambda^2},$$

we shall have finally

$$R=\tfrac{3}{4}a\,(\sin\lambda\cos^5\lambda)^{-\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

Since the radius of curvature is ∞ when $\lambda=0$ and when $\lambda=90°$, there must be an intermediate value for R when it becomes a minimum. For this minimum value, $\tan\lambda=\frac{1}{\sqrt{5}}$.

On the rectification of the involute of the quadrantal hypocycloid.

218.] The projective equation of this curve being $x^{\frac{2}{3}}+y^{\frac{2}{3}}=r^{\frac{2}{3}}$, the tangential equation of its involute, see sec. [136], (b), is

$$[(l+a)\xi^2+(l-a)v^2]=\sqrt{\xi^2+v^2}.$$

Multiplying by P^2, the preceding expression becomes

$$l+a(\sin^2\lambda-\cos^2\lambda)=P,$$

and $\frac{dP}{d\lambda}=2a\sin 2\lambda$; we have also $\int P d\lambda=l\lambda-\frac{a}{2}\sin 2\lambda$; consequently

$$s=\int P d\lambda+\frac{dP}{d\lambda}=l\lambda+\frac{3a}{2}\sin 2\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

It may be remarked that $r=4a$.

219.] *On the rectification of the curve whose tangential equation is*

$$(\xi^2+v^2)^2=\frac{\xi^2}{a^2}+\frac{v^2}{b^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

This is the equation of the curve enveloped by one side of a right angle, which moves along an ellipse, while the other side passes through the centre.

Let the angle λ be measured from the minor axis of the ellipse, and, multiplying by P^4, we shall have

$$1=P^2\left\{\frac{\sin^2\lambda}{a^2}+\frac{\cos^2\lambda}{b^2}\right\} \text{ or } P=\frac{b}{\sqrt{1-e^2\sin^2\lambda}}, \quad . \quad . \quad \text{(b)}$$

and

$$\frac{dP}{d\lambda}=\frac{be^2\sin\lambda\cos\lambda}{\{1-e^2\sin^2\lambda\}^{\frac{3}{2}}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

consequently

$$s=\int P d\lambda+\frac{dP}{d\lambda}=b\int\frac{d\lambda}{\sqrt{1-e^2\sin^2\lambda}}+\frac{be^2\sin\lambda\cos\lambda}{[1-e^2\sin^2\lambda]^{\frac{3}{2}}}. \quad . \quad \text{(d)}$$

This curve gives the simplest geometrical illustration of the first elliptic integral. On the surface of the sphere its true geometrical exponent is the spherical parabola, as will be shown in the second volume of this work.

[*] On the rectification of the inverse curve of the central ellipse.

220.] Ce problème mérite d'être discuté à cause de l'élégance remarquable de sa solution, qui dépend de l'évaluation d'une intégrale elliptique de troisième espèce à paramètre *circulaire*.

On dit que deux courbes sont *inverses* l'une de l'autre lorsque le produit de leurs rayons vecteurs superposés est constant, c'est-à-dire que:

$$Rr=c^2.$$

Soit:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

l'équation de l'ellipse, le centre étant au pôle, et soit $Rr=kab$: on aura, pour la courbe inverse, l'équation:

$$k^2(a^2y^2+b^2x^2)=(x^2+y^2)^2.$$

On peut simplifier la discussion, sans restreindre la généralité, en prenant $k=1$. L'équation de la courbe inverse à l'ellipse, le centre étant au pôle, est alors:

$$a^2y^2+b^2x^2=(x^2+y^2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

---

[*] This demonstration is transcribed from an article in the 'Annali di Matematica pura ed applicata," serie 2, tomo ii. fasc. 1, p. 84.

Si l'on pose :

$$x=r\cos\phi,\ y=r\sin\phi,\quad .\ .\ .\ .\ .\ .\ .\ (2)$$

cette équation devient :

$$a^2\sin^2\phi+b^2\cos^2\phi=r^2,\quad .\ .\ .\ .\ .\ .\ (3)$$

d'où l'on tire, après quelques réductions simples,

$$\frac{ds^2}{d\phi^2}=\frac{a^4\sin^2\phi+b^4\cos^2\phi}{a^2\sin^2\phi+b^2\cos^2\phi}.\quad .\ .\ .\ .\ .\ .\ .\ (4)$$

La substitution :

$$a^2\tan\phi=b^2\tan\lambda,\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ (5)$$

change cette formule en la suivante :

$$\frac{ds^2}{d\phi^2}=\frac{a^2b^2}{a^2\cos^2\lambda+b^2\sin^2\lambda},\quad .\ .\ .\ .\ .\ .\ (6)$$

et puisque on en tire aussi :

$$\frac{d\phi}{d\lambda}=\frac{b^2}{a^2}\frac{\cos^2\phi}{\cos^2\lambda},\ \text{de même que}\ \frac{\cos^2\phi}{\cos^2\lambda}=\frac{a^4}{a^4\cos^2\lambda+b^4\sin^2\lambda},\qquad (7)$$

substituant et simplifiant on obtient :

$$\frac{ds}{d\lambda}=\frac{a^3b^3}{(a^4\cos^2\lambda+b^4\sin^2\lambda)\ \sqrt{a^2\cos^2\lambda+b^2\sin^2\lambda}},$$

ou :

$$\frac{ds}{d\lambda}=\frac{b^3}{a^2}\frac{1}{\left[1-\left(\frac{a^4-b^4}{a^4}\right)\sin^2\lambda\right]\sqrt{1-\left(\frac{a^2-b^2}{a^2}\right)\sin^2\lambda}}.\quad .\ (8)$$

Faisant :

$$\frac{a^2-b^2}{a^2}=c^2\ \text{ et }\ \frac{a^4-b^4}{a^4}=m,\quad .\ .\ .\ .\ .\ .\ (9)$$

on a :

$$m=\frac{a^2+b^2}{a^2}c^2,\quad m>c^2,$$

et par suite, intégrant :

$$s=\frac{b^3}{a^2}\int\frac{d\lambda}{[1-m\sin^2\lambda]\ \sqrt{1-c^2\sin^2\lambda}},\quad .\ .\ .\ (10)$$

intégrale elliptique de troisième espèce à paramètre circulaire, car $m>c^2$.

Imaginons le cylindre droit dont la base est l'ellipse aux demi-axes a et b, et la sphère décrite du centre avec un rayon$=\sqrt{a^2+b^2}$. Cette sphère coupe le cylindre suivant une ellipse sphérique.

Soient α et β les demi-angles principaux de cette ellipse sphérique, alors :

$$\sin^2\alpha=\frac{a^2}{a^2+b^2},\ \sin^2\beta=\frac{b^2}{a^2+b^2},\ \tan^2\alpha=\frac{a^2}{b^2},\ \tan^2\beta=\frac{b^2}{a^2}.\qquad (11)$$

D'ici on tire :

$$\frac{a^4-b^4}{a^4}=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha},\quad \frac{a^2-b^2}{a^2}=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha},$$

et :

$$\frac{b^3}{a^2}=\sqrt{a^2+b^2}\,\frac{\tan\beta}{\tan\alpha}\sin\beta.$$

Effectuant ces substitutions dans l'équation (8) il vient : $\dfrac{s}{\sqrt{a^2+b^2}}=$

$$\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\lambda}{\left[1-\left(\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}\right)\sin^2\lambda\right]\sqrt{1-\left(\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}\right)\sin^2\lambda}}. \quad (12)$$

Or dans les Philosophical Transactions pour 1852, Part II. p. 319, j'ai montré que l'expression d'un arc d'ellipse sphérique qui résulte de l'intersection d'un cône aux demi-angles α et β avec une sphère concentrique est donnée par la formule : $\sigma=$

$$\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\phi}{\left[1-\left(\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha}\right)\sin^2\phi\right]\sqrt{1-\left(\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}\right)\sin^2\phi}}. \quad (13)$$

Soit e l'excentricité de la base plane du cône, savoir $e^2=\dfrac{a^2-b^2}{a^2}$, 2ϵ l'angle des deux *lignes focales*, et 2η l'angle des deux *axes cycliques*, c'est-à-dire des deux droites normales aux sections circulaires du cône ; d'après le même Mémoire on aura :

$$e^2=\frac{\tan^2\alpha-\tan^2\beta}{\tan^2\alpha},\quad \sin^2\eta=\frac{\sin^2\alpha-\sin^2\beta}{\sin^2\alpha}, \quad . \quad (14)$$

et :

$$\tan^2\epsilon=\frac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha},$$

et la précédente expression de l'arc de l'ellipse sphérique deviendra :

$$\sigma=\frac{\tan\beta}{\tan\alpha}\sin\beta\int\frac{d\phi}{[1-e^2\sin^2\phi]\sqrt{1-\sin^2\eta.\sin^2\phi}}. \quad . \quad (15)$$

Désignant par m et n deux paramètres conjugués, on a :

$$(1-m)(1+n)=1+c^2,. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

ainsi qu'on peut le voir dans tout ouvrage élémentaire sur les intégrales elliptiques ; donc si l'on fait $m=e^2$ et $c^2=\sin^2\eta$, on a $n=\dfrac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha}$, ou :

$$n=\tan^2\epsilon.$$

Les trois quantités : e excentricité de la base du cône, 2ϵ angle des lignes focales et 2η angle des lignes cycliques, sont liées par l'équation simple :

$$1-e^2=\cos^2\eta.\cos^2\epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Le coefficient de l'intégrale (12), c'est-à-dire $\frac{\tan\beta}{\tan a}\sin\beta$, est ce qu'on nomme quelquefois le *criterium de circularité,* car :

$$\frac{\tan^2\beta}{\tan^2 a}\sin^2\beta=(1-m)\left(1-\frac{c^2}{m}\right);$$

par suite on peut écrire :

$$\sigma=\sqrt{\left(1-m\right)\left(1-\frac{c^2}{m}\right)}\int\frac{d\phi}{[1-m\sin^2\phi]\ \sqrt{1-c^2\sin^2\phi}}. \quad (15^*)$$

Notre conique sphérique a ses arcs principaux supplémentaires, car puisque $\tan a=\frac{a}{b}$, $\tan\beta=\frac{b}{a}$, on a $\tan a.\tan\beta=1$, d'où $a+\beta=\frac{\pi}{2}$. Elle est égale à sa réciproque tournée d'un angle droit. Les axes focaux de l'une sont les axes cycliques de l'autre : par conséquent la rectification de l'une dépend de la quadrature de l'autre, ainsi qu'on peut le voir dans le Mémoire cité ci-dessus.

II. *De la rectification de la courbe représentée par l'équation :*

$$a^2x^2-b^2y^2=(x^2+y^2)^2. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Posant $x=r\sin\phi$, $y=r\cos\phi$, il vient :

$$a^2\sin^2\phi-b^2\cos^2\phi=r^2, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

d'où :

$$\frac{ds^2}{d\phi^2}=\frac{a^4\sin^2\phi+b^4\cos^2\phi}{a^2\sin^2\phi-b^2\cos^2\phi}. \quad . \quad . \quad . \quad . \quad (20)$$

Prenant :

$$a^2\tan^2\phi=b^2\sec^2\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

et faisant les substitutions nécessaires, on trouvera :

$$\frac{ds}{d\lambda}=\frac{ab\sqrt{a^2+b^2\cos^2\lambda}}{b^2+a^2\cos^2\lambda}, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

ou :

$$\frac{ds}{d\lambda}=\frac{ab[a^2+b^2-b^2\sin^2\lambda]}{[a^2+b^2-a^2\sin^2\lambda]\ \sqrt{a^2+b^2-b^2\sin^2\lambda}}, \quad . \quad (23)$$

ou encore :

$$\frac{ds}{d\lambda}=\frac{ab}{\sqrt{a^2+b^2}}\cdot\frac{\left[1-\frac{b^2}{a^2+b^2}\sin^2\lambda\right]}{\left[1-\frac{a^2}{a^2+b^2}\sin^2\lambda\right]\sqrt{1-\frac{b^2}{a^2+b^2}\sin^2\lambda}}, \quad . \quad (24)$$

ou enfin :

$$\frac{ds}{d\lambda}=\frac{b^3}{a\sqrt{a^2+b^2}}\cdot\frac{\left[\frac{a^2}{b^2}-1+1-\frac{a^2}{a^2+b^2}\sin^2\lambda\right]}{\left[1-\frac{a^2}{a^2+b^2}\sin^2\lambda\right]\sqrt{1-\frac{b^2}{a^2+b^2}\sin^2\lambda}}. \qquad (25)$$

En faisant :

$$\frac{a^2}{a^2+b^2}=m,\quad \frac{b^2}{a^2+b^2}=c^2, \quad . \; . \; . \; . \; . \; . \qquad (26)$$

la dernière équation devient :

$$\frac{ds}{d\lambda}=\frac{b(a^2-b^2)}{a\sqrt{a^2+b^2}}\,\frac{1}{[1-m\sin^2\lambda]\sqrt{1-c^2\sin^2\lambda}}+\frac{b^3}{a\sqrt{a^2+b^2}}\cdot\frac{1}{\sqrt{1-c^2\sin^2\lambda}},$$

ou, intégrant :

$$s=\frac{b}{a}\,\frac{a^2-b^2}{\sqrt{a^2+b^2}}\int\frac{d\lambda}{[1-m\sin^2\lambda]\sqrt{1-c^2\sin^2\lambda}}+\frac{b^3}{a\sqrt{a^2+b^2}}\int\frac{d\lambda}{\sqrt{1-c^2\sin^2\lambda}}. \qquad (27)$$

Si la courbe est la lemniscate, on a $a=b$, et cette expression devient :

$$s=\frac{a}{\sqrt{2}}\int\frac{d\lambda}{\sqrt{1-\frac{1}{2}\sin^2\lambda}}. \quad . \; . \; . \; . \; . \; . \qquad (28)$$

CHAPTER XXIII.

ON THE RELATION BETWEEN TANGENTIAL EQUATIONS, AND THE SINGULAR SOLUTIONS OF THE DIFFERENTIAL EQUATIONS OF PLANE CURVES.

221.] The equation $F(x,y)=0$ is the integral known as the singular solution of the equation $\Phi(\xi,v)=0$. For if between the three equations $\Phi(\xi,v)=0$, $x\xi+yv=1$, and $\frac{dy}{dx}=-\frac{\xi}{v}$ we eliminate ξ and v, we shall find the resulting differential equation

$$\Phi\left(x,y,\frac{dy}{dx}\right)=0;$$

or using the usual notation $\frac{dy}{dx}=p$, we shall have the differential equation of the same curve, $\Phi(x,y,p)=0$.

Since $x\xi+yv=1$, and $p=\frac{-\xi}{v}$, we shall find for ξ and v,

$$\xi=\frac{p}{px-y},\ \text{and}\ v=\frac{-1}{px-y}. \quad . \; . \; . \; . \; . \; . \qquad (a)$$

To illustrate this theory.

(*a*) Let us assume the tangential equation of the evolute of the ellipse $a^2v^2+b^2\xi^2=c^4\xi^2v^2$.

For ξ and v substitute their values as given in (a), and the resulting equation will become

$$a^2+b^2p^2=\frac{c^4p^2}{(y-px)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Differentiating this expression (b), we find

$$\frac{a^2}{c^4}=\frac{y}{(y-px)^3}, \text{ or } p=\frac{c^{\frac{4}{3}}y^{\frac{1}{3}}-a^{\frac{2}{3}}y}{a^{\frac{2}{3}}x}. \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Hence

$$(y-px)^2=\frac{c^{\frac{8}{3}}y^{\frac{2}{3}}}{a^{\frac{4}{3}}};$$

but the equation of the curve (b) gives

$$(y-px)^2=\frac{c^4p^2}{a^2p^2+b^2}.$$

Comparing these two values,

$$a^{\frac{4}{3}}(c^{\frac{4}{3}}-a^{\frac{2}{3}}y^{\frac{2}{3}})p^2=b^2y^{\frac{2}{3}},$$

substituting the value of p derived from (c), we finally obtain

$$(c^{\frac{4}{3}}-a^{\frac{2}{3}}y^{\frac{2}{3}})^3=b^2x^2,$$

or, taking the cube root,

$$a^{\frac{2}{3}}y^{\frac{2}{3}}+b^{\frac{2}{3}}x^{\frac{2}{3}}=c^{\frac{4}{3}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

the projective equation of the evolute of an ellipse.

(β) As another example let us take the tangential equation of the ellipse

$$a^2\xi^2+b^2v^2=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Substituting the values of x and y,

$$a^2p^2+b^2=(y-px)^2.$$

Differentiating this expression, we get

$$p=\frac{-yx}{a^2-x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

Hence

$$y-px=\frac{a^2y}{a^2-x^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Substituting these values in the preceding equation,

$$\frac{a^2y^2x^2}{(a^2-x^2)^2}+b^2=\frac{a^4y^2}{(a^2-x^2)^2},$$

or reducing,

$$a^2y^2+b^2x^2=a^2b^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

the projective equation of an ellipse.

It is beside the object of this work to pursue the subject further.

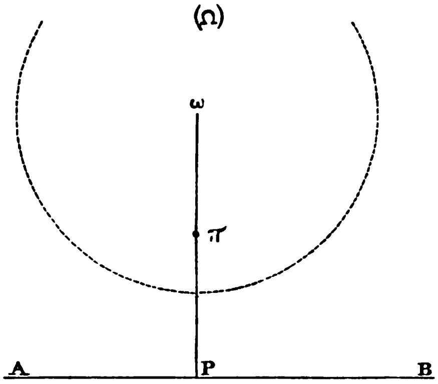
CHAPTER XXIV.

ON THE GEOMETRICAL THEORY OF RECIPROCAL POLARS.

222.] To this simple but very beautiful theory is due the widest development of pure geometry that has been effected in modern times. Nothing can be simpler than the elementary principles on which this theory rests, nothing more beautiful than the results to which it leads.

Let a point ω and a straight line AB be assumed in a plane.

Fig. 41.



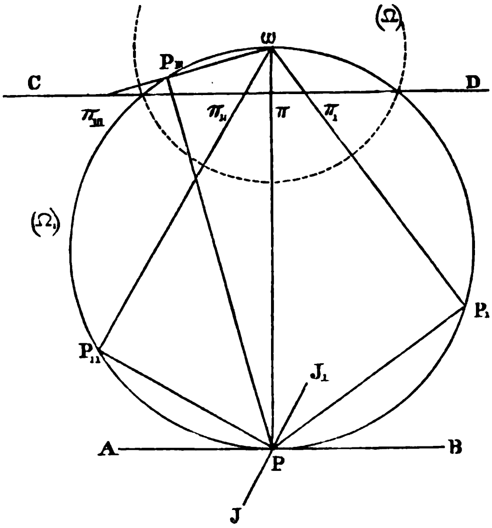
From this point ω let fall a perpendicular ωP on the line A B. With ω as centre, let a circle (Ω) be described, R being the radius. In the line ω P let a point π be assumed such that $\omega\pi \times \omega P$ shall be equal to R^2. π is called the *pole* of the line A B with reference to (Ω); and they are termed *reciprocal pole* and *polar*. The circle whose centre is ω and radius R may be called the *polarizing circle*, as being the instrument by which the polars of given points and lines may be exhibited. The polarizing circle or sphere, a dotted line, will be denoted by the symbol (Ω).

It is obvious that any conic section might be used as a polarizing curve instead of a circle; but no advantage would be gained by complicating this instrument of geometrical discovery.

223.] If we now pass to space of three dimensions, let there be assumed a point and a plane in space. Let a sphere (Ω) be described, having its centre at the given point ω, and from this point let fall a perpendicular ω P upon the plane A B, and let a point π be assumed on this perpendicular, so that $\omega\pi \times \omega P$ shall be equal to R^2. π is the *pole* of the given *plane* A B with reference to the sphere (Ω).

Through the point ω, and touching the given plane at P, let another sphere $(\Omega_{\prime})$ be described. Let ω P $P_{\prime}$ be a great circle of this sphere, and through P let a line J $J_{\prime}$ be drawn at right angles to this plane. Let the common secant plane of the two spheres be drawn, C D being the intersection of this secant plane with the plane of the great circle ω P $P_{\prime}$. Through the line J $J_{\prime}$ let any plane $(\Pi_{\prime})$ be drawn cutting the great circle of the sphere $(\Omega_{\prime})$ in the straight line P $P_{\prime}$. Join ω $P_{\prime}$ meeting the common cord of the two great circles of the two spheres (Ω) and $(\Omega_{\prime})$ in the point $\pi_{\prime}$. It is manifest that $\pi_{\prime}$ is the pole of the plane P $P_{\prime}$ J $J_{\prime}$, since

Fig. 42.



$$\omega\pi_{\prime} \times \omega P_{\prime} = \omega\pi \times \omega P = R^2.$$

Hence it follows that the poles of all the planes drawn through the straight line J P $J_{\prime}$ perpendicular to the plane of the great circle ω P $P_{\prime}$ will range along the common cord of the great circles of the two spheres (Ω) and $(\Omega_{\prime})$.

It is manifest that the angle between the planes (Π) and $(\Pi_{\prime})$ is equal to the angle between the perpendiculars let fall upon them from the centre ω of the polarizing sphere (Ω).

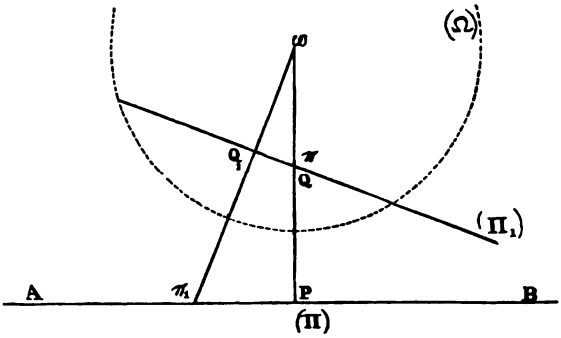
The common chord of the two great circles along which the poles of the planes range is in a plane at right angles to the line J P $J_{\prime}$ through which all the planes pass; and these two straight lines are called *conjugate polars*, one of the other, with reference to the polarizing sphere (Ω).

Indefinite planes and straight lines, or planes and straight lines given only in position, will be denoted by letters enclosed in brackets. Thus (Π) and (*l*) will denote a plane and a straight line given in position only, and otherwise indeterminate.

224.] We shall now proceed to develope some of the results which follow from these elementary and obvious principles.

If any point $\pi_{,}$ *be assumed in a plane* (Π), *the polar plane* ($\Pi_{,}$) *of this point will pass through* π *the pole of* (Π).

Fig. 43.



Join ω and $\pi_{,}$. Let the polar plane ($\Pi_{,}$) of $\pi_{,}$ cut the line $\omega\pi_{,}$ in $Q_{,}$ and the line ω P in Q. Then

$$\omega\pi_{,} \times \omega Q_{,} = \omega Q \times \omega P = R^2 = \omega\pi \times \omega P,$$

or π coincides with Q.

When any number of straight lines are parallel to one another, their conjugate polars will all lie in the same plane.

For the conjugate polar of any one of them will be in a plane passing through the polarizing centre ω, and at right angles to the given straight line. Hence all these planes will be parallel; and as they all pass through the point ω they will be identical.

It will much facilitate the study of this theory if these two principles are firmly grasped, and rendered familiar; that is to say, *the polar plane of any point assumed in a straight line will pass through the conjugate polar of this straight line, and the polar plane of any point assumed in a plane will pass through the pole of this plane.*

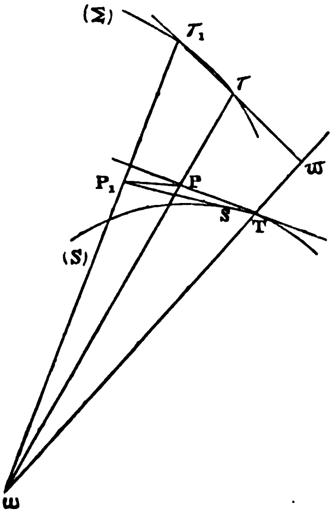
225.] *The reciprocal polar of any plane curve is a cone whose vertex is in the pole of the plane of the curve.*

For if we inscribe a polygon in the given curve, the polar plane of every vertex of the polygon will pass through the pole of the plane of the polygon, as we have just now shown, and the conjugate polar of any side of the polygon will be the line in which two consecutive polar planes of two consecutive vertices of the polygon intersect. Let the sides of the polygon now be indefinitely multiplied and diminished in magnitude, the polygon will ultimately coincide with the plane curve, its limit; and the polygonal pyramid will become a cone.

Throughout the following pages the polarizing circle or sphere will be denoted by the symbol (Ω); the primitive or normal surface, which is to be transformed, by (S); and the reciprocal polar of this surface by (Σ); and if (l) or (m) or (n) be any straight lines, their conjugate polars will be denoted by the symbols (λ), (μ), (ν).

226.] *From the centre* ω *of the polarizing circle* (Ω) *a perpendicular* ω P *is let fall on a tangent drawn to the curve* (S) *at* T. *The radius* ωT *produced will be perpendicular to the corresponding tangent drawn to the reciprocal polar curve* (Σ); *and the perpendicular* ω P *on the tangent to the curve* (S) *at* T *will pass through the point of contact* τ *of the corresponding tangent to the reciprocal polar* (Σ).

Fig. 44.



Let the vector ω T be produced to ϖ, so that $\omega \mathrm{T} \times \omega \varpi = \mathrm{R}^2$. Draw the tangent s $\mathrm{P}_{,}$ indefinitely near to T P, and let the two perpendiculars ω P, $\omega \mathrm{P}_{,}$ be produced to meet in the points τ and $\tau_{,}$, the perpendicular to $\omega\varpi$ erected at ϖ. Now, as by hypothesis (S) and (Σ) are reciprocal curves,

$$\omega \mathrm{T} \times \omega \varpi = \omega \mathrm{P} \times \omega \tau = \mathrm{R}^2,$$

$\mathrm{T} \varpi \mathrm{P} \tau$ is therefore a quadrilateral that may be inscribed in a circle. Hence the angle $\omega \mathrm{T} \mathrm{P} = \omega \tau \varpi = \omega \tau_{,} \varpi$, since at the limit the angle $\tau \omega \tau_{,}$ vanishes. Consequently a circle may also pass through $\mathrm{P} \tau \tau_{,} \mathrm{P}_{,}$, and therefore

$$\omega \mathrm{P} \times \omega \tau = \omega \mathrm{P}_{,} \times \omega \tau_{,} = \mathrm{R}^2;$$

and therefore $\tau_{,}$ must also be a point on the curve (Σ), or $\tau\tau$ an indefinitely small portion of the line $\varpi \tau \tau_{,}$ erected perpendicular to the line $\omega \varpi$ at the point ϖ is a tangent to the reciprocal polar (Σ) of the original curve.

Hence when two curves (S) and (Σ) are reciprocal polars, one of the other, the radius vector through the point of contact of the one will be perpendicular to the corresponding tangent of the other; or if through a point assumed on (S) we draw a tangent, the polar of this point of contact will be a tangent to (Σ), and the pole of the tangent to (S) will be the corresponding point of contact on (Σ); or in other words, the radius vector drawn to a point of the one (S) will coincide with the perpendicular let fall on the corresponding tangent to the other (Σ). If r and $P_{,}$ be these quantities, we shall always have $rP_{,}=R^2$, a constant quantity.

227.] *To find the conjugate polar of the normal to a tangent plane applied to the primitive surface* (S).

As the normal passes through the point of contact of the tangent plane *to* (S), its conjugate polar will lie *in* the tangent plane to (Σ); and as the normal and tangent plane to (S) are at right angles, one to the other, the vector line from the origin to the point of contact of the tangent plane to (Σ) will be perpendicular to the vector plane drawn from the origin through the conjugate polar of the normal, which lies, as we have shown, in the tangent plane to (Σ).

Hence the conjugate polar of the normal to a curved surface (S) is the straight line in the tangent plane to (Σ), through which and the origin if a plane be drawn, it will be perpendicular to the radius vector drawn from the origin to the point of contact of the tangent plane to (Σ).

228.] (a) A plane (Π) passes through a given point $\pi_{,}$ and a given straight line (l). The pole π of this plane (Π) is the point in which the polar plane ($\Pi_{,}$) of $\pi_{,}$ is pierced by the conjugate polar (λ) of the given straight line. See fig. in sec. [224].

For as this latter point is in the straight line (λ), its polar plane will pass through (l); and as it is in the plane ($\Pi_{,}$), its polar plane will pass through $\pi_{,}$. Hence, conversely, the plane passing through $\pi_{,}$ and (l) must have for its pole the intersection of ($\Pi_{,}$) with (λ).

(β) A straight line (l) and a plane (Π) are at right angles one to the other. The plane drawn through the pole π of (Π) and (λ), the conjugate polar of (l), will be perpendicular to the line drawn from ω, the centre of the polarizing sphere, to $\varpi_{,}$, the foot of the perpendicular (l).

Let the given plane (Π) be the plane of the paper, suppose; then (l) will be vertical, and therefore (λ) will be horizontal. Let the line (l) pierce the plane (Π) in $\varpi_{,}$. Then, as $\varpi_{,}$ is a point in (Π), the polar plane of $\varpi_{,}$ will pass through π the pole of (Π); and as $\varpi_{,}$ is a point in the straight line (l), the polar plane of $\varpi_{,}$ will pass through (λ). Hence the line drawn from ω to $\varpi_{,}$ will be at right angles to the plane which passes through π and (λ).

(γ) Or this proposition may be proved more simply thus: join the point $\varpi_{,}$, assumed as the intersection of the line (l) with the plane (Π), to ω, the centre of the polarizing sphere. Hence ($\Pi_{,}$), the polar plane of $\varpi_{,}$, will be at right angles to the line $\varpi_{,}\omega$; and as $\varpi_{,}$ is in the plane (Π) its polar plane ($\Pi_{,}$) will pass through π, the pole of (Π); and as $\varpi_{,}$ is a point in the straight line (l), its polar plane ($\Pi_{,}$) will pass through (λ), the conjugate polar of (l).

If a straight line (l) be perpendicular to a given plane (Π), the conjugate polar (λ) of this straight line will lie in a plane parallel to (Π) passing through ω the centre of (Ω), which therefore may be called the central plane. It will also lie in the polar plane of the point in which (l) intersects (Π), therefore (λ) will be the intersection of these two planes.

When the line (l) makes an acute angle with the plane (Π), the central plane will cut (l) still at right angles, and (λ) will be the intersection of this central plane with the polar plane of the intersection of (l) with (Π).

(δ) A straight line (l) lies in a plane (Π), the conjugate polar (λ) of (l) will pass through the pole π of (Π), and lie in a plane at right angles to (l), passing through ω the centre of (Ω). For if through (l) we draw two tangent planes to the polarizing sphere (Ω), the line (λ) which joins the points of contact with the sphere will be the conjugate polar of (l) and will be in a plane at right angles to it, and as the line (l) lies in the plane (Π) its conjugate polar (λ) will pass through π the pole of (Π).

229.] When points are assumed along a straight line (l) passing through ω the centre of the polarizing sphere (Ω), the polar planes of all these points will be parallel, seeing that they must all pass through the conjugate polar (λ) of (l), which is at infinity, since (l) its polar passes through ω the centre of (Ω).

CHAPTER XXV.

THE GEOMETRICAL THEORY OF RECIPROCAL POLARS APPLIED TO THE DEVELOPMENT OF A NEW METHOD OF DERIVING THE PROPERTIES OF SURFACES OF THE SECOND ORDER WITH THREE UNEQUAL AXES FROM THOSE OF THE SPHERE.

230.] M. Chasles and other geometers have shown how the properties of surfaces of revolution of the second order may be derived from those of the sphere, by the method of reciprocal polars. But they have not extended their researches so as to include the case of surfaces with three unequal axes.

In a memoir entitled "Recherches de géometrie pure sur les lignes et les surfaces du second degré," published in 1829, M. Chasles has shown how the properties of surfaces of revolution may be derived with singular simplicity from those of the sphere. But this great geometer has omitted to apply the method to obtain the corresponding properties of surfaces with three unequal axes*. This omission may be supplied from the following considerations.

---

* "Quand on emploie une sphère pour surface auxiliaire, s'il se trouve une autre sphère dans la figure qu'on vent transformer, il lui correspondra dans la

By the help of one reciprocation we may pass from the sphere to surfaces of revolution having two equal axes. By a second reciprocation we may proceed from the properties of these surfaces to those of surfaces with three unequal axes.

Thus this method of dual transformation enables us to obtain the reciprocal polars of properties which are themselves reciprocal polars. The second reciprocal surface exhibits like properties to those of the sphere. This may need some explanation. Suppose, for example, that n points lie in the same plane with respect to a sphere; the reciprocal of this property would be that in a surface of revolution we should have n planes passing through the same point, and we should have for the reciprocal of this, again, n points lying in the same plane, with reference to a surface with three unequal axes.

231.] It is a well-known theorem that if a point be taken in the plane of a circle (S), and made the centre ω of the polarizing circle (Ω), the reciprocal polar of the circle (S) will be a conic section having one of its foci at the centre ω of the polarizing circle (Ω), and its axis in the line joining the centres of (S) and (Ω).

Of the several proofs that may be given of this cardinal theorem, the following is perhaps the most elegant, as it is certainly the most simple.

Since the product of the segments of a chord of a circle passing through a fixed point is constant, the product of the reciprocals of these segments will be constant also; and as these reciprocals are coinciding perpendiculars let fall from the polarizing centre ω on two tangents to the reciprocal curve (Σ), these tangents will be parallel, because the extremities of the cord of the circle (S) and the polarizing centre ω, are in the same straight line. Consequently we obtain this property of the reciprocal curve (Σ), that the product of two perpendiculars let fall from a point on parallel tangents to the curve is constant. But this we know to be a property of the conic sections, that the product of perpendiculars let fall from a focus on parallel tangents is constant. Hence the truth of the proposition.

When the point of intersection of the chords is on the circumference of the circle, a chord of the circle passing through this point is $=2a\cos\phi$, ϕ being the angle which this chord makes with the diameter $2a$ passing through this point; hence $\frac{\mathrm{R}^2}{2a}\sec\phi$ is the length of the perpendicular let fall from this point on the corresponding tangent of the reciprocal curve; but this we know to be the expression for a focal perpendicular on a tangent to a parabola.

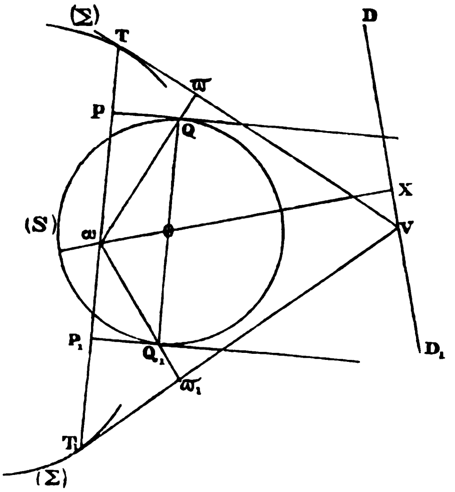
---

nouvelle figure une surface du second degré de révolution ; on n'aura donc point les propriétés générales d'un surface du second degré qnelconque."—Chasles, 'Aperçu Historique,' p. 233.

232.] *The polar of the centre* O *of* (S) *is* $D\,D_{,}$, *the directrix of the conic section* (Σ).

Let ω be the centre of the polarizing circle (Ω), and O the centre of (S). Let the diameter $Q\,Q_{,}$ be drawn and the parallel tangents $Q\,P$ and $Q_{,}\,P_{,}$. Let the polars of the points Q and $Q_{,}$ be taken, which will be tangents $\varpi\,T$, $\varpi_{,}\,T_{,}$ to the curve (Σ) : also let the poles of the tangents $P\,Q$ and $P_{,}\,Q_{,}$ to the circle be taken; these will be the points T and $T_{,}$ of contact of the tangents $\varpi\,T$ and $\varpi_{,}\,T_{,}$ to the reciprocal curve (Σ). These points T and $T_{,}$ will be on the same straight line passing through ω, since the tangents $P\,Q$, $P_{,}\,Q_{,}$ to the circle are parallel. Take $D\,D_{,}$ the polar of the centre O; and as $T\,\varpi$, $T_{,}\,\varpi_{,}$, and $D\,D_{,}$ are the polars of the three poles Q, $Q_{,}$, and O respectively, which all lie in the same straight line, the diameter of the circle, these three lines must meet in a point, or the point V in which the tangents meet must be on the polar of O the centre of the circle. But it has been shown in [231] that ω is the focus of the conic section (Σ) ; consequently $D\,D_{,}$ is the directrix.

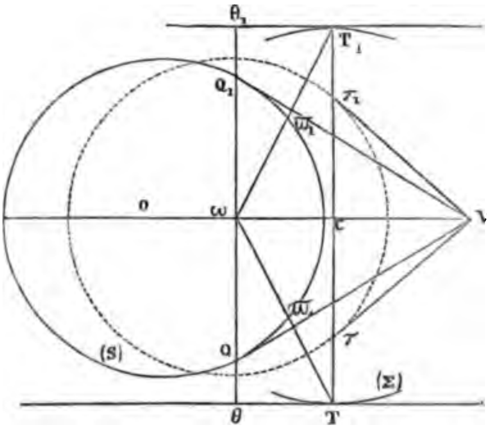
Fig. 45.



233.] *If in the transverse axis of a surface of revolution* (Σ) *we assume a point* V *as the common vertex of two cones of revolution circumscribing the sphere* (S) *and the polarizing sphere* (Ω) *whose centre is at* ω *the focus of* (Σ), *if the base of the former cone pass through the focus of* (Σ), *the base of the latter will pass through the centre of* (Σ).

Let a principal plane be drawn through the centres of (S), (Ω),

Fig. 46.



and (Σ). Let parallel tangents be drawn to the principal section of (Σ) at the points T and T_l; the diameter TT_l will of course pass through C the centre of (Σ).

Let the reciprocal polars $Q\varpi$, $Q_l\varpi_l$ of the points T and T_l be taken; they will touch the circle in the points Q and Q_l; and as by supposition the line QQ_l passes through ω, the centre of the polarizing circle (Ω), the two tangents $Q\varpi$, $Q_l\varpi_l$ will meet in a point V. And as the point C the centre of (Σ) lies in the diameter TT_l, the polars of the three points T, T_l, and C will meet in V. Hence $V\omega : R :: R : C\omega$. But as $V\tau$ and $V\tau_l$ are by hypothesis tangents to the polarizing circle (Ω), we shall have also $V\omega : R :: R : C\omega$.

Consequently the base of the triangle $V\tau\tau_l$ passes through C the centre of (Σ).

As the force of the preceding demonstration may not be at once obvious to every reader, it may be observed that the point V is the pole of the line TCT_l by the method of reciprocal polars, and therefore $V\omega : R :: R : C\omega$, R being the radius of the polarizing circle. But in the circle $\tau\omega\tau_l$ whose radius is R_l, we have by common geometry $V\omega : R_l :: R_l : C\omega$; but when $R = R_l$ the two proportions become identical.

We may give another demonstration of this important proposition.

Let a and b be the semiaxes of (Σ), and e the eccentricity. Then

$$V\omega \times \omega C = R^2; \quad \ldots\ldots\ldots \quad (a)$$

and we have shown in [232] that the polar of O the centre of (S) was the directrix; therefore

$$\omega O = \frac{R^2 e}{a(1-e^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(b)}$$

But

$$V\omega \times \omega O = \overline{Q\omega}^2 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(c)}$$

or

$$V\omega = \frac{\overline{Q\omega}^2 a(1-e^2)}{R^2 e} : \text{ and } \overline{Q\omega}^2 = \frac{R^4}{b^2} ; \quad . \quad . \quad . \quad . \qquad \text{(d)}$$

consequently

$$V\omega = \frac{R^2}{b^2 e} a(1-e^2) = \frac{R^2}{ae}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(e)}$$

But $V\omega \times \omega C = R^2$; hence $\omega C = ae$, the distance between the centre and the focus.

Therefore, while the base of the cone $VQQ_{\prime}$ determines the focus of (Σ), the base of the convertical cone will determine the centre of (Σ).

234.] Let a sphere be assumed as the polarizing surface (Ω), and a *prolate* spheroid or ellipsoid as the primitive or normal surface (S), from which the properties of the reciprocal surface (Σ) are to be derived. We have therefore three surfaces to consider :—the primitive or normal surface (S), which is to supply the properties that are to be transformed; the auxiliary or polarizing sphere (Ω), the instrument of transformation; and the derived or polar surface (Σ).

Let A and C be the semiaxes of the primitive surface (S), the prolate spheroid or elongated ellipsoid of revolution having its principal circular section whose radius is A in the horizontal plane or plane of XY, and its third semiaxis C, greater than A, vertical. Let D be the distance between O the centre of the prolate spheroid (S) and ω the centre of the polarizing sphere (Ω). Let a, b, c, in the order of magnitude be the semiaxes of the derived surface (Σ).

We shall now establish the following relations between a, b, c and A, C, D, R—R being the radius of the polarizing sphere (Ω), which is represented by the circle whose centre is ω :—

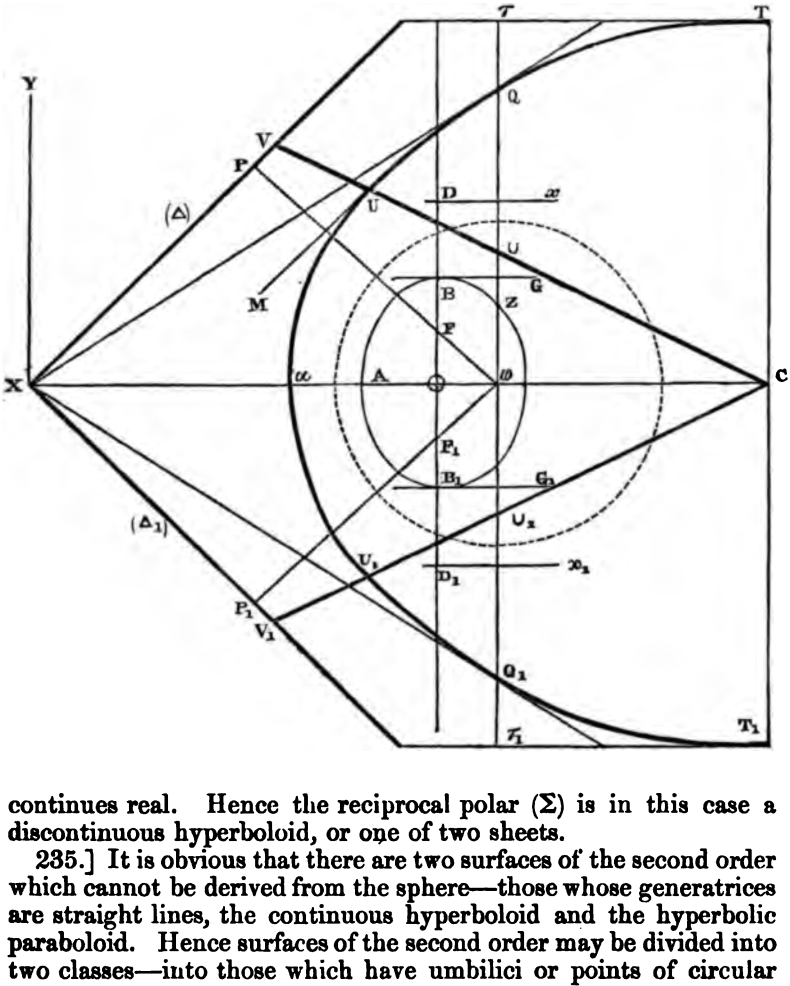
$$a = \frac{R^2 A}{A^2 - D^2}; \; b = \frac{R^2}{\sqrt{A^2 - D^2}}; \text{ and } c = \frac{R^2 A}{C\sqrt{A^2 - D^2}}. \quad . \qquad \text{(a)}$$

Thus we shall be enabled to express the semiaxes a, b, c of the reciprocal polar (Σ) in terms of the semiaxes A, C of the primitive surface (S), and the distance D between its centre O and the centre ω of (Ω) the polarizing sphere.

We may observe that when ω the centre of (Ω) is on the surface of (S), $D = A$, and the above values of a, b, and c become infinite, or the surface becomes a paraboloid, of which the semiparameters of the principal sections $\frac{b^2}{a}, \frac{c^2}{a}$ are $\frac{R^2}{A}$ and $\frac{R^2 A}{C^2}$, which are finite quantities.

When D > A, the values of *b* and *c* become imaginary, while *a* continues real. Hence the reciprocal polar (Σ) is in this case a discontinuous hyperboloid, or one of two sheets.

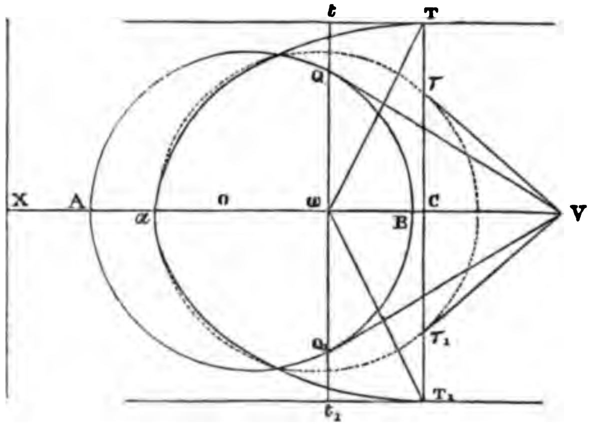
Fig. 47.



235.] It is obvious that there are two surfaces of the second order which cannot be derived from the sphere—those whose generatrices are straight lines, the continuous hyperboloid and the hyperbolic paraboloid. Hence surfaces of the second order may be divided into two classes—into those which have umbilici or points of circular contact with tangent planes, and those which admit of contact only along straight lines.

236.] Let A B Q $Q_{,}$ (fig. 48) be the circular section of the ellipsoid whose radius is A, and which for clearness we assume to be in the plane of XY. Let a point ω be assumed at the distance D from the centre O of this circle, and let this be the centre of the polarizing sphere (Ω). Through ω let the perpendicular $Q\,Q_{,}$ be drawn, and at these points $Q\,Q_{,}$ let tangents drawn to the circle meet in V. From V let two tangents $V\tau$, $V\tau_{,}$ be drawn to $\omega\tau\tau_{,}$, the great circle of the polarizing sphere (Ω) in the plane of X Y. Through τ, $\tau_{,}$ let a straight line be drawn meeting the diameter $O\omega$ in C; C will be the centre of the surface (Σ) by [233].

Fig. 48.



Through T and $T_{,}$ let parallel tangents be drawn to the reciprocal polar of the circle; this will be a conic section whose focus will be at ω by the proposition in sec. [231]. Take the reciprocal polars of the points A and B, the extremities of the diameter of the base circle; let one of them pass through *a*, it will be a tangent to the conic section at *a* the extremity of the major axis. Let *a* and *b* be the semiaxes of this conic section in the plane of X Y, then $\omega t = CT = b$; and as Q is the pole of the tangent T *t*, we shall have $\omega Q \times \omega t = R^2$; or as $\omega t = b$, and $\omega Q = \sqrt{A^2 - D^2}$, the result becomes

$$b = \frac{R^2}{\sqrt{A^2 - D^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(a)}$$

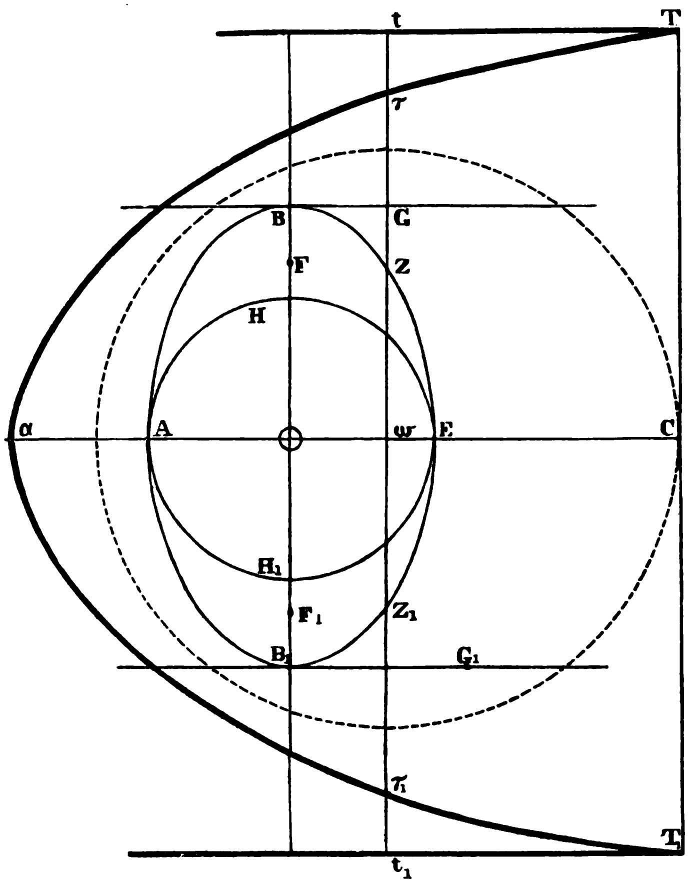
The distances of the polar focus ω from the vertices of the transverse axis of the principal section of (Σ) in the plane of XY are manifestly

$$\frac{R^2}{A+D} \text{ and } \frac{R^2}{A-D}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(b)}$$

Half the sum of these expressions will be equal to *a* the transverse axis of (Σ), or

$$a = \frac{R^2 A}{A^2 - D^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(c)}$$

Fig. 49.



237.] To determine the third semiaxis *c*.

On the base AHH_1E let us conceive one half of the prolate spheroid to be erected, and an ordinate ωz drawn through ω in the principal plane of the spheroid (S) whose semiaxes are A and C; then $\omega z : OB :: \sqrt{A^2-D^2} : A$, or $\omega z = \frac{C\sqrt{A^2-D^2}}{A}$. But the extremity z of ωz is the pole of the tangent plane to (Σ) parallel to the plane of XY; consequently $c \times \omega z = R^2$; or, putting for ωz its value,

$$c = \frac{R^2 A}{C\sqrt{A^2-D^2}}. \quad \dots\dots\dots \quad (a)$$

Hence the values of the three semiaxes of the polar surface (Σ) in terms of the semiaxes A and C of the primitive surface (S), the radius R of the polarizing sphere (Ω), and the distance D between the centres of (S and (Ω) are, as we have shown,

$$a=\frac{R^2A}{A^2-D^2},\ b=\frac{R^2}{\sqrt{A^2-D^2}},\ \text{and}\ c=\frac{R^2A}{C\sqrt{A^2-D^2}}. \quad . \quad \text{(b)}$$

Conversely, we may express the constants of (S) in terms of a, b, c, the semiaxes of (Σ)—

$$A=\frac{R^2a}{b^2},\ C=\frac{R^2a}{cb},\ D^2=R^4\frac{(a^2-b^2)}{b^4}. \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Hence

$$\frac{C}{A}=\frac{b}{c}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

or the ratio of the axes of the primitive (S) is equal to the ratio of the axes of the reciprocal polar (Σ) in the plane at right angles to the transverse axis $2a$.

Let

$$\frac{a^2-b^2}{a^2}=e^2,\ \frac{a^2-c^2}{a^2}=\epsilon^2,\ \text{and}\ \frac{b^2-c^2}{b^2}=\eta^2; \quad . \quad . \quad \text{(e)}$$

then e, ϵ, and η are the eccentricities of the three principal sections of (Σ) in the planes of XY, XZ, and YZ.

Since $\frac{C^2-A^2}{C^2}=\frac{b^2-c^2}{b^2}=\eta^2$, it follows that the eccentricity of the primitive surface (S) is equal to that of the principal section of (Σ) in the plane of YZ.

We find also $$e=\frac{D}{A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

Accordingly, therefore, as $D<A$, or $D=A$, or $D>A$, the principal section of (Σ), in the plane of XY, will be an ellipse, parabola, or hyperbola.

We may express the eccentricities of the three principal sections of the reciprocal polar (Σ) in terms of the semiaxes A and C of (S) and of D the distance between the centre O of (S) and ω the centre of the polarizing sphere (Ω),

$$e^2=\frac{D^2}{A^2},\ \eta^2=\frac{C^2-A^2}{C^2},\ \epsilon^2=\frac{C^2+D^2-A^2}{C^2}.$$

We have also the simple relation between the three eccentricities

$$1-\epsilon^2=(1-e^2)(1-\eta^2). \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

238.] We shall now proceed to develope some very beautiful and general properties of umbilical surfaces of the second order having three unequal axes.

The point ω, the centre of the polarizing sphere (Ω), has been shown to be a focus of the principal section of (Σ) in the plane of XY.

This point ω may be called the *polar focus*.

Let us take the polar planes of the two foci F and $F_{\prime}$ of (S), see fig. 47. The distance of one of the foci of (S) from the polar focus ω is $F\omega = \sqrt{D^2+C^2-A^2}$; and the length of the perpendicular ϖ let fall from ω on the polar plane of F, will be

$$\varpi = \frac{R^2}{\sqrt{D^2+C^2-A^2}};$$

computing the value of this expression from (c) in the preceding section,

$$\varpi = \frac{bc}{a\epsilon}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

We should find the same value for $\varpi_{\prime}$, the perpendicular on the polar plane of $F_{\prime}$, the other focus of (S).

239.] Now, as the two foci F, $F_{\prime}$, and O the centre of (S) are on the same straight line (the major axis of (S) perpendicular to the plane of XY), the polar planes of these three points will all meet in the same straight line; and as the polar of O, the centre of (S), has been shown, see [232], to be the directrix of the conic section in the plane of XY, whose focus is ω, and which is a principal plane of (Σ), the two polar planes of the foci F and $F_{\prime}$ of (S) will intersect in the directrix of the principal section of (Σ) in the plane of XY.

As these planes are of the primest importance in this theory of surfaces of the second order, and as we shall show that they are parallel to the circular sections of the surface (Σ) in every case, we shall denote them by the symbols (Δ), $(\Delta_{\prime})$, and call them *the conjugate umbilical directrix planes* of a surface of the second order.

It is obvious that as the point ω, the centre of the polarizing sphere (Ω), may be taken at the distance D on the other side of O the centre of (S), there will be in general *four* umbilical planes passing two by two through the directrices of the principal section of (Σ) in the plane of XY. The lines in which the umbilical directrix planes intersect two by two on the plane of XY may be called *the polar directrices*.

240.] The inclination i of the umbilical directrix plane may be thus found, $\sin i = \dfrac{\varpi}{\frac{a}{e} - ae} = \dfrac{\varpi ae}{b^2}$; but ϖ, as has been shown above, is $= \dfrac{bc}{a\epsilon}$.

Consequently $\sin i=\dfrac{ce}{b\epsilon}$, . (a) or $\tan^2 i=\dfrac{\left(\dfrac{1}{b^2}-\dfrac{1}{a^2}\right)}{\left(\dfrac{1}{c^2}-\dfrac{1}{b^2}\right)}$. . (b)

and $\cos i=\dfrac{\eta}{\epsilon}$, a simple expression for the cosine of i.

Now it is shown in every elementary work on this subject, that the inclination θ of the planes of the circular sections of an ellipsoid to the trifocal plane is given by the same formula,

$$\tan^2\theta=\frac{\left(\dfrac{1}{b^2}-\dfrac{1}{a^2}\right)}{\left(\dfrac{1}{c^2}-\dfrac{1}{b^2}\right)}.$$

Since the sum of the squares of any three conjugate diameters is equal to the sum of the squares of the axes, if we put u for the umbilical semidiameter, b and b being the two other conjugate diameters, we shall have

$$u^2+2b^2=a^2+b^2+c^2, \text{ or } u^2=a^2+c^2-b^2. \quad . \quad . \quad \text{(c)}$$

241.] The angle which a diameter of (Σ) passing through an umbilicus makes with an umbilical tangent plane is thus found. From the centre let fall a perpendicular on the umbilical tangent plane which is parallel to a circular section. Hence, as generally, $a_\prime b_\prime \mathrm{P}=abc$; but in this case $a_\prime=b_\prime=b$, we shall have $\mathrm{P}=\dfrac{ac}{b}$; but ϕ being the angle which the diameter $2u$ makes with the umbilical tangent plane,

$$\sin\phi=\frac{ac}{bu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

or

$$\tan^2\phi=\frac{a^2c^2}{(a^2-b^2)(b^2-c^2)}. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

There are therefore two cases in which the umbilical diameter is perpendicular to the umbilical directrix plane (Δ)—when $a=b$, or $b=c$.

In the former case the derived surface (Σ) is an oblate spheroid; in the latter it is an ellipsoid of revolution round the transverse or major axis.

It may easily be shown that this umbilical angle ϕ is a minimum when the umbilical semidiameter $u=b$, or when the sphere which passes through the central circular sections of the surface passes also through the umbilicus.

In this case $\sin\phi=\frac{ac}{b^2}$.

Let u be prolonged to meet the umbilical directrix plane (Δ); its length U measured from the centre C is $=\frac{u}{\epsilon}$; for

$$\frac{a}{e}:\mathrm{U}::\sin\phi:\sin i.\quad \text{But} \sin i=\frac{ce}{b\epsilon},\ \text{and}\ \sin\phi=\frac{ac}{bu};$$

therefore

$$\mathrm{U}=\frac{u}{\epsilon}.\quad \dots\dots\dots \quad \text{(c)}$$

242.] In every umbilical surface of the second order there are *four foci*, independently of the foci of the principal sections. They are, two by two, on the umbilical diameters of the surface, and are also to be found on the vertical which passes through the polar focus ω at right angles to the plane of XY.

In fig. 47 v and $v_{,}$ on the vertical axis ω Q $\mathrm{Q}_{,}$ and on the umbilical semidiameters C U and C $\mathrm{U}_{,}$ are the umbilical foci of the surface (Σ).

Let θ be the angle which the umbilical semidiameter u makes with the plane of XY; then, as

$$\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{c^2}=\frac{1}{u^2},$$

we shall have

$$\cos^2\theta=\frac{a^2(u^2-c^2)}{u^2(a^2-c^2)}=\frac{a^2(a^2-b^2)}{u^2(a^2-c^2)}=\frac{a^2e^2}{u^2\epsilon^2}.$$

Hence

$$\mathrm{C}v=u\epsilon.\quad \dots\dots\dots \quad \text{(a)}$$

Consequently the point v is the pole of (Δ) with respect to the surface (Σ).

The distance of the umbilical focus v from the plane of XY is thus found. Let this distance be z, then $z=ae\tan\theta$.

But $\tan\theta=\frac{cb\eta}{a^2e}$; consequently $z=\frac{cb\eta}{a}$. . . . . (b)

The segment of this axis z passing through the polar focus, and cut off by the umbilical directrix plane (Δ), is $z_{,}=\frac{bc}{a\eta}$;

for $\cos i=\frac{\eta}{\epsilon}$; consequently $\tan i=\frac{ce}{b\eta}$, and

$$z_{,}=\left(\frac{a}{e}-ae\right)\tan i=\frac{bc}{a\eta}.\quad \dots\dots \quad \text{(c)}$$

243.] The ordinate through ω in the plane of XY is $=\frac{b^2}{a}$; but the ordinate through ω in the plane of XZ is $=\frac{bc}{a}$.

This follows at once by putting ae for x in the equation $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$.

This line $\frac{2bc}{a}$, passing through the polar focus and the two umbilical foci, may be called the principal parameter of the polar reciprocal surface (Σ).

The segment of the axis of Z cut off by the umbilical plane (Δ) is

$$=\frac{a}{e}\tan i=\frac{a}{e}\frac{ce}{b\eta}=\frac{ac}{b\eta}. \quad \ldots\ldots \quad \text{(a)}$$

The distance between the points on the axes of X and Z (passing through the centre) in which they are cut by the umbilical plane may be thus found: putting H for this distance, $\text{CX}=\frac{a}{e}$; but $\cos i=\frac{\eta}{\epsilon}$; hence

$$\text{H}=\frac{a\epsilon}{e\eta}, \quad \ldots\ldots\ldots \quad \text{(b)}$$

a simple expression, in which the three eccentricities of the principal sections are involved.

The length of the perpendicular from the centre on the umbilical directrix plane (Δ) is $\text{P}_{\prime}=\frac{a}{e}\sin i$, or

$$\text{P}_{\prime}=\frac{ac}{b\epsilon}. \quad \ldots\ldots\ldots \quad \text{(c)}$$

The perpendicular $\text{P}_{\prime\prime}$ from the polar focus ω on the umbilical directrix plane (Δ) is $=\frac{a}{e}(1-e^2)\sin i$; or, putting for $\sin i$ its value $\frac{ce}{b\epsilon}$,

$$\text{P}_{\prime\prime}=\frac{bc}{a\epsilon}. \quad \ldots\ldots\ldots \quad \text{(d)}$$

In like manner we may find an expression for the perpendicular let fall from the umbilical focus υ on the umbilical directrix plane (Δ). Let this perpendicular be $\text{P}_{\prime\prime\prime}$; then it may easily be shown that

$$\text{P}_{\prime\prime\prime}=\frac{c^3}{ab\epsilon}. \quad \ldots\ldots\ldots \quad \text{(e)}$$

The perpendicular $\text{P}_{\prime\prime\prime\prime}$ let fall from the polar focus ω on the tangent plane through the polar directrix touching the surface (Σ) at the extremity of the principal parameter is given by the formula

$$\frac{1}{\mathrm{P}_{\prime\prime\prime\prime}{}^2}=\frac{a^2}{b^2}\left(\frac{1}{c^2}+\frac{1}{b^2}-\frac{1}{a^2}\right). \quad \dots\dots \quad \text{(f)}$$

The distance of B, the extremity of C, the major axis of (S) from ω is $\sqrt{\mathrm{C}^2+\mathrm{D}^2}$; but $\mathrm{P}_{\prime\prime\prime\prime}\sqrt{\mathrm{C}^2+\mathrm{D}^2}=\mathrm{R}^2$ (see fig. 47). Now

$$\mathrm{C}^2=\frac{\mathrm{R}^4a^2}{c^2b^2}, \text{ and } \mathrm{D}^2=\frac{\mathrm{R}^4(a^2-b^2)}{b^4}.$$

Substituting these values of C and D in the preceding equation, we get

$$\frac{1}{\mathrm{P}_{\prime\prime\prime\prime}{}^2}=\frac{a^2}{b^2}\left(\frac{1}{c^2}+\frac{1}{b^2}-\frac{1}{a^2}\right).$$

If we add together these three perpendiculars on the umbilical plane (Δ), $\mathrm{P}_\prime$ from the centre C, $\mathrm{P}_{\prime\prime}$ from the polar focus ω, and $\mathrm{P}_{\prime\prime\prime}$ from υ, the umbilical focus, we shall have

$$\frac{ab\epsilon}{c}\{\mathrm{P}_\prime+\mathrm{P}_{\prime\prime}+\mathrm{P}_{\prime\prime\prime}\}=a^2+b^2+c^2. \quad \dots \quad \text{(g)}$$

244.] Let (D) be the directrix plane of the primitive surface (S) whose major axis is C, and eccentricity η. Hence the distance of the pole δ of (D) (which is parallel to the plane of XY) from it is $\omega\delta=\frac{\mathrm{R}^2\eta}{\mathrm{C}}$. But $\mathrm{C}=\frac{\mathrm{R}^2a}{cb}$; see (c), [237]: consequently $\omega\delta=\frac{bc\eta}{a}$; or the extremity δ, the pole of (D), coincides with υ, the umbilical focus.

The radius of the circular section of (Σ) whose plane passes through the umbilical focus is equal to the semiparameter $\frac{bc}{a}$ of the surface (Σ).

For we manifestly have $u^2:b^2::u^2(1-\epsilon^2):r^2$, or

$$r^2=b^2(1-\epsilon^2)=\frac{b^2c^2}{a^2}; \quad \dots\dots \quad \text{(a)}$$

when $c=b$, the radius is, as we know, $=\frac{b^2}{a}$ in the ellipsoid of revolution.

The radius of the circular section of the surface (Σ) which passes through the polar focus ω is

$$r^2=\frac{b^4}{a^2}\left(\frac{e^2+\eta^2}{\epsilon^2}\right). \quad \dots\dots \quad \text{(b)}$$

245.] In the major axis of (S) there are seven remarkable points—

the centre O, the two foci F and $F_{\prime}$, the extremities B and $B_{\prime}$ of the major axis of (S), the points D, $D_{\prime}$ in which this major axis meets the directrix planes (D) and ($D_{\prime}$). Now these seven points, O, F, $F_{\prime}$, B, $B_{\prime}$, D, and $D_{\prime}$, all range along the major axis of (S); hence the seven reciprocal polar planes will all intersect in the same straight line, the polar directrix of the surface (Σ), or the directrix of its principal section in the plane of X, Y (see fig. 47).

The polar planes of F and $F_{\prime}$ are the umbilical directrix planes (Δ) and ($\Delta_{\prime}$), passing through the same polar directrix; the polar planes of the points B and $B_{\prime}$, the extremities of the major axis of (S), will touch the reciprocal surface (Σ) at the extremities Q $Q_{\prime}$ of its principal parameter, while the polar planes of the points D and $D_{\prime}$ will pass through the umbilical foci v and $v_{\prime}$ of the surface (Σ).

A tangent plane is drawn to the vertex of (Σ) cutting the umbilical directrix plane (Δ) in a straight line (t). The plane drawn through this line (t) and the polar focus ω will make with the plane of XY an angle whose sine is $=\frac{c}{b}$.

The principal parameter of the surface (Σ) is a mean proportional between the parameters of the principal sections in the planes of XY and XZ.

For $\frac{bc}{a}$ is a mean proportional between $\frac{b^2}{a}$ and $\frac{c^2}{a}$.

246.] Let $L=\frac{bc}{a}$ be the principal semiparameter of the surface (Σ); and if we substitute for a, b, c their values as given in terms of A, C, D, and R, see [237], (c), we shall find $L=\frac{R^2}{C}$. This is a very remarkable result. The value of L is independent of A and D. Consequently, if we assume the minor axis A of the surface (S) to vary, while the major axis C remains constant, the reciprocal polar surfaces (Σ), thus generated, will all touch in the same point; and if we further assume A=C, (S) will become a sphere, and (Σ) will be an ellipsoid of revolution round the transverse axis, so that the series of reciprocal polar surfaces (Σ) derived from the variation of A in (S) will all touch the ellipsoid of revolution the reciprocal polar of the sphere whose radius is C, and also the tangent plane to all these surfaces which passes through the polar directrix in the plane of XY and touches all the reciprocal surfaces at the extremity of their common semiparameter L. This plane may be called the parametral tangent plane. If we assume the polar focus ω, whose distance from the centre of (S) is D, to range along the transverse axis, while the semiaxes of (S) continue unchanged, then, as the distance of the vertex of the principal semiparameter from the plane of XY is constant, the vertices of the semiparameters of all the reciprocal polars (Σ) derived from the variation of D will range along a straight line parallel to the plane of XY, and at the distance $\frac{\mathrm{R}^2}{\mathrm{C}}$ from it; and the distance of the umbilical focus v from the plane of XY is $\frac{cb\eta}{a}$ or $\frac{\mathrm{R}^2\sqrt{\mathrm{C}^2-\mathrm{A}^2}}{\mathrm{C}^2}$. As this is a constant expression independent of D, the umbilical focus will also range along a straight line parallel to the plane of XY.

The principal parameter of the elliptic paraboloid is $2\sqrt{kk_{\prime}}$, $2k$ and $2k_{\prime}$ being the parameters of the principal sections intersecting in the transverse axis; and the principal semiparameter is the ordinate z through the focus ω of the section in the plane of XY. k is greater than $k_{\prime}$.

The coordinates of the umbilicus are

$$\frac{\bar{x}^2}{a^2}=\frac{a^2-b^2}{a^2-c^2} \text{ and } \frac{\bar{z}^2}{c^2}=\frac{b^2-c^2}{a^2-c^2}. \quad . \quad . \quad . \quad . \quad . \qquad \text{(a)}$$

The length of the umbilical normal from the umbilicus to the axis of X is $=\frac{cb}{a}$.

Hence the length of the umbilical normal is equal to that of the principal semiparameter of the surface (Σ); and as the length of the perpendicular from the centre of (Σ) on the umbilical tangent plane is $=\frac{ac}{b}$, consequently the rectangle under the normal and perpendicular will be equal to c^2, the square of the least semiaxis.

247.] *To find the polar plane of the extremity of the major axis of* (S). Since this point B is on the major axis of (S), its polar plane will pass through the conjugate polar of the axis of (S)—that is, through the polar directrix of the surface (Σ); and as this point is on *the tangent plane* to (S) at B, its polar plane will *meet* the surface (Σ); and as this point is on *the surface* of (S), its polar plane will *touch* the surface of (Σ); and as the tangent plane at B to (S) is parallel to the plane of XY, which contains the polar focus, the point of contact will be on the perpendicular to the plane of XY passing through the polar focus. (See fig. 47.)

This point so found on the surface (Σ) at the extremity of the vertical ordinate passing through the polar focus ω, is the extremity of L, the principal semiparameter of the surface; and the tangent plane passing through this point and the polar directrix in the plane of XY is the parametral tangent plane to the surface (Σ), a surface having three unequal axes.

248.] We may justly infer from the simplicity of the values found for the quantities which determine the umbilical directrix planes, the umbilical foci, as also the lines and points connected with them, that they have a real existence with relation to surfaces of the second order (Σ) having three unequal axes, as much as the corresponding magnitudes in spheres and surfaces of revolution, especially as the properties of these latter surfaces may be derived from those of (Σ) by simply changing c into b.

The reader may desire to see those simple expressions brought together under one view.

Let a, b, c be the semiaxes of the umbilical surface (Σ) having three unequal axes; and let $\frac{a^2-b^2}{a^2}=e^2$, $\frac{a^2-c^2}{a^2}=\epsilon^2$, and $\frac{b^2-c^2}{b^2}=\eta^2$. These may be called the first, second, and third eccentricities of the principal sections of the surface (Σ). Let u be the umbilical semidiameter. Then $u^2=a^2+c^2-b^2$.

(α) The perpendicular from centre on umbilical directrix plane (Δ) is
$P_{,}=$ . . . . $\frac{ac}{b\epsilon}$.

(β) The perpendicular from centre on umbilical tangent plane is $P=$ $\frac{ac}{b}$.

(γ) The perpendicular from umbilical focus on umbilical directrix plane is $P_{,,}=$ . . . . $\frac{c^3}{ab\epsilon}$.

$(\gamma_{,})$ The perpendicular from polar focus on umbilical directrix plane is $P_{,,,}=$ . . . . $\frac{bc}{a\epsilon}$.

(δ) Let i be the inclination of umbilical directrix plane to plane of XY,
$\cos i=$ . . . . $\frac{\eta}{\epsilon}$.

(ε) Inclination ϕ of umbilical diameter to umbilical plane (Δ),
$\tan^2\phi=$ . . . . $\frac{a^2c^2}{(a^2-b^2)(b^2-c^2)}$.

(ζ) Distance of polar focus ω from the centre = ... ae.

Distance of polar directrix from the centre = ... $\frac{a}{e}$.

(η) The ordinate passing through the polar focus ω is . . . . . $\frac{bc}{a}=L$, which is called the principal semiparameter of (Σ).

(θ) The portion of the umbilical diameter u between the centre and umbilical directrix plane $(\Delta)=$ . . . . $\frac{u}{\epsilon}$.

(ι) The portion of the umbilical diameter u between the centre and the umbilical focus $v=$ . . . . $u\epsilon$.

(κ) Distance of umbilical focus v from the plane of XY, or from ω the polar focus $=$ . . . . $\frac{bc\eta}{a}$.

(λ) The portion between the plane of XY and the umbilical plane (Δ) measured along the semiparameter is $=$ . . . . $\frac{bc}{a\eta}$.

(μ) Inclination of umbilical diameter u to plane of XY being ψ, $\tan\psi=$ $\frac{cb\eta}{a^2e}$.

(ν) Inclination to plane of XY of the tangent plane passing through polar directrix and touching (Σ) at extremity of principal parameter being j, $\tan j=$ . . . . $\frac{ce}{b}$.

(o) The coordinates of the umbilicus are . . . . . . $\bar{x}=\frac{ae}{\epsilon}$, $\bar{z}=\frac{bc\eta}{\epsilon}$.

(π) The length of the umbilical normal $=$ . . . . $\frac{cb}{a}$, and is therefore equal to the principal semiparameter L of (Σ).

(ρ) The radius of circular section of (Σ) whose plane passes through the umbilical focus is $=$ . . . . $\frac{bc}{a}$, and is therefore equal to the principal semiparameter L.

(σ) The radius of circular section of (Σ) whose plane passes through the polar focus ω is . . . . $r^2=\frac{b^4}{a^2}\left[\frac{e^2+\eta^2}{\epsilon^2}\right]$.

(τ) The squared reciprocal of perpendicular from the polar focus ω on the tangent plane at the extremity of the principal parameter L is . . . . $\frac{a^2}{b^2}\left[\frac{1}{c^2}+\frac{1}{b}-\frac{1}{a^2}\right]$.

(υ) Distance between the points on the axes of X and Z in which they are cut by the umbilical plane (Δ) $=$ . . . . $\frac{a\epsilon}{e\eta}$.

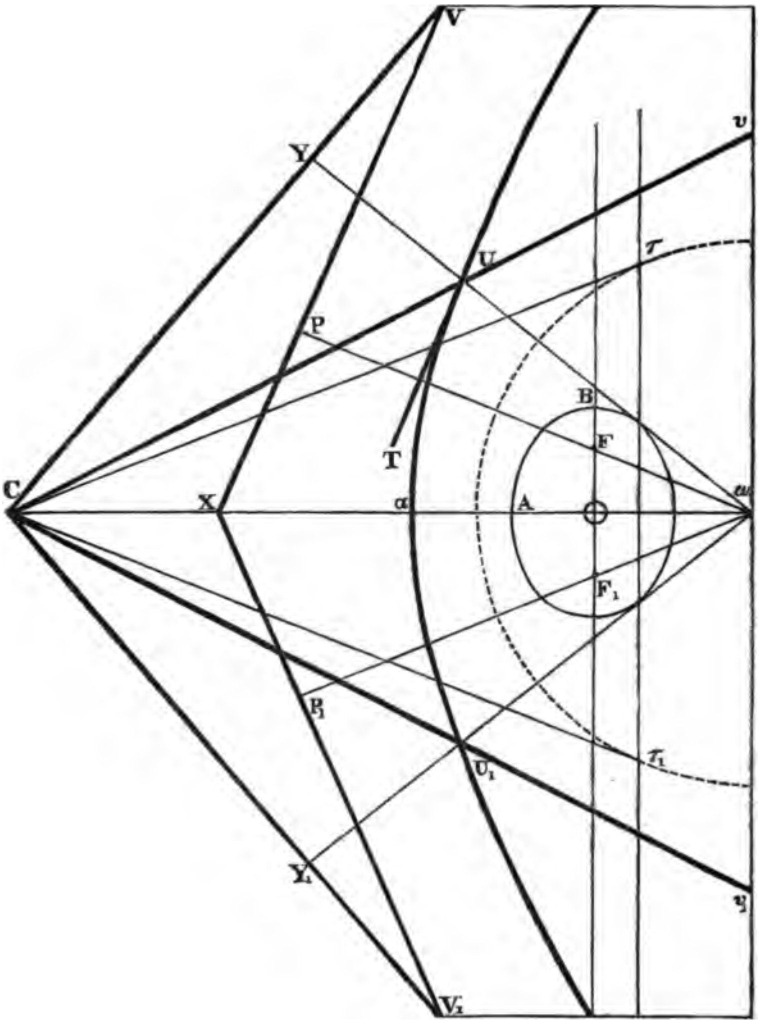
(ϕ) A plane being drawn through the umbilical focus v parallel to the plane of XY, the principal axes of this section will be u and $\frac{bu}{a}$

(ψ) The distance of the foot of the umbilical normal from the centre of (Σ) is $ae\epsilon$.

(ω) The distances of the feet of the principal normals to the vertices of the principal sections of (Σ) passing through the axis of X are $a\epsilon^2$ and ae^2 respectively. Therefore the distance of the feet of the three normals from the centre are in geometrical progression, or the distance of the foot of the umbilical normal from the centre is a mean proportion between the distances from the centre of the feet of the normals of the vertices of the principal sections.

249.] ON THE HYPERBOLOID OF TWO SHEETS AND ITS ASYMPTOTIC CONE.

Fig. 50.



When ω, the centre of the polarizing sphere (Ω), is outside the primitive surface (S), D is greater than A, and the reciprocal polar (Σ) becomes a discontinuous hyperboloid, while the reciprocal of the plane curve of contact of the cone whose vertex is ω, with the surface (S) becomes the asymptotic cone of the reciprocal polar (Σ).

The cone whose vertex is at ω, and whose base is the common section of (S) with it, is manifestly supplemental to the cone the reciprocal polar of this section, whose vertex is at C. The former may be called the *polar cone*, the latter the *asymptotic cone*.

When a cone envelopes a surface of revolution (S), the lines drawn from the vertex of this cone to the foci of the surface (S) are called the *focal lines*[*] of this cone; therefore the planes of the circular sections of the supplemental cone are at right angles to these focals; hence the planes of the circular sections of the asymptotic cone are at right angles to these focals: but the umbilical directrix planes (Δ) and ($\Delta_{,}$) are also at right angles to these focal lines; and therefore the planes of the circular sections of (Σ) which are parallel to the umbilical directrix planes are also perpendicular to the focal lines of (S); consequently the planes of the circular sections of the asymptotic cone are parallel to the circular sections of (Σ).

250.] The centre C of the surface (Σ) may be thus found. Generally, *the polar plane of* ω, *the centre of* (Ω) *with respect to* (S), *is the polar plane of* C, *the centre of* (Σ) *with respect to* (Ω).

For the plane $\tau\tau_{,}$, fig. 50, is the polar plane of ω with respect to (S), while it is the polar plane of C with respect to (Ω).

The plane of contact of the polar cone with the surface (S) divides it into two segments, the *remoter* one of which is the reciprocal polar of the sheet of (Σ), of which ω is the focus, while the *nearer* segment of (S) is the reciprocal polar of the *remoter* sheet of (Σ).

It is worthy also of remark that when ω, the polar focus, is *within* the surface (S), see fig. 48, the point V is the common vertex of two cones circumscribing (S) and (Ω). The plane of contact of the former will determine the polar focus ω; and the plane of contact of the latter with (Ω) will determine C, the centre of (Σ). But when ω is *without* the surface (S), the two cones will have a common base, and their vertices will be at the centre and focus of (Σ).

251.] *The reciprocal polar of the continuous hyperboloid is also a continuous hyperboloid.*

---

[*] That the lines drawn from the vertex of a cone to the foci of a surface of revolution which it circumscribes are the *focals* of the cone may be thus simply shown. Let f and $f_{,}$ be the lines drawn from the vertex of the cone to these foci; let p and $p_{,}$ be the perpendiculars let fall from these foci on the tangent plane to the cone. Let ϕ and $\phi_{,}$ be the angles which the lines f and $f_{,}$ make with the tangent plane. Then we shall evidently have $ff_{,} \sin\phi \sin\phi_{,} = pp_{,} = b^2$. So also for any other like tangent plane,

$$ff_{,} \sin\psi \sin\psi_{,} = pp_{,} = b^2;$$

consequently $\sin\phi \sin\phi_{,} = \sin\psi \sin\psi_{,}$, or the product of the sines of the angles which $f, f_{,}$ make with any tangent plane to the cone is constant—a well-known property of the *focals* of a cone.

For if we take the continuous hyperboloid whose principal section in the plane of XY is a circle, and if we assume in this plane a point ω the centre of the polarizing sphere (Ω), the reciprocal polar of this circle will be a conic section having its focus at the point ω; and the conjugate polars of every pair of generatrices of the primitive hyperboloid (U) will be a pair of generatrices on the reciprocal polar (T), a continuous hyperboloid also, whose principal section in the plane of XY will be a conic section.

When the polar focus ω is taken on the circumference of the circle which is the principal section of (U) in the plane of XY, then the reciprocal section of this circle is a parabola, and the polar reciprocal (T) becomes a hyperbolic paraboloid.

On the oblate spheroid.

252.] When the polar focus ω is assumed to coincide with C the centre of (S), then $D=0$, and consequently $a=\frac{R^2}{A}$, $b=\frac{R^2}{A}$, and $c=\frac{R^2}{C}$; and also $e=0$, $\epsilon^2=\eta^2=\frac{a^2-c^2}{a^2}$.

Since $\cos i=\frac{\eta}{\epsilon}=1$, $i=0$; in this case therefore the umbilical planes are parallel to the plane of XY, the plane of the principal circular section, at the distance $\frac{c}{\epsilon}$ from the origin; the polar directrix in which the two umbilical directrix planes intersect vanishes to infinity, and the two umbilical foci are on the axis of Z, at the distance $c\epsilon$ from the plane of XY.

CHAPTER XXVI.

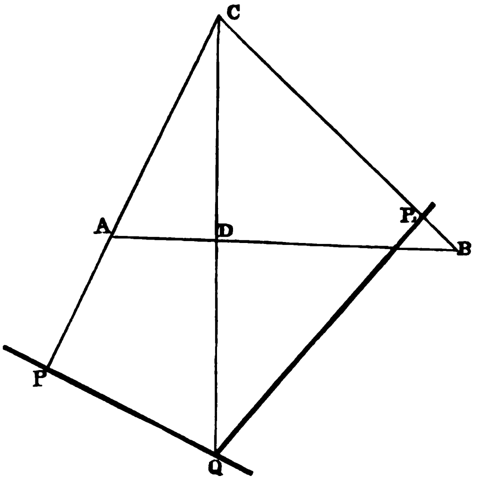
ON THE APPLICATION OF THE THEORY DEVELOPED IN THE PRECEDING CHAPTER TO THE DISCUSSION OF SOME THEOREMS AND PROBLEMS.

253.] It will be well to premise the following lemmas before we proceed to apply the principles established in the foregoing pages.

LEMMA I.

To express the base A B of the triangle A B C in terms of the perpendiculars on the polars of the points A, B, and the distance of the pole of A B from C. Let the angle A C B be $=\theta$; let C be the polarizing centre. Then

Fig. 51.



$$AB=\frac{CA\,.\,CB\,.\sin\theta}{CD};\ \text{but}\ CA=\frac{R^2}{p},\ CB=\frac{R^2}{p_{\prime}},\ \text{and}\ CD=\frac{R^2}{r};$$

consequently

$$AB=\frac{R^2 r\sin\theta}{pp_{\prime}}.$$

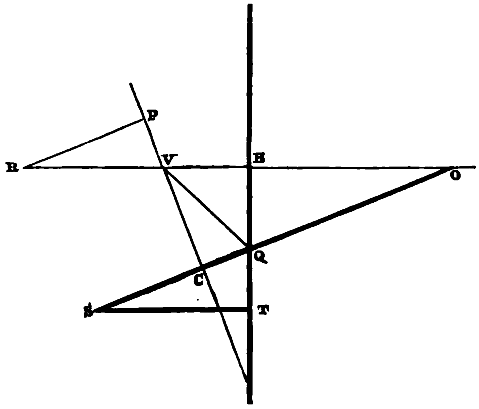
LEMMA II.

From a given point R let a perpendicular R P be let fall on the given straight line C V, and let the point O be assumed as the centre of the polarizing circle whose radius is R.

Let B Q be the polar of the point R, and S the pole of the given straight line C V. From S let fall a perpendicular S T on the polar of R. It may easily be shown that the ratio of the lines drawn to the poles R, S from the centre O of the polarizing circle will be the same as the ratio of the perpendiculars let fall from these points on the polar straight lines; for

$$\frac{\mathrm{OR}}{\mathrm{OS}}=\frac{\mathrm{R}^2}{\mathrm{OB}}\div\frac{\mathrm{R}^2}{\mathrm{OC}}=\frac{\mathrm{OC}}{\mathrm{OB}}=\frac{\mathrm{OV}}{\mathrm{OQ}}=\frac{\mathrm{OR}-\mathrm{OV}}{\mathrm{OS}-\mathrm{OQ}}=\frac{\mathrm{RV}}{\mathrm{SQ}}=\frac{\mathrm{RP}}{\mathrm{ST}} \text{ or } \frac{\mathrm{OR}}{\mathrm{RP}}=\frac{\mathrm{OS}}{\mathrm{ST}}.$$

Fig. 52.



The proposition is equally true when C V P and T Q B are planes. For through O R P draw a plane, and through C V and Q B planes perpendicular to the plane O R P, the above demonstration will still hold.

254.] *If from any point on an umbilical surface of the second order perpendiculars be let fall on two conjugate umbilical directrix planes, the rectangle under these perpendiculars will have to the square of the distance of this point from the polar focus a constant ratio.*

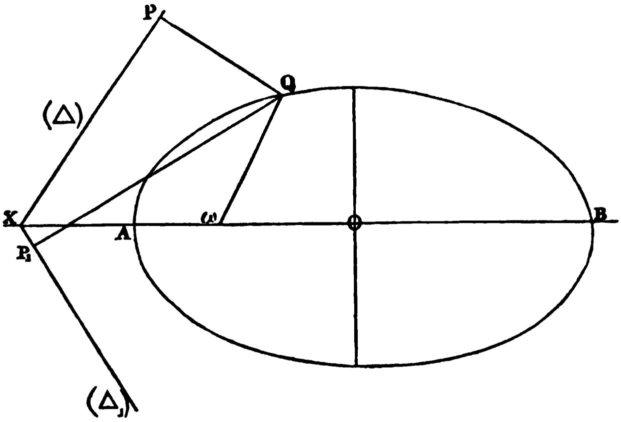
It has been shown by M. Chasles and other geometers—indeed it follows obviously from the corresponding property *in plano*—that if from the foci of an ellipsoid of revolution perpendiculars be let fall on a tangent plane, their product will be equal to the square of the semi- minor axis of the ellipsoid.

Let the reciprocal polar of this property be taken. The pole of the tangent plane (T) to (S) will be a point Q on (Σ); the polar planes of the two foci F, $F_{\prime}$ of (S) will be the conjugate umbilical directrix planes (Δ), ($\Delta_{\prime}$) to (Σ); hence the perpendiculars are let fall from *one* point Q on the surface (Σ) to its two directrix planes (Δ) and ($\Delta_{\prime}$).

But the distance of the focus of (S) from the polar centre ω is

$\sqrt{C^2+D^2-A^2}$, see sec. [238]; and the product of the perpendiculars is, as we have assumed, equal to A^2.

Fig. 53.



Consequently the ratio of the product of the two perpendiculars from the foci of (S) on a tangent plane to it, to the product of the distances of the two foci of (S) from the polar centre, is $=\frac{A^2}{C^2+D^2-A^2}$.

But from the preceding Lemma II. it follows that this must also be the ratio of the product of two perpendiculars let fall from the point Q on the directrix planes (Δ), $(\Delta_{,})$ to the square of the radius r drawn from Q to the polar focus ω. Putting p and $p_{,}$ for these perpendiculars, we shall have

$$\frac{pp_{,}}{r^2}=\frac{A^2}{C^2+D^2-A^2}. \quad \text{. . . . .} \quad \text{(a)}$$

We may determine the value of this expression in terms of a, b, c by the help of the formulæ in sec. [237], (c); hence

$$\frac{A^2}{C^2+D^2-A^2}=\frac{a^2c^2}{b^2(a^2-c^2)}=\frac{c^2}{b^2\epsilon^2}=\frac{a^2c^2}{b^2\epsilon^2.a^2}.$$

Now it has been shown in sec. [243], (c), that the perpendicular from the centre on one of the conjugate umbilical directrix planes is $\frac{ac}{b\epsilon}$. Let this perpendicular be P.

Then we shall have, finally, $\frac{pp_{,}}{r^2}=\frac{P^2}{a^2}$. . . . . (b) (α) Resuming the equation $\frac{pp_{\prime}}{r^2}=\frac{a^2c^2}{b^2(a^2-c^2)}$, let the surface be the elliptic paraboloid; let the semiparameters of the principal sections be $2k$ and $2k_{\prime}$, then $b^2=2ak$, $c^2=2ak_{\prime}$. Substituting these values in the preceding expression, we shall have

$$\frac{a^2c^2}{b^2(a^2-c^2)}=\frac{k_{\prime}}{k-\frac{kk_{\prime}}{2a}}=\frac{k_{\prime}}{k},$$

since a in this case is infinite.

Therefore in the elliptic paraboloid $\frac{pp_{\prime}}{r^2}=\frac{k_{\prime}}{k}$. . . . . . (c)

(β) When $b=c$ the surface (Σ) becomes an ellipsoid of revolution round the transverse axis, and $\frac{a^2c^2}{b^2(a^2-c^2)}$ becomes $\frac{a^2}{a^2-b^2}=\frac{1}{e^2}$, a well-known property of the ellipsoid of revolution.

(γ) From the polar focus ω let a sphere be described with a constant radius r, it will cut the ellipsoid (Σ) in a curve of double curvature; so that if from any point on this curve perpendiculars be let fall on the conjugate umbilical directrix planes, their product will be constant, since r is constant.

(δ) If we assume the theorem as established, $\frac{pp_{\prime}}{r^2}=\frac{a^2c^2}{b^2(a^2-c^2)}$, and make this latter expression $=1$, we shall have $c=\frac{ab}{\sqrt{a^2+b^2}}$.

Hence it follows that if the three semiaxes of an ellipsoid be a, b, and $\frac{ab}{\sqrt{a^2+b^2}}$, the product of the pair of perpendiculars let fall from any point on the surface to the two umbilical directrix planes will be equal to the square of the distance of this point from the polar focus.

(ε) The above relation is equivalent to $\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}=0$. When b is also $=c$, $\frac{1}{a^2}=0$, or $a=\infty$; or in order that this property may hold in a surface of revolution, it must be a paraboloid*.

---

* A simple algebraical proof may be given of this very general and important theorem.

It has been shown in sec. [57], (c), that the length of a perpendicular let fall on a plane whose tangential coordinates are ξ, υ, ζ, from a given point (x, y, z), is

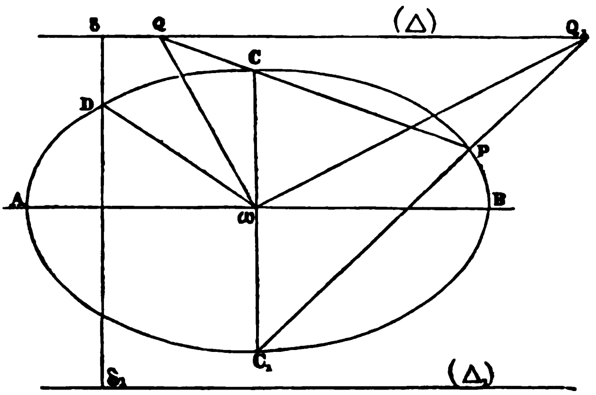
$$p=\frac{1-x\xi-y\upsilon-z\zeta}{\sqrt{\xi^2+\upsilon^2+\zeta^2}}.$$

Now the tangential coordinates of one of the conjugate directrix planes (Δ) are

$$\xi=-\frac{ae}{b^2},\ \upsilon=0,\ \zeta=\frac{a\eta}{cb}.$$

255.] When $a=b$, $D=0$, since $D^2=\frac{R^4}{b^4}(a^2-b^2)$; therefore the polar focus ω becomes the centre of (Σ), which becomes an oblate spheroid having its principal circular sections coincident and in the plane of XY. Consequently the four umbilical directrix planes coalesce, two by two, parallel to the plane of XY, and therefore parallel to one another. These planes may be called the *minor directrix planes*, seeing that they are perpendicular to the minor axis of the oblate spheroid. The perpendiculars let fall from any point on this surface on the minor directrix planes are evidently in the same straight line. Assuming the general equation

Fig. 54.



$$\frac{pp_{\prime}}{r^2}=\frac{a^2c^2}{b^2(a^2-c^2)}=\frac{c^2}{a^2-c^2} \text{ or } \frac{pp_{\prime}}{r^2}=\frac{c^2}{a^2\epsilon^2}, \text{ or } \frac{D\delta\times D\delta_{\prime}}{D\omega^2}=\frac{h^2}{a^2}.$$

Substituting these values in the preceding expression,

$$p=\frac{b^2c+acex-ab\eta z}{b\sqrt{a^2-c^2}}.$$

In the same way we shall find for the perpendicular let fall on the other conjugate directrix plane,

$$p_{\prime}=\frac{b^2c+acex+ab\eta z}{b\sqrt{a^2-c^2}}.$$

Hence

$$b^2(a^2-c^2)pp_{\prime}=b^4c^2+2b^2c^2aex+a^2c^2e^2x^2-a^2b^2\eta^2z^2.$$

The equation of the surface, the origin being at the focus in the plane of XY, or at the *polar focus*, when multiplied by $a^2b^2c^2$, is

$$b^2c^2x^2+a^2c^2y^2+a^2b^2z^2-2b^2c^2aex-b^4c^2=0;$$

adding this expression to the preceding, we get

$$\frac{pp_{\prime}}{x^2+y^2+z^2}=\frac{a^2c^2}{b^2(a^2-c^2)}.$$

Now $\frac{c}{\epsilon}$ is the distance of the minor directrix plane from the centre $=h$ suppose; consequently $\frac{pp_{\prime}}{r^2}=\frac{h^2}{a^2}$.

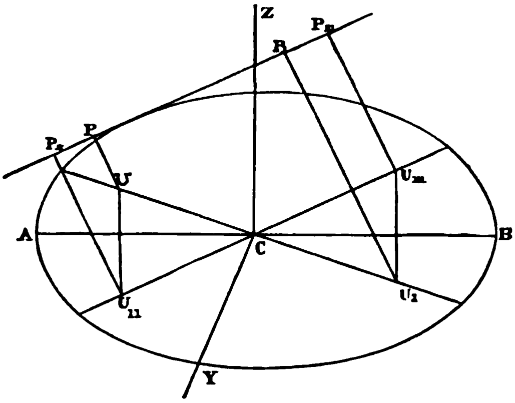
When $c^2=a^2-c^2$, or $2c^2=a^2$, we shall have $pp_{\prime}=r^2$.

256.] *The sum of the products, taken two by two, of the perpendiculars let fall from the four umbilical foci on a tangent plane to* (Σ) *is equal to* $2b^2\sin^2\nu+\frac{2c^2u^2}{a^2}\cos^2\nu$.

We have shown in sec. [57], (c), that if P be the perpendicular let fall from a point whose projective coordinates are x, y, z on a plane whose tangential coordinates are ξ, v, ζ, we shall have

$$\mathrm{P}=\frac{1-x\xi-yv-z\zeta}{\sqrt{\xi^2+v^2+\zeta^2}}.$$

Fig. 55.



Now the projective coordinates of the four umbilical foci are

$$\left.\begin{aligned}x&=-ae,\\ y&=\ \ 0,\\ z&=\ \ \frac{bc\eta}{a},\end{aligned}\right\}\quad\left.\begin{aligned}x&=\ \ ae,\\ y&=\ \ 0,\\ z&=-\frac{bc\eta}{a},\end{aligned}\right\}\quad\left.\begin{aligned}x&=-ae,\\ y&=\ \ 0,\\ z&=-\frac{bc\eta}{a},\end{aligned}\right\}\quad\left.\begin{aligned}x&=ae,\\ y&=0,\\ z&=\frac{bc\eta}{a}.\end{aligned}\right\}\quad .\ .\quad \text{(a)}$$

Consequently the perpendiculars are

$$\left.\begin{aligned}a\mathrm{P}&=\frac{a+a^2e\xi-bc\eta}{\sqrt{\xi^2+v^2+\zeta^2}},\quad a\mathrm{P}_{\prime}=\frac{a-a^2e\xi+bc\eta}{\sqrt{\xi^2+v^2+\zeta^2}},\\ ap&=\frac{a+a^2e\xi+b^2c\eta}{\sqrt{\xi^2+v^2+\zeta^2}},\quad ap_{\prime}=\frac{a-a^2e\xi-bc\eta\zeta}{\sqrt{\xi^2+v^2+\zeta^2}}.\end{aligned}\right\}\quad .\ .\quad \text{(b)}$$

Therefore $$a^2\mathrm{PP}_{\prime}+a^2pp_{\prime}=\frac{2a^2-2a^4e^2\xi^2-2b^2c^2\eta^2}{\xi^2+\upsilon^2+\zeta^2}\,; \quad . \quad . \quad (c)$$

and if we add to the numerator of this fraction the tangential equation of the surface multiplied by $2a^2$,

$$2a^4\xi^2+2a^2b^2\upsilon^2+2a^2c^2\zeta^2-2a^2=0,$$

we shall finally, on reducing, have

$$\mathrm{PP}_{\prime}+pp_{\prime}=2\left[b^2\sin^2\nu+\frac{c^2u^2}{a^2}\cos^2\nu\right]. \quad . \quad . \quad . \quad . \quad (d)$$

In this formula u is the umbilical semidiameter, and ν is the angle which the parallel perpendiculars make with the axis of Z. When $b=c$, $u=a$, $\mathrm{P}=p$, $\mathrm{P}_{\prime}=p_{\prime}$, and the formula becomes $\mathrm{PP}_{\prime}=b^2$, a well-known expression for the product of the perpendiculars from the foci of a surface of revolution on a tangent plane. It is remarkable that the sum of the products of these perpendiculars varies only with the value of ν, and is entirely independent of the values of λ and μ; λ, μ, and ν being the angles which the perpendicular from the centre on the tangent plane makes with the axis of coordinates. Consequently, if round the axis of Z passing through the centre of (Σ) we describe a right cone whose vertical angle is 2ν, and the four perpendiculars from the umbilical foci on a tangent plane be all drawn parallel to a side of this central cone, the sum of their products, two by two, will be constant, since ν is constant. See (d).

257.] The locus of the feet of perpendiculars let fall from the foci of (S), a surface of revolution, on a tangent plane to this surface is a sphere.

Taking the reciprocal polar of this property, we may infer that if through the polar focus of (Σ), a surface of the second order, a plane and a straight line be drawn at right angles to each other, the line meeting the surface in the point τ, and the umbilical directrix plane (Δ) in the straight line (δ), the plane which passes through the point τ and the straight line (δ) will envelop a surface of revolution ($\Sigma_{\prime}$) whose focus will coincide with the polar focus of (Σ), and whose directrix plane will pass through the polar directrix of the polar focus ω in the principal section in the plane of XY.

Let (T) be the tangent plane to (S); and let ϖ, $\varpi_{\prime}$ be the feet of the perpendiculars let fall from the foci F, $\mathrm{F}_{\prime}$ on the tangent plane (T), aud let (Π), the polar plane of ϖ, cut the umbilical directrix plane in the straight line (δ).

Now the pole τ of (T) the tangent plane *to* (S), will be a point *on* the surface (Σ); and as ϖ is a point on the tangent plane (T), its polar plane (Π) will pass through τ, the pole of (T); and as ϖ is a point on one of the focal perpendiculars, the polar plane (Π) of ϖ will pass through the conjugate polar of this perpendicular, which is the intersection of the planes (Π) and (Δ), the polar planes of ϖ and F. But as the point ϖ is always found on the surface of a sphere, its polar plane $(\delta)\tau$ must envelop a surface of revolution $(S_{\prime})$ whose focus coincides with ω, the polar focus. Since $\varpi_{\prime}$, the foot of the other perpendicular, is also on the tangent plane (T) to (S), its polar plane $(\Pi_{\prime})$ will also pass through τ and intersect the conjugate umbilical directrix plane $(\Delta_{\prime})$ in $(\delta_{\prime})$; and as the plane which passes through $(\delta_{\prime})$ and ω is also perpendicular to the line $\omega\tau$, it will follow that the two planes through ω and the lines (δ), $(\delta_{\prime})$ are identical.

When (Σ) becomes an oblate spheroid, the polar focus ω becomes the centre of (Σ); and hence follows this theorem:—

Through the centre ω of an oblate spheroid let a diameter and a diametral plane be drawn at right angles one to the other, the diameter meeting the surface (Σ) in the point τ, while the diametral plane meets the directrix plane (Δ) in the straight line (δ), the plane $\tau(\delta)$ will envelop a concentric sphere whose radius is c.

258.] The algebraical proof of this theorem is somewhat complicated. To afford a means of comparing it with the geometrical method, it may be thus given.

The origin being taken at the polar focus, and the axes of coordinates parallel to the axes of the figure, let the projective equation of the surface be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{2ex}{a}=\frac{b^2}{a^2};$$

or if $(x_{\prime}y_{\prime}z_{\prime})$ be the projective coordinates of the point on the surface, and we multiply the preceding equation by $a^2b^2c^2$, we shall have

$$b^2c^2x_{\prime}^2+a^2c^2y_{\prime}^2+a^2b^2z_{\prime}^2=b^4c^2+2b^4c^2aex_{\prime}. \quad . \quad . \quad \text{(a)}$$

Let $$\xi=\mu\zeta+\alpha, \quad \upsilon=\nu\zeta+\beta \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

be the tangential equations of the straight line in which the planes (Π) and (Δ) intersect; then, as this straight line (δ) lies in a plane passing through the origin, we shall have

$$\mu=\frac{\xi}{\zeta}=\frac{x_{\prime}}{z_{\prime}}, \text{ and } \nu=\frac{\upsilon}{\zeta}=\frac{y_{\prime}}{z_{\prime}}; \text{ see sec. [59].}$$

Consequently $z_{\prime}\xi=x_{\prime}\zeta+\alpha z_{\prime}$, and $z_{\prime}\upsilon=y_{\prime}\zeta+\beta z_{\prime}$.

Now, as the line (b) lies in the umbilical directrix plane (Δ), of which the coordinates are

$$\xi=-\frac{ae}{b^2}, \quad \upsilon=0, \quad \zeta=\frac{a\eta}{bc},$$

substituting these values of ξ, υ, and ζ in (b) and reducing, we shall find

$$\left.\begin{aligned} x_{\prime}b(bc\zeta-a\eta)&=z_{\prime}c(b^2\xi+ae), \\ \text{and}\quad y_{\prime}(bc\zeta-a\eta)&=z_{\prime}cb\upsilon. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

As the plane, whose tangential equations are required, passes through the point of which the projective coordinates are $x_{\prime}\, y_{\prime}\, z_{\prime}$, and whose tangential coordinates are ξ, υ, ζ, we must have

$$x_{\prime}\xi + y_{\prime}\upsilon + z_{\prime}\zeta = 1. \quad \ldots\ldots \quad \text{(d)}$$

Now substituting in this equation the values of $x_{\prime}\, y_{\prime}\, z_{\prime}$ given in the preceding equations, we shall have, for the values of $z_{\prime}$, $x_{\prime}$, and $y_{\prime}$,

$$\left.\begin{aligned} z_{\prime} &= \frac{b(bc\zeta - a\eta)}{b^2c(\xi^2+\upsilon^2+\zeta^2)+ace\xi-ab\eta\zeta}, \\ x_{\prime} &= \frac{c(b^2\xi + ae)}{b^2c(\xi^2+\upsilon^2+\zeta^2)+ace\xi-ab\eta\zeta}, \\ y_{\prime} &= \frac{b^2c\upsilon}{b^2c(\xi^2+\upsilon^2+\zeta^2)+ace\xi-ab\eta\zeta}. \end{aligned}\right\} \quad \ldots\ldots \quad \text{(c)}$$

If we now substitute these values of $x_{\prime}\, y_{\prime}\, z_{\prime}$ in (a), the projective equation of the surface (Σ), we shall have, first writing

$$\mathrm{V} = b^2c^2(\xi^2+\upsilon^2+\zeta^2) + 2c^2ae\xi - a^2\eta^2 - c^2, \quad \ldots \quad \text{(f)}$$

and making these substitutions, for the resulting equation

$$\left.\begin{aligned} &b^4c^4\xi^2 + 2b^2c^4ae\xi + c^4a^2e^2 + a^2c^4b^2\upsilon^2 + a^2b^4c^2\zeta^2 - 2b^3ca^3\eta\zeta + a^4b^2\eta^2 \\ &= b^2\mathrm{V}^2 + 2b^2\mathrm{V}(a^2\eta^2 + c^2 - c^2ae\xi - abc\eta\zeta) + a^4b^2\eta^4 + 2a^2b^2c^2\eta^2 + b^2c^4 \\ &- 2b^2c^2a^3\eta^2e\xi - 2b^2c^4ae\xi + a^2b^2c^4e^2\xi^2 - 2a^3b^3c\eta^3\zeta - 2b^3c^3a\eta\zeta \\ &+ 2a^2b^3c^3e\eta\xi\zeta + a^2b^4c^2\eta^2\zeta^2 + 2\mathrm{V}b^2c^2ae\xi + 2b^2c^2a^3e\eta^2\xi \\ &+ 2b^2c^4ae\xi - 2b^2c^4a^2e^2\xi^2 - 2b^3c^3a^2e\eta\xi\zeta + 2\mathrm{V}a^2c^2e^2 + 2a^4c^2e^2\eta^2 \\ &+ 2a^2c^4e^2 - 2a^3c^4e^3\xi - 2a^3bc^3e^2\eta\zeta. \end{aligned}\right\} \text{(g)}$$

Eliminating, combining, and reducing, the preceding equation becomes

$$\mathrm{V}(2bc\eta\zeta - c^2e\xi - a\epsilon^2) = 0; \quad \ldots\ldots \quad \text{(h)}$$

and this is satisfied by putting $\mathrm{V}=0$, or

$$b^2c^2(\xi^2+\upsilon^2+\zeta^2) + 2ac^2e\xi = a^2\eta^2 + c^2. \quad \ldots \quad \text{(i)}$$

If we refer to sec. [105], (i), and apply the principles there laid down, we shall find, putting A and B for the principal semiaxes of this surface,

$$\mathrm{A} = \frac{abc}{a^2\eta^2 + c^2} \text{ and } \mathrm{B}^2 = \frac{b^2c^2}{a^2\eta^2+c^2}; \quad \ldots \quad \text{(j)}$$

consequently $\frac{\mathrm{B}^2}{\mathrm{A}} = \frac{bc}{a}$, or the principal parameters of the enveloped surface ($\mathrm{S}_{\prime}$) and the reciprocal polar (Σ) of (S) are equal.

Let e be the eccentricity of this enveloped surface of revolution, we shall find

$$\mathrm{e} = \frac{c}{b}e, \text{ or } \frac{\mathrm{e}^2}{c^2} = \frac{\epsilon^2}{b^2} - \frac{\eta^2}{a^2}, \quad \ldots\ldots \quad \text{(k)}$$

e being the eccentricity of the principal section of (Σ) in the plane of XY.

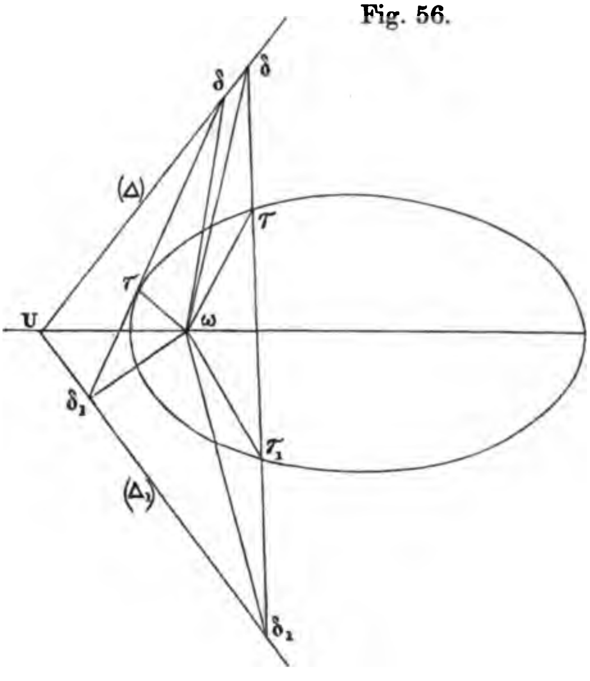
When $a=b$, or the surface is an oblate spheroid, $A=B$, or the enveloped surface is a sphere.

259.] Through a given straight line four planes are drawn, two as tangents to a surface of revolution (S); the other two are drawn through the foci of this surface; it is known that the angles between each tangent plane and focal plane are equal.

The conjugate polar of the straight line in which the four planes intersect will be a chord of the surface (Σ) on which the poles of these four planes will lie; and as two of the planes pass through the foci of (S), their poles δ and $\delta_{,}$ will lie on the two umbilical directrix planes (Δ) and $(\Delta_{,})$; and as the two other planes are tangent planes to (S), their poles τ and $\tau_{,}$ will be on the surface of (Σ).

Hence, *if in any surface* (Σ) *having three unequal axes a straight line be drawn meeting the surface in two points* τ *and* $\tau_{,}$ *and the umbilical directrix planes* (Δ) *and* $(\Delta_{,})$ *in two points also,* δ *and* $\delta_{,}$, *the angles which each pair of points subtend at the polar focus* ω *will be equal to each other, or the angle* $\tau\omega\delta$ *will be equal to the angle* $\tau_{,}\omega\delta_{,}$.

Fig. 56.



When the secant line $\tau\tau_{,}\delta\delta_{,}$ becomes a tangent to the surface (Σ), the two points τ and $\tau_{\prime}$ coalesce into one τ, and we obtain this other remarkable theorem:—

If we draw any straight line touching the surface (Σ) *and meeting the umbilical directrix planes* (Δ) *and* ($\Delta_{\prime}$) *in two points* δ *and* $\delta_{\prime\prime}$ *the segments of this line between the point of contact* τ *and the umbilical directrix planes will subtend equal angles at the polar focus* ω.

When (Σ) becomes an oblate spheroid, the umbilical directrix planes (Δ) and ($\Delta_{\prime}$) become parallel to the plane of XY, which contains the principal circular section of (Σ). The polar focus ω coincides with its centre. Therefore, if a straight line be drawn touching an oblate spheroid, the segments of this line between the point of contact and the directrix planes (Δ) and ($\Delta_{\prime}$) will subtend equal angles at the centre of (Σ).

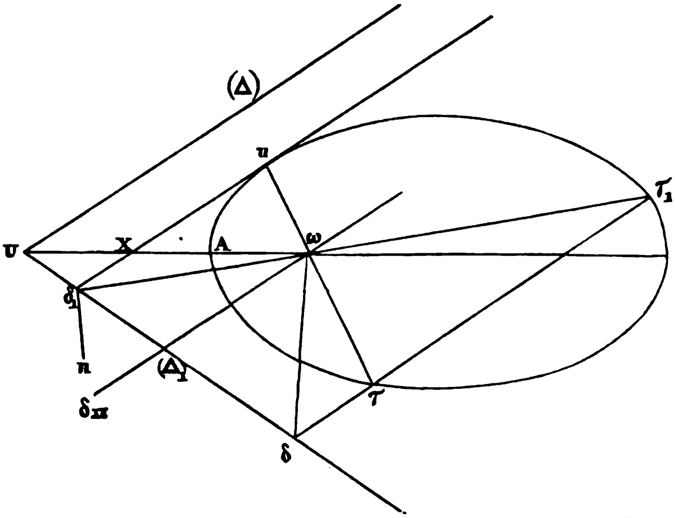
260.] Should the secant line be drawn parallel to one of the umbilical directrix planes, (Δ) suppose, and meeting the other umbilical directrix plane ($\Delta_{\prime}$) in δ, and the surface in the points τ and $\tau_{\prime\prime}$ the rectangle

$$\delta\tau \times \delta\tau_{\prime} = \overline{\delta\omega}^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

ω being the polar focus. See fig. 57.

Hence this theorem:—*If through any point* δ *assumed on one of the directrix planes* ($\Delta_{\prime}$) *a secant be drawn parallel to the other directrix plane* (Δ), *and meeting the surface in the points* τ *and* $\tau_{\prime\prime}$ *the rectangle under these segments will be equal to the square of the distance of* δ *from the polar focus* ω.

Fig. 57.



If the secant be drawn through the umbilicus, it will become a tangent to the surface. Consequently if through u the umbilicus of the surface (Σ) a tangent be drawn to meet the umbilical directrix plane $(\Delta_{\prime})$ in $\delta_{\prime}$, the distances of the point $\delta_{\prime}$ from the umbilicus u and the polar focus ω will be equal.

A direct geometrical proof of this theorem may be given. Let $\omega\tau\tau_{\prime}$ be a section of (Σ), a surface of the second order having three unequal axes. Let $U\Delta$ and $U\Delta_{\prime}$ be the umbilical directrix planes cutting the plane of XY in the polar directrix UV. Let un be a tangent plane through the umbilicus u parallel to (Δ) and cutting $(\Delta_{\prime})$ in the line $n\delta_{\prime}$. Let the secant $\tau\,\tau_{\prime}$ be drawn parallel to the plane (Δ), and therefore meeting this plane in $\delta_{\prime\prime}$, a point at infinity, and the other umbilical plane $(\Delta_{\prime})$ in the point δ.

Now, as by the preceding part of the proposition the lines $\omega\delta$ and $\omega\delta_{\prime\prime}$ drawn to meet the secant $\tau\tau_{\prime}\delta\delta_{\prime}$ in the umbilical directrix planes (Δ) and $(\Delta_{\prime})$ make equal angles with the lines $\omega\tau$ and $\omega\tau_{\prime}$, we shall have the angle $\delta\omega\tau=$ angle $\delta\tau_{\prime}\omega$; or the triangles $\delta\omega\tau$ and $\delta\omega\tau_{\prime}$ are similar, or $\delta\tau : \delta\omega :: \delta\omega : \delta\tau_{\prime}$, or $\delta\tau \times \delta\tau_{\prime} = \overline{\delta\omega}^2$. . . . . . (b)

When the secant $\tau\tau_{\prime}\delta\delta_{\prime}$ is drawn in the umbilical tangent plane $u\delta_{\prime}n$, then $\delta\tau \times \delta\tau_{\prime}$ evidently becomes $\overline{nu}^2$; but it is also equal to $\overline{n\omega}^2$; hence $nu = n\omega$. . . . . (c)

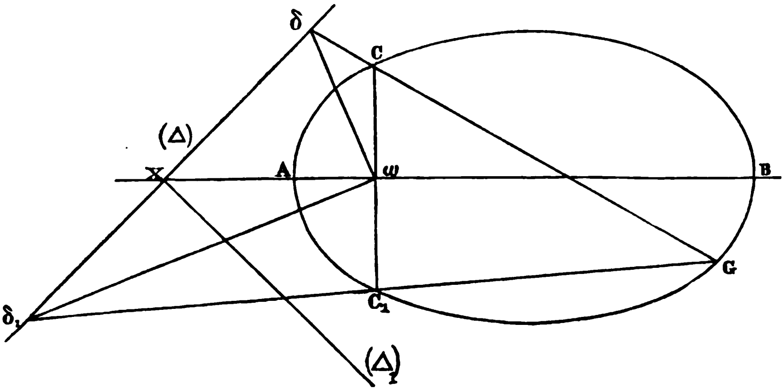
We may also from this theorem conclude that when any point δ is taken on one of the umbilical directrix planes of a cone, *i. e.* the planes drawn through the vertex parallel to the circular sections of the cone, and from this point a line be drawn parallel to the other circular section, and meeting the cone in the points τ and $\tau_{\prime}$, the rectangle $\delta\tau \times \delta\tau_{\prime}$ will be equal to $\overline{\delta V}^2$, V being the vertex of the cone.

261.] Tangent planes (T) and $(T_{\prime})$ are drawn at the extremities of the major axis of (S). They are cut by a third tangent plane $(T_{\prime\prime})$ in two straight lines (l) and $(l_{\prime})$. The planes (V) and $(V_{\prime})$ passing through the focus F and the lines (l), $(l_{\prime})$ are at right angles, as is very generally known. We may thus polarize this theorem.

The poles of the tangent planes (T), $(T_{\prime})$ to the extremities of the major axis of (S) are the extremities τ, $\tau_{\prime}$ of the principal parameter L of (Σ). The pole of the third tangent plane $(T_{\prime\prime})$ is a point $\tau_{\prime\prime}$ on the surface of (Σ); and δ, the pole of (V), is a point on the umbilical directrix plane (Δ). But as the three planes (T), $(T_{\prime\prime})$, and (V) all pass through the same straight line (l), their poles τ, $\tau_{\prime\prime}$, and δ will all range on the same straight line $\tau_{\prime\prime}\,\tau\,\delta$, the conjugate polar of (l). In the same way the poles $\tau_{\prime\prime}$, $\tau_{\prime}$, and $\delta_{\prime}$ of the three planes $(T_{\prime})$, $(T_{\prime\prime})$, and $(V_{\prime})$ will all range on the same straight line $\tau_{\prime\prime}$, $\tau_{\prime}$, and $\delta_{\prime}$, the conjugate polar of $(l_{\prime})$. But as the focal planes are at right angles, their poles δ and $\delta_{\prime}$ on the umbilical plane (Δ) will subtend right angles at the polar focus ω. Hence we obtain this very general theorem :—

(α) *Let the principal parameter* $CC_{\prime}$ *of a surface of the second order* (Σ) *having three unequal axes be the base of a triangle whose vertex is the point* G *anywhere on the surface; and if the sides of this triangle be produced to meet the umbilical directrix plane* (Δ) *in two points* δ *and* $\delta_{\prime}$, *the points* δ *and* $\delta_{\prime}$ *will subtend a right angle at the polar focus* ω.

Fig. 58.



(β) When the surface (Σ) becomes an oblate spheroid, the umbilical directrix planes (Δ) and ($\Delta_{\prime}$) become parallel to the plane of XY, which contains the principal circular section of the oblate spheroid, and the polar focus coincides with the centre.

Hence this theorem, see fig. 54:—

If the shorter axis of an oblate spheroid be taken as the fixed base of a triangle $CC_{\prime}P$ *inscribed in the spheroid, whose sides are produced to meet one of the directrix planes* (Δ) *in two points* Q *and* $Q_{\prime}$, *these points,* Q *and* $Q_{\prime}$, *will subtend a right angle at the centre of the oblate spheroid* (Σ).

(γ) When the surface (Σ) becomes a surface of revolution (S) round the transverse axis, the polar focus becomes the focus of (S), the umbilical directrix plane becomes the directrix plane (D), whence we may derive this theorem:—

If the parameter of a surface of revolution (S) *be the base of a triangle inscribed in it, and the sides be produced to meet the directrix plane* (D) *in* d *and* $d_{\prime}$, *the points* d, $d_{\prime}$ *will subtend a right angle at the focus* F *of* (S).

(δ) We may extend this theorem to any angle. If two fixed tangent planes (T) and ($T_{\prime}$) be drawn to a surface of revolution (S), and a third tangent plane ($T_{\prime\prime}$) movable and which intersects the two fixed tangent planes in the straight lines (l) and ($l_{\prime}$), the vector planes (V) and ($V_{\prime}$) drawn through the focus F and the straight lines in which the tangent planes intersect are inclined at a constant angle to each other. This is a known theorem.

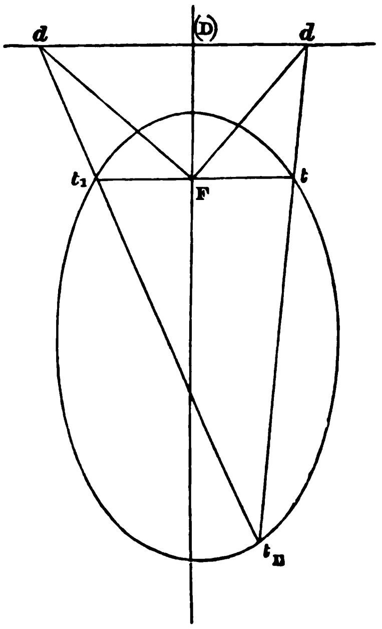
The poles of the vector planes (V) and $(V_{\prime})$, since they pass through F, are on the umbilical directrix plane (Δ); and as the planes (V), (T), and $(T_{\prime\prime})$ all intersect in the same straight line (l), their poles will be on the conjugate polar of this straight line—that is, on the chord which joins the poles τ and $\tau_{\prime\prime}$ on the surface (Σ); and as the pole of (V) is on this straight line, it must be the point δ where this line $\tau\tau_{\prime\prime}$ pierces the umbilical directrix plane (Δ). The same may be said of the other chord $\tau_{\prime\prime}\tau_{\prime}\delta_{\prime}$. Therefore the angle $\delta\omega\delta_{\prime}$, the angle between the plane (V) and $(V_{\prime})$, is constant. We may therefore infer that,

If a fixed chord be taken in a surface (Σ), *and this fixed chord* $c\,c_{\prime}$ *be made the base of a triangle whose vertex* G *(see fig.* 58*) is anywhere on this surface, the sides of this triangle* Gc, G$c_{\prime}$ *being produced to meet the umbilical plane* (Δ) *in two points* δ *and* $\delta_{\prime}$, *the points* δ *and* $\delta_{\prime}$ *will subtend a constant angle at the polar focus* ω.

262.] If we take the reciprocal polar of (γ) in the preceding section, we may derive another theorem equally new.

263.] Since the two points t and $t_{\prime}$, the extremities of the parameter of (S), and the focus F of (S) in fig. 59 are all three on the same straight line, their polar planes (Θ), $(\Theta_{\prime})$, and (Δ) will all meet in the same straight line (ϑ); but the two former planes are tangent planes to (Σ), while the latter is the umbilical plane (Δ). Since d is in the directrix plane (D) of (S), the polar plane of d will pass through the umbilical focus v, the pole of (D), in (Σ); and as d is a point on the chord of contact, t, $t_{\prime\prime}$ the polar plane of d will pass through the intersection $(\vartheta_{\prime})$ of the two tangent planes (Θ) and $(\Theta_{\prime\prime})$ to (Σ). Hence the polar plane of the point d in the directrix plane (D) to (S) is the plane which passes through v, the umbilical focus of (Σ), and the straight line $(\vartheta_{\prime})$ the intersection of the tangent planes (Θ) and $(\Theta_{\prime\prime})$ to (Σ); and therefore the conjugate polar

Fig. 59.



of the line Fd will be the intersection of this plane with the umbilical directrix plane (Δ) of (Σ). In the same way it may be shown that the conjugate polar of F$d_{,}$ is the line in which the plane through v and the intersection ($\vartheta_{,,}$) of the two tangent planes ($\Theta_{,}$) and ($\Theta_{,,}$) cuts the umbilical directrix plane (Δ). But the lines Fd and F$d_{,}$ are at right angles; therefore the planes passing through the polar focus ω and their conjugate polars are at right angles also, whence we may derive the following theorem:—

If through a straight line (ϑ) *in the umbilical directrix plane* (Δ) *of a surface* (Σ) *of the second order, two tangent planes* (Θ) *and* ($\Theta_{,}$) *be drawn meeting a third tangent plane* ($\Theta_{,,}$) *in the straight lines* ($\vartheta_{,}$) *and* ($\vartheta_{,,}$), *the planes through these two lines and the umbilical focus v will cut the umbilical directrix plane* (Δ) *in two straight lines, through which and the polar focus ω if two planes be drawn, they will be at right angles to each other.*

264.] The vector plane (V) drawn through the focus F, and the line of intersection (l) of two tangent planes (T) and ($\mathrm{T}_{,}$) to a surface of revolution (S), is at right angles to the line drawn from the focus F to the point d in which the chord of contact t, $t_{,}$ of the two tangent planes (T) and ($\mathrm{T}_{,}$) meets the directrix plane (D) of (S).

Let us reciprocate this property.

Since the three planes (T), ($\mathrm{T}_{,}$), and (V) all intersect in the same straight line (l), their poles τ, $\tau_{,}$, the points of contact of two tangent planes (Θ) and ($\Theta_{,}$) to (Σ), and the point ν, the pole of (V), will all range on the same straight line; and as this latter plane (V) passes through the focus F, its pole ν will be on the umbilical directrix plane (Δ).

Again, as d is a point on the directrix plane (D) of (S), its polar plane will pass through v, the umbilical focus; and as d is a point on the chord of contact t, $t_{,}$, its polar plane will pass through the conjugate polar of t, $t_{,}$, namely (ϑ) the intersection of the tangent planes (Θ) and ($\Theta_{,}$); and as d is a point on the focal line Fd, its conjugate polar (δ) will lie in the umbilical directrix plane (Δ); consequently the conjugate polar of Fd is the line in which the plane that passes through v and (ϑ) meets the umbilical directrix plane (Δ). But as Fd is at right angles to the plane (V), the vector line $\omega\nu$ will be at right angles to the plane $\omega(\delta)$. See last figure.

Hence this theorem:—

If two tangent planes (Θ) *and* ($\Theta_{,}$) *are drawn to a surface of the second order* (Σ), *having three unequal axes, meeting in the straight line* (ϑ), *and if τ and $\tau_{,}$ be the points of contact of these tangent planes, let the chord $\tau\,\tau_{,}$ meet the umbilical directrix plane* (Δ) *in ν; and let the plane drawn through v, the umbilical focus, and* (ϑ) *the intersection of the tangent planes* (Θ) *and* ($\Theta_{,}$) *meet the umbilical directrix plane* (Δ) *in the straight line* (δ), *the line $\omega\nu$ will be at right angles to the plane drawn through ω, the polar focus and* (δ).

265.] If the surface become an oblate spheroid, then the polar focus ω will coincide with the centre of this surface (Σ), and its umbilical directrix plane (Δ) will be parallel to the plane of XY which contains the principal circular section of (Σ). Therefore the preceding theorem may be thus modified:—

If two tangent planes (Θ) *and* ($\Theta_{,}$) *be drawn to an oblate spheroid, and their chord of contact be produced to meet the directrix plane in* v, *and if a plane be drawn through the umbilical focus* v *and the line* (ϑ) *in which these two tangent planes intersect and meeting the directrix plane in the line* (δ), *the diametral plane* C(δ) *will be at right angles to the diameter* Cv.

266.] Let a tangent plane (T) be drawn to a surface of revolution (S) touching it in the point t, and cutting the directrix plane (D) in (d). The vector plane (V) drawn through the focus F and the straight line (d) will be perpendicular to the focal line drawn from F to t. Let us take the reciprocal polar of this theorem.

Since the three planes (D), (T), and (V) all pass through the same straight line (d), their poles will be on the straight line (δ), the conjugate polar of (d); and as v is the pole of (D), (δ) will pass through v. As the pole of (T) will be a point τ on (Σ), (δ) will pass through this point; and as (V) passes through F, its pole will be the intersection of (δ) with the umbilical directrix plane (Δ). Again, the conjugate polar of the line joining F and t will be the straight line in which the tangent plane (Θ) to (Σ) intersects the directrix plane (Δ); but the line drawn from the polar focus ω to the pole δ of (V) will be perpendicular to the plane passing through the intersection of the tangent plane (Θ) with (Δ).

Hence this theorem:—

If a tangent plane (Θ) *be drawn to* (Σ), *touching it in the point* τ, *and cutting the umbilical directrix plane in the line* (ϑ), *and if through the umbilical focus* v *and the point* τ *a straight line be drawn meeting the directrix* (Δ) *in the point* δ, *the line* $\omega\delta$ *drawn from the polar focus* ω *to* δ *will be at right angles to the plane drawn from* ω *through the straight line* (ϑ).

When the surface becomes an oblate spheroid, the resulting theorem is as follows:—

If a tangent plane be drawn to an oblate spheroid cutting one of the directrix planes (Δ) *in a straight line* (ϑ), *the diametral plane drawn through* (ϑ) *will be at right angles to the diameter* Oτ *drawn through the point of contact* τ.

267.] Two tangent planes, (T) and ($T_{,}$), are drawn to the primitive surface (S). The focal vectors (f) and ($f_{,}$) drawn from F to the points of contact t and $t_{,}$ are equally inclined to the vector plane (V) drawn through the focus F and the intersection (l) of the tangent planes (T) and (T).

Now, since the vector plane (V) passes through F, its pole will be on the directrix plane (Δ); and as it passes through the straight line (l), its pole will be on the polar of (l)—that is, the chord $\tau\,\tau_{\prime}$ which joins the points of contact of the tangent planes (Θ) and ($\Theta_{\prime}$) to (Σ), which are the polar planes of the points t and $t_{\prime}$—and therefore is the point λ in which the directrix plane (Δ) is pierced by the line $\tau\,\tau_{\prime}$. The conjugate polar of the line Ft is the straight line in which the tangent plane (Θ) cuts the directrix plane (Δ). So also for F$t_{\prime}$. The resulting theorem is therefore as follows :—

Let two tangent planes, (Θ) *and* ($\Theta_{\prime}$), *be drawn to a surface* (Σ) *of three unequal axes, cutting the umbilical planes in two straight lines* (δ) *and* ($\delta_{\prime}$). *Let a straight line be drawn through the points of contact meeting the directrix plane* (Δ) *in* λ. *The line drawn from the polar focus* ω *to* λ *is equally inclined to the planes* $\omega(\delta)$ *and* $\omega(\delta_{\prime})$.

When the surface becomes an oblate spheroid, the foregoing theorem is thus modified:—

If two tangent planes are drawn to an oblate spheroid, and cutting the directrix plane (Δ) *in two straight lines* (δ) *and* ($\delta_{\prime}$), *while the chord through the points of contact meets it in* λ, *the diametral planes* O(δ) *and* O$(\delta_{\prime})$ *are equally inclined to the diameter* Oλ.

268.] Two tangent planes, (T) and (T$_{\prime}$), being drawn to the primitive surface (S), the focal vector lines (f) and ($f_{\prime}$) drawn to the points of contact t and $t_{\prime}$ are equally inclined to the focal vector $f_{\prime\prime}$ drawn to the point d where the chord of contact $t\,t_{\prime}$ meets the directrix plane (D) of (S).

The conjugate polars of the focal vectors (f) and ($f_{\prime}$) are the straight lines (j) and ($j_{\prime}$), in which two tangent planes (Θ) and ($\Theta_{\prime}$) to the polar surface (Σ) intersect the umbilical directrix plane (Δ); and as the point d in which the chord of contact $t\,t_{\prime}$ pierces the directrix plane (D) is on this plane (D), its polar plane (U) will pass through v; and as this point d is on the chord $t\,t_{\prime}$, its polar plane (U) will pass through (ϑ), the intersection of the tangent planes (Θ) and ($\Theta_{\prime}$). Hence the conjugate polar of ($f_{\prime\prime}$) is the straight line in which this plane (U) intersects (Δ).

Hence the following theorem :—

If two tangent planes, (Θ) *and* ($\Theta_{\prime}$), *are drawn to a surface* (Σ) *meeting the directrix plane* (Δ) *in two straight lines* (δ) *and* ($\delta_{\prime}$), *and if through the intersection of these tangent planes* (ϑ) *and the umbilical focus* v *a plane* (U) *be drawn cutting the directrix plane* (Δ) *in a straight line* (χ), *the planes* $\omega(\delta)$ *and* $\omega(\delta_{\prime})$ *will be equally inclined to the plane* $\omega(\chi)$.

When the surface becomes an oblate spheroid, the theorem is thus modified :—

When two tangent planes to an oblate spheroid are drawn meeting in the line (ϑ), *and the directrix plane* (Δ) *in the straight lines* (δ) *and* ($\delta_{\prime}$)—*and if a plane be drawn through the umbilical focus* v *and the intersection* (ϑ) *of the tangent planes, and cutting the directrix plane*

(Δ) in the straight line (χ), the diametral planes $O(\delta)$ and $O(\delta_{\prime})$ will make equal angles with the diametral plane $\omega(\chi)$.

269.] Two tangent planes, (T) and $(T_{\prime})$, are drawn touching the primitive surface (S). From any point p in the line (l) of their intersection two tangents to (S) are drawn touching the surface (S) in the points of contact t and $t_{\prime}$. The vector planes (V), $(V_{\prime})$ through the focus F and these tangents $p\,t$ and $p\,t_{\prime}$ will be equally inclined to the vector plane $(V_{\prime\prime})$ which passes through F and (l), the intersection of the tangent planes (T) and $(T_{\prime})$. Let us take the dual of this property.

The polar plane (Π) of the point p, since it is in the intersection (l) of the tangent planes (T) and $(T_{\prime})$, will pass through its conjugate polar τ and $\tau_{\prime}$, the chord which joins the points of contact of the tangent planes (Θ) and $(\Theta_{\prime})$ drawn to (Σ); and as p is a point in the line $p\,t$, its conjugate polar will be the intersection of the plane (Π) with the tangent plane (Θ). Hence the pole of the plane (V) will be the point on the umbilical directrix plane (Δ) where it is pierced by the intersection of the planes (Π) and (Θ). In the same way the pole of the plane $(V_{\prime})$ is the point in which (Δ) is pierced by the intersection of the planes (Π) and $(\Theta_{\prime})$, and the pole of the plane $(V_{\prime\prime})$ is the point in which the directrix plane (Δ) is pierced by the chord of contact $\tau\,\tau_{\prime}$; but these poles subtend equal angles at the polar focus ω.

Hence the following theorem :—

If two tangent planes, (Θ) and $(\Theta_{\prime})$, to a surface of the second order be cut by another plane (Π) passing through the points of contact $\tau\,\tau_{\prime}$, and cutting the tangent planes in two straight lines $\tau\,\pi$ and $\tau_{\prime}\,\pi$, if the three sides of the triangle $\tau\,\tau_{\prime}$, $\tau\,\pi$, and $\tau_{\prime}\,\pi$ be produced to meet the directrix plane in the points λ, δ, and $\delta_{\prime}$, the angles $\lambda\omega\delta$ and $\lambda\omega\delta_{\prime}$ will be equal.

270.] If two tangents are drawn from any point p to two points t and $t_{\prime}$ on the primitive surface (S), the focal vector planes (V) and $(V_{\prime})$ drawn through the tangents pt and $pt_{\prime}$ are equally inclined to the vector plane $(V_{\prime\prime})$ drawn through the chord of contact $t\,t_{\prime}$.

Since the three vector planes (V), $(V_{\prime})$, and $(V_{\prime\prime})$ all pass through the focus F of (S), the poles of these three planes will lie on the umbilical directrix plane (Δ). Draw the plane (Π), the polar plane of p. Now the conjugate polar of the line $p\,t$ will be the intersection of the planes (Π) and (Θ), namely (ϖ), and the conjugate polar of the line $p\,t_{\prime}$ will be the intersection $(\varpi_{\prime})$ of the planes (Π) and $(\Theta_{\prime})$, while the conjugate polar of the line $t\,t_{\prime}$ will be the line (ϑ) in which the tangent planes (Θ) and $(\Theta_{\prime})$ intersect; therefore the poles of the three vector planes (V), $(V_{\prime})$, and $(V_{\prime\prime})$ will be the three points in which the lines (ϖ), $(\varpi_{\prime})$, and (ϑ) meet the directrix plane (Δ). From these relations we may obtain the following theorem :—

Two tangent planes, (Θ) and $(\Theta_{\prime})$, are drawn to a surface having *three unequal axes. A third plane* (Π) *is drawn through the chord of contact cutting the tangent planes in the lines* (ϖ) *and* ($\varpi_{\prime}$). *These lines and the line* (ϑ), *in which the tangent planes intersect, are produced to meet the directrix plane* (Δ) *in the three points* π, $\pi_{\prime}$, *and* δ. *The angle* $\pi\omega\delta = \pi_{\prime}\omega\delta$.

When the surface is an oblate spheroid, we obtain the following theorem:—

Two tangent planes, and a secant plane through the chord of contact, are drawn to an oblate spheroid, the diameter drawn through the point in which the common intersection of the two tangent planes meets the directrix plane (Δ) *is equally inclined to the diameters which pass through the points in which the common intersections of the secant plane with the tangent planes meet the directrix plane.*

271.] Through the same straight line (l) two tangent planes (T), ($T_{\prime}$) and a plane (V) are drawn to the surface and focus F of (S). From the other focus $F_{\prime}$ perpendiculars P, $P_{\prime}$ are let fall on the tangent planes (T) and ($T_{\prime}$); the line joining the feet of these perpendiculars will be at right angles to the plane (V).

This may easily be shown, as the tangent planes make equal angles with the focal planes passing through the same straight line.

Let us now take the polar of this theorem.

Since the two tangent planes (T), ($T_{\prime}$), and the focal plane (V), all three pass through the same straight line (l), and (V) also through the focus F, their poles τ, $\tau_{\prime}$, and ν will range along the same chord of (Σ), τ and $\tau_{\prime}$ being points on the surface, and ν the point of intersection of this chord with the umbilical directrix plane (Δ); and as P, $P_{\prime}$ are perpendiculars to the tangent planes (T) and ($T_{\prime}$), their conjugate polars will lie in the planes drawn through ω at right angles to $\omega\tau$ and $\omega\tau_{\prime}$, sec. [257]; and as these perpendiculars P and $P_{\prime}$ pass through the second focus $F_{\prime}$ of (S), the conjugate polars of P and $P_{\prime}$ will also lie in the second umbilical plane ($\Delta_{\prime}$).

Consequently the lines in which the planes through ω at right angles to $\omega\tau$ and $\omega\tau_{\prime}$ meet the second umbilical directrix plane ($\Delta_{\prime}$) are the conjugate polars (ϖ) and ($\varpi_{\prime}$) of the perpendiculars P and $P_{\prime}$. Therefore the polar planes of the feet of the perpendiculars P and $P_{\prime}$ on the tangent planes (T) and ($T_{\prime}$) are the planes $\tau(\varpi)$ and $\tau_{\prime}(\varpi_{\prime})$; therefore the line which joins the feet of these perpendiculars P and $P_{\prime}$ must be the line in which the two planes intersect. Let this line be (λ); hence the plane $\omega(\lambda)$ will be at right angles to the line $\omega\nu$.

Hence we may derive the following theorem:—

If through a surface (Σ) with three unequal axes a chord be drawn meeting this surface in the points τ and $\tau_{\prime}$ and the umbilical directrix plane (Δ), and if through ω the polar focus planes be drawn at right angles to $\omega\tau$ and $\omega\tau_{\prime}$ cutting the second directrix plane ($\Delta_{\prime}$) in (ϖ) and ($\varpi_{\prime}$), and planes be drawn through $\tau(\varpi)$ and

$\tau_{\prime}(\varpi_{\prime})$ intersecting in the straight line (λ), the plane $\omega(\lambda)$ will be at right angles to the straight line $\omega\nu$.

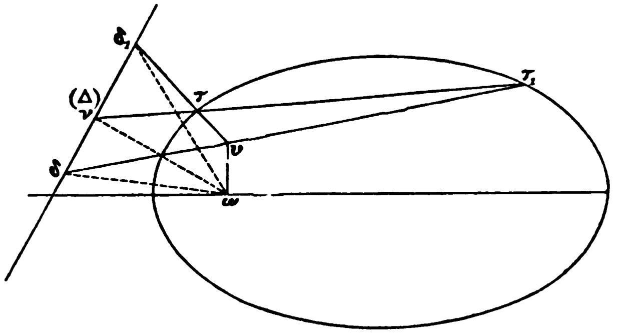
272.] The vector planes (V) and ($V_{\prime}$) passing through the focus F of a surface of revolution (S) and the straight lines (d) and ($d_{\prime}$), in which two tangent planes (T) and ($T_{\prime}$) cut its directrix plane (D), are equally inclined to the plane (G) drawn from the focus F through the intersection (l) of the tangent planes (T) and ($T_{\prime}$).

Hence, in the reciprocal polar (Σ) of this surface (S), the poles of the tangent planes (T) and ($T_{\prime}$) are two points τ and $\tau_{\prime}$ on the surface of (Σ); the conjugate polar of the line (l) in which these tangent planes intersect is the chord τ, $\tau_{\prime}$; and as the plane (G) passes through (l) and F, the pole γ of this plane (G) will be the point in which the chord $\tau\,\tau_{\prime}$ meets the umbilical directrix plane (Δ). Since (V) passes through the focus F, its pole will be on the umbilical directrix plane (Δ); and as the line (d) is the intersection of the three planes (D), (T), and (V), consequently (δ), the conjugate polar of (d), will pass through the poles of (D), (T), and (V)—that is, through the umbilical focus v, the point τ on the surface of (Σ), and δ the intersection of the line (δ) with the plane (Δ). In the same way we may find the point $\delta_{\prime}$, the intersection of the line ($\delta_{\prime}$) with the directrix plane (Δ). Now as the plane (G) is equally inclined to the planes (V) and (V), the poles of these planes will subtend equal angles at the polar focus.

Hence we derive the following theorem:—

(a) Let there be a triangle whose vertex is at the umbilical focus, and whose base is a chord of the surface (Σ). If the three sides be produced to meet the umbilical directrix plane (Δ) in three points ν, δ, and $\delta_{\prime}$, the line drawn from the polar focus ω to ν will be equally inclined to the focal lines $\omega\delta$ and $\omega\delta_{\prime}$.

Fig. 60.



(β) Should (Σ) become a surface of revolution (S), we may derive the following theorem:—If in a surface of revolution (S) a chord be drawn meeting the directrix plane in d, and the surface in the points c and $c_{\prime}$, the angles cFd and $c_{\prime}Fd_{\prime}$ are equal.

When the chord $c\,c_{\prime}$ passes through the focus F, the angles cFd and $c_{\prime}Fd_{\prime}$ are right angles.

(γ) When (Σ) becomes an oblate spheroid, let a chord to the surface $\tau\,\tau_{\prime}$ be drawn. Let υ be the umbilical focus on the minor axis at the distance $c\epsilon$ from the centre, measured along the vertical axis OZ; let the sides $\upsilon\,\tau$ and $\upsilon\,\tau_{\prime}$ of the triangle $\upsilon\,\tau\,\tau_{\prime}$ be produced to meet the minor directrix plane (Δ) in the points δ and $\delta_{\prime}$, and let $\tau\,\tau_{\prime}$ meet the same plane in the point ν. The diameter $O\nu$ will be equally inclined to the diameters $O\delta$ and $O\delta_{\prime}$.

273.] Conjugate polar straight lines polarized by a sphere ($\Omega_{\prime}$) are always at right angles. When they are on the surface of the sphere (that is, on a tangent plane to the sphere), they are still at right angles. Hence, if we polarize this sphere ($\Omega_{\prime}$) by the polarizing sphere (Ω), we shall obtain a surface of revolution (S), whose focus will be ω, the centre of the polarizing sphere (Ω); and as the conjugate tangents pass through a point *on the surface* of the sphere ($\Omega_{\prime}$), their conjugate polars will lie *on* a tangent plane to (S); and as the conjugate tangents lie on a tangent plane to ($\Omega_{\prime}$), their conjugate polars will pass through a point *on* the surface of (S); and as the conjugate tangents to the sphere ($\Omega_{\prime}$) are at right angles, the planes passing through the focus of (S), and these conjugate tangents to the surface of ($\Omega_{\prime}$), will contain the conjugate polars which are tangents to (S), and these planes will be at right angles to each other.

If we now polarize this theorem, the umbilical directrix plane (Δ) to the surface (Σ) will be the polar plane of the focus F of (S). Now the poles of these two rectangular planes which pass through the focus F of (S) will be found on the corresponding umbilical directrix plane (Δ); and as these planes pass through two straight lines which are found on a tangent plane to (S), and touching the same, their poles will be found on the reciprocal conjugate polars, see [247], which touch the surface (Σ), and are in a tangent plane to it. Therefore these poles are the points in which the two conjugate tangents to the surface (Σ) pierce the umbilical directrix plane (Δ). Hence the following theorem may be derived:—

Let any pair of conjugate tangents to the surface (Σ) *be produced to meet the umbilical directrix plane* (Δ) *in two points* α *and* β, *the lines drawn from the polar focus* ω *to these points will be at right angles.*

When these conjugate tangents are at right angles, they are tangents to the lines of greatest and least curvature of the surface (Σ), as we shall now proceed to show.

274.] *Conjugate tangents are parallel to a pair of conjugate diameters of the central plane of the surface* (Σ) *parallel to the tangent plane containing the conjugate tangents.*

A very elegant and at the same time a purely geometrical proof of this proposition may be established by the method of limits.

Join Q, the point of contact of the tangent plane, to C the centre of (Σ). Through the centre C, and parallel to the tangent plane at Q, let a plane section of the surface be drawn; then the centres of all the sections parallel to the tangent plane at Q will have their centres on the straight line QC. Let a plane section of the surface be drawn parallel and indefinitely near to the tangent plane at Q, and let κ be the centre of this infinitesimal section. Through the diameter $Q\kappa C$ let two planes be drawn cutting the diametral plane parallel to the tangent plane in the conjugate diameters CA and CB, and the infinitesimal section in the diameters $\kappa\alpha$ and $\kappa\beta$. Let two tangents be drawn parallel to the diameter $\kappa\alpha$ in the infinitesimal section, the centre of the line joining the points of contact of these tangents will be on the line QC; and as these parallels lie in the plane of the variable section, the two tangent planes to the surface (Σ) through these tangents will intersect in a straight line, which (Euclid, XI. 19) will be also parallel to the plane of the variable section, and to the two tangents which it contains, and therefore parallel to the diameter of the surface CA.

Let this variable section be conceived to move parallel to itself, it will have its centre always on the diameter QC; and when at length it comes ultimately to coincide with the fixed tangent plane at Q, the two movable tangent planes to the variable section, or *indicatrix** as named by Dupin, will continue always parallel to the diameter CA drawn in the parallel diametral plane, until they at length coincide, and we shall have their ultimate intersection parallel to the diameter CA along the tangent $Q\alpha$. These planes will therefore be the constituent elements of a *developable surface* whose edges lie along the lines $Q\alpha$, $Q_{\prime}\alpha_{\prime}$, and which touch the surface (Σ) along the line $Q\beta$, $Q_{\prime}\beta_{\prime}$.

If we now take the normal radii of curvature along the indicatrix and perpendicular to the tangent plane at Q, these radii of curvature, multiplied by the perpendicular distances between the tangent plane at Q and the plane of the *indicatrix* parallel to it, will be ultimately equal to the squares of the corresponding semidiameters of the indicatrix; but as these perpendicular distances are all equal, the normal radii of curvature round the point Q will be ultimately as the squares of the corresponding semidiameters of the *indicatrix*. Now, as the axes are the greatest and the least of all the semidiameters of the indicatrix, the greatest and the least radii of curvature will be in normal planes at right angles to each other, and passing through the point Q. We may also infer that, since the sum of the squares of any pair of conjugate diameters is equal to the sum of the squares of the axes, the sum of the radii of curvature along any pair of conjugate tangents will be equal to the sum of the radii of greatest and least curvature. And since, moreover, the sum of the squares of the reciprocals of the axes of the indicatrix is equal to the sum of the squares of the reciprocals of any other two diameters at right angles to each other, and as the curvature of any curve is inversely proportional to its radius of curvature, it will follow that the sum of the curvatures of any two normal sections passing through Q and at right angles to each other is constant.

---

* Dupin, 'Développements de Géométrie,' p. 48.

275.] From these principles we may derive a simple method of determining the lines of greatest and least curvature at the point Q. Through this point Q let us conceive two planes to be drawn parallel to the umbilical directrix planes (Δ) and ($\Delta_{\prime}$). These planes will be parallel to the central circular sections of the surface (Σ); and as the central circular sections parallel to them cut any central plane section in two equal straight lines, making equal angles with the axes of this section, it will follow that the lines drawn through Q on the tangent plane to (Σ) bisecting the angles between the traces on this tangent plane of the two planes parallel to the umbilical directrix planes drawn through the same point Q will be tangents to the sections of greatest and least curvature*.

From the theorem established in [273], we may derive the following very elegant geometrical method of determining the position of the lines of greatest and least curvature.

Let the lines of greatest and least curvature, which are at right angles, be conceived to meet the umbilical directrix plane (Δ) in the points α and β, which lie in the straight line in which the tangent plane at Q meets the umbilical plane (Δ). Now, as the angles $\alpha Q\beta$ and $\alpha\omega\beta$ are right angles, a sphere described on $\alpha\beta$ as diameter would pass through the points Q and ω, since the angles in a hemisphere are right angles. Hence if we draw a tangent plane to (Σ) at Q meeting the umbilical plane (Δ) in the straight line (δ), and if we join Q the point of contact, and ω the polar focus, and bisect this line $Q\omega$ in the point π, and through π conceive a plane to be drawn at right angles to the line ωQ, and meeting the line (δ) in the point γ, the sphere described with this point as centre and through the point ω will cut the line (δ) in the points α and β equally remote from γ, and will also pass through Q by the construction. Now as $\alpha\omega\beta$ is a right angle, being the angle in a semicircle, α and β must be points in which a pair of conjugate tangents to (Σ) at Q meet the umbilical plane in the line (δ) on (Δ). Therefore the lines αQ, βQ are a pair of conjugate tangents to (Σ) at Q. But as Q is a point on the surface of the sphere, the angle $\alpha Q\beta$ is a right angle; therefore the conjugate tangents at Q are at right angles, and are therefore tangents along the lines of greatest and least curvature.

---

* This very elegant construction is due to M. Chasles: see 'Recherches de Géométrie pure,' p. 78.

There is a remarkable case in which this construction fails. When the point Q is an umbilicus, the tangent plane through Q is parallel to the umbilical directrix plane (Δ). The line (δ) therefore in which they intersect recedes to infinity; so also do the points α and β in which the conjugate tangents at Q are supposed to meet it. How is this to be interpreted? The solution is as follows. Let a pair of conjugate tangents $Q\alpha$, $Q\beta$ be drawn in the tangent plane at the umbilicus Q tending to meet the lines $\omega\alpha$, $\omega\beta$ drawn from the polar focus ω in a focal plane parallel to the tangent plane and umbilical directrix plane in the points α and β; but as these points have receded to infinity, the lines $Q\alpha$ and $\omega\alpha$ will be parallel, so also will the lines $Q\beta$ and and $\omega\beta$. Hence the angles $\alpha Q\beta$ and $\alpha\omega\beta$ will be equal. But the lines $Q\alpha$ and $Q\beta$ were assumed as conjugate tangents. Hence the angle $\alpha\omega\beta$ is a right angle. But $\alpha\omega\beta = \alpha Q\beta$. Hence $\alpha Q\beta$ is a right angle. Consequently any pair of conjugate tangents at the umbilicus will be at right angles.

276.] The vertex k of a cone circumscribing (S), a surface of revolution, is on the directrix plane (D) of this surface. The plane of contact (K) will pass through the focus F, and the line drawn from k to F will be perpendicular to the plane of contact (K). Consequently, as the plane of contact (K) passes through F, its pole κ will be on the umbilical directrix plane (Δ); and as the vertex k of the circumscribing cone to (S) is on the directrix plane (D) of (S), its polar plane (G) will pass through v, the umbilical focus. Therefore the conjugate polar of the line kF is the straight line in which (G), the plane of contact of the cone circumscribing (Σ), meets the umbilical plane (Δ).

This theorem consequently follows:—

If a cone whose vertex κ *is on the umbilical directrix plane* (Δ) *be circumscribed to a surface* (Σ), *the plane of contact* (G) *of this cone with* (Σ) *will pass through* v, *the umbilical focus, and will cut* (Δ) *in a straight line* (γ). *The line drawn from the polar focus* ω *to the vertex* κ *of the cone will be at right angles to the plane drawn from* ω *through the line* (Υ) *in which* (G), *the plane of contact of the cone circumscribing* (Σ), *meets the umbilical directrix plane* (Δ).

277.] The lines (f), $(f_{\prime})$ drawn from the foci of a surface of revolution (S) to a point t on its surface make equal angles with the tangent plane (T) at that point. Taking the dual of this theorem,

The pole τ of the tangent plane (T) to (S) is a point on the surface (Σ), and the polar plane of the point t is the tangent plane (Θ) through τ; and the conjugate polars of the focal lines (f) and $(f_{\prime})$ in (S) are the lines in which the tangent plane (Θ) to (Σ), the polar plane of the point t, intersects the umbilical planes (Δ) and $(\Delta_{\prime})$.

We may therefore infer that

If a tangent plane (Θ) *be drawn to* (Σ) *touching it in the point* τ, *and cutting the conjugate umbilical directrix planes* (Δ) *and* ($\Delta_{\prime}$) *in the lines* (δ) *and* ($\delta_{\prime}$), *the planes drawn through these straight lines and the polar focus* ω *will be equally inclined to the line* $\omega\tau$, *drawn from the polar focus* ω *to the point of contact* τ.

When the surface becomes an oblate spheroid it will hence follow that, *If a tangent plane be drawn to an oblate spheroid cutting the parallel directrix planes* (Δ) *and* ($\Delta_{\prime}$) *in the straight lines* (δ) *and* ($\delta_{\prime}$), *the diameter through the point of contact will be equally inclined to the diametral planes* O(δ) *and* O($\delta_{\prime}$).

When one of the secant planes is drawn parallel to one of the umbilical directrix planes (Δ), the line in which it meets this umbilical plane will recede to infinity. Hence the tangent plane drawn through the umbilicus will meet the other directrix plane ($\Delta_{\prime}$) in the straight line ($\delta_{\prime}$), so that any point in this straight line will be equally distant from the umbilicus and the polar focus ω—a result already obtained by a different method in sec. [260].

When τ coincides with the extremity of the principal parameter L of the surface (Σ), the tangent plane at τ will cut the umbilical planes (Δ), ($\Delta_{\prime}$) in their common intersection, the polar directrix which lies in the plane of XY; or the principal parameter of the surface (Σ) is at right angles to the plane of XY—a result long since obtained.

278.] It has been shown, in note to sec. [249],

That if a cone be circumscribed to (S), a surface of revolution round the major or transverse axis, the lines drawn from the vertex of the circumscribing cone to the foci F, $F_{\prime}$ of (S) will be (f), ($f_{\prime}$), the focals of the cone.

Taking the reciprocal polar of this theorem, the polar plane of k, the vertex of the cone circumscribing (S), will be a plane section (K) of (Σ), and the conjugate umbilical directrix planes (Δ) and ($\Delta_{\prime}$) of (Σ) are the polar planes of the foci of (S), namely F and $F_{\prime}$; and the straight lines in which the plane (K) cuts the umbilical directrix planes (Δ) and ($\Delta_{\prime}$) will be the conjugate polars of (f) and ($f_{\prime}$), the focals of (S). And if through the polar focus ω and these two lines we draw planes, these planes will be perpendicular to the focals of (S), and therefore parallel to the circular sections of the cone whose vertex is at ω, and whose base is the plane section (K) of (Σ). Therefore we may infer that

If a secant plane be drawn cutting the surface (Σ) *in a plane section* (K), *and the conjugate umbilical directrix planes* (Δ) *and* ($\Delta_{\prime}$) *in the straight lines* (δ) *and* ($\delta_{\prime}$), *the planes drawn through the polar focus* ω *and these straight lines* (δ) *and* ($\delta_{\prime}$) *will be parallel to the circular sections of the cone whose vertex is at* ω *and whose base is* (K).

(a) When the surface is one of revolution round the transverse axis, the two umbilical directrix planes (Δ) and $(\Delta_{,})$ coalesce into (D), and therefore the lines in which the secant plane cuts the directrix planes coalesce into one; therefore the circular sections of the cone coalesce into one, or the cone is a right cone. Consequently it follows that the cone whose vertex is at the focus of a surface of revolution (S), and whose base is any plane section of this surface, is a right cone; and its circular section is parallel to the plane drawn through the focus F and the straight line in which the base of the cone cuts the directrix plane (D) of (S).

(β) When (Σ) becomes an oblate spheroid, ω coincides with C, the centre of (Σ); the umbilical directrix planes (Δ) and $(\Delta_{,})$ become parallel to the plane of XY. Hence we derive this very elegant theorem:—

The circular sections of a cone whose vertex is at the centre of an oblate spheroid, and whose base is any plane section of this surface, are parallel to the two diametral planes which pass through the straight lines in which this base intersects the minor directrix planes (Δ) *and* $(\Delta_{,})$ *of the oblate spheroid* (Σ).

(γ) If in a surface (Σ) having three unequal axes a secant plane be drawn which shall pass through the line in which the umbilical planes (Δ) and $(\Delta_{,})$ intersect, *i. e.* the directrix of the principal section in the plane of XY; then, as the secant plane cuts the directrix planes (Δ) and $(\Delta_{,})$ in the same straight line, there can be but one plane drawn from the polar focus ω parallel to the circular sections of the cone, or, in other words, this cone is a right cone. Hence we obtain this other remarkable theorem:—

The cone, whose vertex is at the polar focus ω *of a surface* (Σ) *having three unequal axes, and whose base is a plane section of this surface which passes through the polar directrix, is a right cone whose circular section is parallel to the plane of* XY.

279.] A simple algebraical proof of this theorem may be given. The equation of the ellipsoid when the origin is at the focus ω and the axes of coordinates are parallel to the axes of the figure, is

$$b^2c^2x^2+a^2c^2y^2+a^2b^2z^2-2b^2c^2aex-b^4c^2=0. \quad . \quad . \quad . \quad \text{(a)}$$

The equation of the secant plane passing through the directrix is

$$\mathrm{A}x+\mathrm{B}z=1; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

but $\mathrm{A}=-\frac{ae}{b^2}$, and $\mathrm{B}=\frac{1}{h}$, h being the distance measured along the axis of z, at which it is cut by the secant plane. Hence the equation of the secant plane becomes

$$\frac{b^2z}{h}=b^2+aex; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

or, squaring this equation and multiplying it by c^2,

$$b^4c^2+2b^2c^2aex+a^2c^2e^2x^2-\frac{b^4c^2z^2}{h^2}=0. \quad . \quad . \quad . \quad . \quad \text{(d)}$$

If we add this expression to the preceding equation, we shall have

$$x^2+y^2-\frac{b^2}{a^2}\left(\frac{b^2}{h^2}-\frac{a^2}{c^2}\right)z^2=0. \quad . \quad . \quad . \quad . \quad . \qquad \text{(e)}$$

The equation of a right cone of which the axis of Z is the axis. When $h=\frac{bc}{a}$, the equation becomes $x^2+y^2=0$, the equation of a vertical straight line. This we might have anticipated; for when $h=\frac{cb}{a}$, $h=\mathrm{L}$, the semiparameter of the surface (Σ); and we have shown in sec. [246] that a tangent plane to (Σ) which passes through the polar directrix touches the surface at the extremity of the semiparameter.

When $h>\frac{bc}{a}$ or greater than L, the secant plane falls outside the surface, or the cone becomes imaginary.

280.] In a surface of revolution (S), the sum of the reciprocals of the segments of any focal chord is constant. This is a well-known theorem. Let C and A be the semiaxes of (S), C being greater than A; and if f and $f_{,}$ be the focal chords, we shall have

$$\frac{1}{f}+\frac{1}{f_{,}}=\frac{2\mathrm{C}}{\mathrm{A}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(a)}$$

Now as $f+f_{,}$ is a line drawn through the focus F of (S), and meeting the surface (S) in the points t and $t_{,}$, the polar planes of F, t, and $t_{,}$ will be the umbilical directrix plane (Δ) and the two tangent planes (Θ) and $(\Theta_{,})$ to (Σ), all meeting in the same straight line (δ) on (Δ). But by the lemma established in sec. [253],

$$f=\frac{\mathrm{R}^2 r\sin\theta}{\mathrm{P}\varpi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(b)}$$

Now P is the perpendicular on the umbilical plane (Δ) let fall from ω, ϖ, and $\varpi_{,}$, the perpendiculars from the same point on the tangent planes (Θ) and $(\Theta_{,})$; θ and $\theta_{,}$ are the angles between the umbilical plane (Δ), and the tangent planes (Θ) and $(\Theta_{,})$ to (Σ). Through ω and the perpendiculars P, ϖ, and $\varpi_{,}$ on the umbilical and tangent planes let a focal plane (Π) be drawn cutting the line (δ) in the point κ. From κ let a straight line be drawn to ω; this will be r, and the preceding expression, adding to it that for the other segment $f_{,}$, becomes

$$\frac{1}{f}+\frac{1}{f_{,}}=\frac{\mathrm{P}}{\mathrm{R}^2 r}\left\{\frac{\varpi}{\sin\theta}+\frac{\varpi_{,}}{\sin\theta_{,}}\right\}=\frac{2\mathrm{C}}{\mathrm{A}^2}. \quad . \quad . \quad . \quad . \qquad \text{(c)}$$

From secs. [238] and [237] we find that $\mathrm{P}=\frac{bc}{a\epsilon}$, and $\frac{\mathrm{C}}{\mathrm{A}^2}=\frac{b^3}{\mathrm{R}^2 ac}$.

Consequently

$$\frac{1}{r}\left[\frac{\varpi}{\sin\theta}+\frac{\varpi_{,}}{\sin\theta_{,}}\right]=\frac{2b^2}{c^2}\epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{d})$$

Now, if through the polar focus ω we draw a plane parallel to the umbilical directrix plane (Δ), this plane will cut (Σ) in a circular section, and the line in which the secant plane (Π) and the tangent plane (Θ) intersect will meet the plane of this circular section in a point; and if h be the distance of this point from ω, we shall have $h\sin\theta=\varpi$. Making the same substitution for the other tangent plane, we shall have

$$\frac{h+h_{,}}{r}=\frac{2b^2}{c^2}\epsilon. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{e})$$

Hence we may derive this theorem.

If through any straight line on the umbilical directrix plane (Δ) *two tangent planes be drawn to* (Σ), *and if through the three perpendiculars let fall from* ω *on these three planes a secant plane* (Π) *be drawn cutting the tangent planes in lines which produced meet the plane drawn through* ω *parallel to the umbilical directrix plane* (Δ) *in the points* τ *and* $\tau_{,}$, *and if* $\omega\tau=h$, $\omega\tau_{,}=h_{,}$, *and* $\omega\kappa=r$ *in the triangle* $\tau\kappa\tau_{,}$, *the base* $\tau\omega\tau_{,}$ *will have to the line* $\omega\kappa$ *a constant ratio, or*

$$\frac{h+h_{,}}{r}=\frac{2b^2\epsilon}{c^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{f})$$

When the surface (Σ) becomes a surface of revolution (S) round the transverse axis, $c=b$ and $\epsilon=e$, hence

$$\frac{h+h_{,}}{r}=2e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{g})$$

We may also derive this other theorem in the conic sections.

If from any point in the directrix a pair of tangents be drawn to the curve, they will cut off equal segments from the ordinate passing through the focus.

Let the equation of the tangent to the curve in rectangular coordinates be $\frac{xx_{,}}{a^2}+\frac{yy_{,}}{b^2}=1$, and as the tangent passes through a point whose coordinates are Y and $-ae$, $\mathrm{Y}=\frac{b^2(a+ex_{,})}{ay_{,}}$; and as the tangent passes also through a point whose coordinates are U and $-\frac{a}{e}$, $\mathrm{U}=\frac{b^2(ae+x_{,})}{aey_{,}}$; and if D be the distance from the focus to the directrix, $\mathrm{D}=\frac{b^2}{ae}$.

Squaring, adding, and taking the square root,

$$r=\sqrt{\mathrm{U}^2+\mathrm{D}^2}=\frac{b^2(a+ex_{,})}{aey_{,}}\,;\ \text{consequently}\ \frac{\mathrm{Y}}{r}=e.$$

So also for the negative side, $\frac{Y_{\prime}}{r}=e$, or $\frac{Y+Y_{\prime}}{r}=2e$.

281.] In a surface of revolution (S) the sum of any two focal vectors f and $f_{\prime}$ drawn to any point t on the surface is constant and equal to the major axis of (S); or

$$f+f_{\prime}=2C. \quad \dots\dots\dots \quad \text{(a)}$$

Now the polar planes of the two foci F and $F_{\prime}$ of (S) are the umbilical directrix planes (Δ) and ($\Delta_{\prime}$) of the surface (Σ); and the polar plane of the point t on (S), to which the focal vectors are drawn, is a tangent plane (Θ) to (Σ), cutting the umbilical directrix planes (Δ) and ($\Delta_{\prime}$) in the straight lines (δ) and ($\delta_{\prime}$), which are the conjugate polars of (f) and ($f_{\prime}$).

Let θ and $\theta_{\prime}$ be the angles which the tangent plane (Θ) makes with the umbilical planes (Δ) and ($\Delta_{\prime}$). Let P and ϖ be the perpendiculars let fall from the polar focus ω on the planes (Δ) and (Θ). Let (Π) be a secant plane passing through ω, the polar focus, and through P and ϖ, cutting (δ) the line of intersection of the planes (Δ) and (Θ) in the point τ. Join τ and ω, and let r be equal to $\tau\omega$.

Now, assuming the proposition established in lemma I. p. 228, we shall have $f=\frac{R^2 r\sin\theta}{P\varpi}$.

Let ϕ be the angle which the plane drawn through ω and (δ), the conjugate polar of (f), makes with the vertical ordinate through ω the polar focus, and let l be the distance from the polar directrix to the foot of r; then $r : l :: \sin i : \cos\phi$, or $r=\frac{l\sin i}{\cos\phi}$, and $l\sin\theta=\varpi_{\prime}$.

Therefore $r\sin\theta=\frac{\varpi_{\prime}\sin i}{\cos\phi}$. Now $\sin i=\frac{ce}{b\epsilon}$, see (a), sec. [240], and $P=\frac{bc}{a\epsilon}$, see (a), sec. [238]; consequently $\frac{\sin i}{P}=\frac{ae}{b^2}$.

Making these substitutions in (a), we shall have

$$f=\frac{R^2ae}{b^2}\frac{\varpi_{\prime}}{\varpi}\sec\phi;\ \text{and for } f_{\prime} \text{ we shall find } f_{\prime}=\frac{R^2ae}{b^2}\frac{\varpi_{\prime}}{\varpi}\sec\phi_{\prime}.$$

But $f+f_{\prime}=2C=\frac{2R^2a}{bc}$, see sec. [237]; consequently

$$\frac{\varpi_{\prime}}{\varpi}[\sec\phi+\sec\phi_{\prime}]=\frac{2b}{ce}. \quad \dots\dots \quad \text{(b)}$$

Let h and $h_{\prime}$ be the segments into which the tangent plane (Θ) divides the distance between the polar focus and the polar directrix, then $\frac{h}{h_{\prime}}=\frac{\varpi}{\varpi_{\prime}}$.

$$\text{Consequently } \frac{h}{h_{\prime}}(\sec\phi+\sec\phi_{\prime})=\frac{2b}{ce}. \quad \dots\dots\dots \quad \text{(c)}$$

Therefore the reciprocal polar of the theorem that in a surface of revolution the sum of the focal vectors is constant, is as follows.

To a surface (Σ) *having three unequal axes let a tangent plane* (Θ) *be drawn, cutting the umbilical planes* (Δ) *and* $(\Delta_{\prime})$ *in the straight lines* (δ) *and* $(\delta_{\prime})$, *and the line joining the polar focus and the polar directrix in the segments* h *and* $h_{\prime}$. *Let the planes that are drawn from* ω *through the straight lines* (δ) *and* $(\delta_{\prime})$ *make the angles* ϕ, $\phi_{\prime}$ *with the vertical ordinate passing through the polar focus* ω, *the resulting expression becomes*

$$\frac{h}{h_{\prime}}(\sec\phi+\sec\phi_{\prime})=\frac{2b}{ce}.$$

CHAPTER XXVII.

282.] It may interest the reader to learn the method by which M. Chasles has derived the properties of surfaces of revolution of the second order (S) from those of the sphere, and how these properties may be extended to surfaces of the second order (Σ) having three unequal axes. By this method of double polarization the countless theorems that have been established with reference to the sphere (including the whole of spherical trigonometry) may be extended to umbilical surfaces (Σ) of the second order having three unequal axes.

The following propositions are extracted from M. Chasles's work, entitled "Recherches de géométrie pure sur les lignes et les surfaces du second degré," Bruxelles, 1829.

(*a*) Deux plans tangens à la sphère font des angles égaux avec le plan mené du centre à leur droite d'intersection; donc

Les rayons vecteurs menés d'un foyer aux extrémités d'une corde d'une surface de révolution (S), sont également inclinés sur le rayon vecteur mené au point où cette corde rencontre le plan directeur.

We may extend this proposition of M. Chasles to a surface with three unequal axes, and conclude that if any two tangent planes (Θ) and $(\Theta_{\prime})$ be drawn to the polar surface (Σ) having three unequal axes, they will cut the umbilical directrix plane (Δ) in two straight lines (δ) and $(\delta_{\prime})$, and one another in a third straight line (ϑ), and these three straight lines will be the conjugate polars of the two focal vectors and of the chord of the surface (S); and if a plane be drawn through the straight line (ϑ) in which the tangent planes to (Σ) intersect, and through the corresponding umbilical focus υ, this plane will cut the umbilical plane in a fourth straight line (γ), which will be the conjugate polar of the line drawn from the focus of (S) to the point where the chord meets the directrix plane (D), and which make equal angles with its focal vectors;

hence the planes drawn through the polar focus and these three lines (δ), (γ), $(\delta_{,})$ in the umbilical directrix plane (Δ) will make equal angles, the one with the other. Hence we may infer

That if two tangent planes, (Θ) *and* $(\Theta_{,})$, *be drawn to a surface* (Σ) *having three unequal axes cutting the umbilical plane* (Δ) *in two straight lines* (δ) *and* $(\delta_{,})$, *and through the line* (ϑ) *in which they intersect, and the corresponding umbilical focus* v *a plane* (U) *be drawn cutting the directrix plane* (Δ) *in the straight line* (γ), *the planes drawn through the polar focus* ω *and the straight lines* (δ), $(\delta_{,})$ *will make equal angles with the plane passing through* ω *and the straight line* (γ).

(β) Deux rayons de la sphère font des angles égaux avec la corde qui joint leur extrémités; donc

Les plans vecteurs menés d'un foyer d'une surface de révolution aux droites suivant lesquelles deux plans tangens rencontrent le plan directeur sont également inclinés sur le plan vecteur mené à la droite d'intersection des deux plans tangens.

Since the tangent plane (T), the vector plane (U), and the directrix plane (D), all three meet in the same straight line (d), the conjugate polar of this straight line (d) will pass through the poles of these three planes, that is to say, through v the pole of (D), τ the pole of (T); and as (U) passes through the focus F, its pole will be in the umbilical directrix plane (Δ)—that is, in the point δ where it is pierced by the line $v\tau$.

In the same way we may show that $\delta_{,}$ is the pole of the plane $(U_{,})$, and is also on the line $v\tau_{,}$; and as the vector plane (V) passes through F and the intersection (l) of the tangent planes (T) and $(T_{,})$, its pole will lie on the directrix plane (Δ) and *in* the line which joins the points τ, $\tau_{,}$ on (Σ). Now as (U) and $(U_{,})$ make equal angles with (V), the lines drawn from ω to the points δ and $\delta_{,}$ the poles of (U) and $(U_{,})$ will be equally inclined to the line $\omega\nu$ drawn from ω to ν the pole of (V).

We may hence infer that

If through the umbilical focus v *two straight lines be drawn meeting the surface* (Σ) *in the points* τ *and* $\tau_{,}$, *and the umbilical directrix plane* (Δ) *in the points* δ *and* $\delta_{,}$, *while the line* $\tau\,\tau_{,}$ *meets the same plane in* v, *the lines drawn from* ω *to the two points* δ, $\delta_{,}$ *on the umbilical plane will be equally inclined to the third line* $\omega\nu$.

(γ) Le plan mené par le centre d'un sphère et par la droite d'intersection de deux plans tangens est perpendiculaire à la corde qui joint les deux points de contact, et passe par son milieu; donc

Le plan vecteur mené d'un foyer d'une surface de révolution à la droite d'intersection de deux plans tangens est perpendiculaire au rayon vecteur mené de ce foyer au point où la droite qui joint les deux points de contact des plans tangens rencontre le plan directeur.

Let us now take the reciprocal polar of this theorem.

The pole of the vector plane which passes through the focus of (S) and the intersection of the tangent planes (T) and $(T_{\prime})$ to (S) is the point in which the umbilical plane (Δ) is pierced by the chord of contact of two tangent planes (Θ) and $(\Theta_{\prime})$ drawn to (Σ).

Let d be the point on the directrix plane (D) to (S) in which the chord of contact of the two tangent planes (T) and $(T_{\prime})$ meets this plane; then, as d is a point on (D), its polar plane will pass through υ the umbilical focus; and as d is a point in the chord of contact of the tangent planes (T) and $(T_{\prime})$ to (S), its polar plane will pass through (ϑ) the intersection of (Θ) and $(\Theta_{\prime})$; and as d is also a point in the line passing through (F), the conjugate polar of this straight line dF will be the line in which the polar plane of d meets the umbilical plane (Δ); and as the point d and the intersection of (T) and $(T_{\prime})$ subtend a right angle at the focus F of (S), their reciprocals will subtend a right angle at the polar focus ω.

Hence this theorem:—

If two tangent planes (Θ) and $(\Theta_{\prime})$ be drawn to a surface (Σ) with three unequal axes, and if through the umbilical focus and their line of intersection a plane be drawn meeting the umbilical directrix plane (Δ) in a straight line, the plane drawn through this straight line and the polar focus will be perpendicular to the line drawn from the polar focus to the point in which the chord of contact of the tangent planes (Θ) and $(\Theta_{\prime})$ pierces the corresponding umbilical plane (Δ).

(δ) La droite qui va du centre d'une sphère au sommet d'un cône circonscrit, est perpendiculaire au plan du cercle de contact du cône et de la sphère; donc

Le rayon vecteur mené d'un foyer d'une surface de révolution au sommet d'un cône circonscrit à la surface est perpendiculaire au plan vecteur mené par la droite d'intersection du plan de la courbe de contact et du plan directeur.

Now taking the reciprocal polars of the preceding lines and surfaces,

The conjugate polar of the line joining v the vertex of the cone with the focus F of (S), is the line (k) in which the base (K) of the cone circumscribing (Σ) cuts (Δ) the umbilical directrix plane; and as the straight line in which the plane of contact (C) of the cone circumscribing (S) cuts its directrix plane (D) is the conjugate polar of the line which joins the umbilical focus with the vertex κ of the cone circumscribing (Σ), and as the pole of a plane which passes through a given point and a given straight line is the point in which the polar plane of the given point is pierced by the conjugate polar of the given straight line; hence the pole of the vector plane which passes through the focus of (S) and the intersection of the plane of contact with its directrix plane (D) will be the point in the umbilical directrix plane (Δ) where it is pierced by the line passing through the vertex κ of the cone circumscribing (Σ) and the umbilical focus v of the surface.

Hence this theorem :—

If a cone circumscribe a surface (Σ), *the line drawn through the umbilical focus of the surface and the vertex of the cone will pierce the umbilical directrix plane* (Δ) *in a point from which, if a straight line be drawn to the polar focus, this line will be perpendicular to the plane drawn through the polar focus and the intersection of the plane of contact* (K) *with the corresponding umbilical plane* (Δ).

(ϵ) Un cylindre circonscrit à une sphère la touche suivant un grand cercle, dont le plan est perpendiculaire aux arêtes du cylindre; donc

Tout cône circonscrit à une surface de révolution, suivant une courbe dont le plan passe par un foyer, a son sommet sur le plan directeur, et la droite menée du foyer à ce sommet est perpendiculaire au plan de la courbe.

The line which joins the vertex of the cone (C) circumscribing (S) with its focus F is the conjugate polar of the line in which the plane of contact of the cone (K) circumscribing (Σ) meets the umbilical directrix plane (Δ). The pole of the plane which passes through the focus of (S), and the line in which the base of (C) cuts the directrix plane (D), is the point in which the line joining the vertex κ of the cone (K) and the umbilical focus v of (Σ) meets the umbilical directrix plane (Δ). As the plane of contact of the cone (C) circumscribing (S) passes through its focus, the vertex of the cone (K) circumscribing (Σ) will be on the umbilical directrix plane (Δ) of (Σ).

Hence this theorem :—

The plane of contact of a cone circumscribing a surface (Σ), *and having its vertex on the umbilical directrix plane* (Δ), *will pass through the umbilical focus v, and will cut the umbilical directrix plane in a straight line. The vector plane drawn through this line and the polar focus will be at right angles to the line drawn from the polar focus to the vertex of the cone.*

(ζ) Tous les plans tangens à un cône circonscrit à une sphère, sont également inclinés sur le plan du cercle de contact; donc

Le cône qui a pour sommet un foyer d'un surface de révolution, et pour base une section plane quelconque de la surface, est de révolution, et a pour axe, le rayon vecteur mené au sommet du cône circonscrit à la surface suivant sa section plane.

Let v be the vertex of the circumscribing cone, F that of the inscribed, and Q a point on the two cones and the surface of (S).

Since the cones whose vertices are v and F have one common plane of intersection with the surface (S), the polar cone will have one vertex and two bases, one the plane of contact of the cone (K) enveloping (Σ), the other the section of this cone made by the umbilical directrix plane (Δ); and as F is the focus of (S), its polar plane will be the umbilical plane (Δ); and as Q is a point on the plane of contact of the cone circumscribing (S), its polar plane will pass through the vertex κ of the cone circumscribing (Σ); and as Q is also a point on the surface (S), its polar plane will be a tangent plane to (Σ); and as Q is a point on the line FQ, its polar plane will cut the umbilical plane (Δ) in the line (l). Hence the conjugate polar of the line FQ will be the line (l) in which the tangent plane (Θ) to the cone circumscribing (Σ) cuts the umbilical plane (Δ), and the plane which passes through this line and the polar focus will be perpendicular to the line FQ: hence this plane will envelope a cone (supplemental to that of which FQ is the side) whose vertex is at the polar focus and whose base is the section in the umbilical plane (Δ) made by the cone circumscribing (Σ); and as the former is a right cone, so must also be the latter.

The conjugate polar of the line joining F, the focus of (S) with v the vertex of the cone circumscribing (S), is the line in which the umbilical plane (Δ) and the plane of contact of the cone (K) circumscribing (Σ) intersect; hence the plane through the polar focus ω and this straight line is parallel to the circular section of the cone.

When the section of the cone (K) circumscribing (Σ) is parallel to the directrix plane (Δ), the line in which they intersect is at infinity, the plane of contact of the cone circumscribing (Σ) is parallel to the umbilical directrix (Δ), and is therefore a circular section of the surface; the section of the cone circumscribing (Σ) on the umbilical directrix plane is also a circle.

(η) Tous les plans tangens au cône qui a pour base un cercle tracé sur une sphère, et pour sommet le centre de la sphère, sont également inclinés sur le plan du cercle; donc

Le cône qui a pour sommet un foyer d'une surface de révolution, et pour base la courbe d'intersection d'un cône circonscrit à la surface par le plan directeur est de révolution, et a pour axe la droite menée du foyer au sommet du cône circonscrit.

Let Q be a point on the directrix plane (D) of (S) through which passes a side (s) of the cone (C) whose vertex is c, and which circumscribes (S). Join Q with F the focus of (S). Now, as Q is a point on the directrix plane (D), its polar plane (Π) will pass through υ, the umbilical focus of (Σ); and as Q is a point on the cone (C) which circumscribes (S), its polar plane will pass through a tangent (λ) to the curve which lies in the plane (K), the polar plane of c, and is the reciprocal of the cone (C), sec. [225]; and as this point Q is on (s), a side of the cone which is a tangent to (S), the conjugate polar (λ) will touch the plane section of (Σ) made by (K), therefore the curve touched by (λ) is a plane section of (Σ); and again, as Q is in the line FQ, the conjugate polar of FQ will be the line in which (Π), the polar plane of Q, cuts the umbilical directrix plane (Δ) in the straight line (δ).

Hence the polar plane (Π) passes through v, the umbilical focus of (Σ), and through (λ), the tangent to the plane section of (Σ) made by the plane (K), and through the straight line (δ) in the umbilical directrix plane (Δ); and the cone whose vertex is v and base the plane section (K) of (Σ), will cut the umbilical directrix plane (Δ) in a conic section to which (δ) is always a tangent; but as (δ) is the conjugate polar of FQ, the plane through the polar focus ω and the straight line (δ) will be at right angles to the line FQ; and as FQ is a side of a right cone, the plane through ω and (δ) will envelope the supplemental cone to that of which FQ is a side; and as the conjugate polar of the line which joins the vertex of the cone (C) with F is the straight line in which the secant plane (K) meets (Δ), this plane will be perpendicular to the line Fc, the axis of the cone circumscribing (S), and will therefore be parallel to the circular section of the cone whose vertex is at the polar focus ω, and whose base is the conic section on the umbilical directrix plane (Δ).

This is equivalent to the following theorem :—

Let a plane section (K) *of a surface of the second order* (Σ) *having three unequal axes cut the umbilical directrix plane* (Δ) *in a straight line* (*k*). *The cone whose vertex is at v, and whose base is the section* (K) *of* (Σ), *will cut the umbilical plane* (Δ) *in another conic section, which will be the base of a right cone having its vertex at the polar focus ω, and its circular section parallel to the plane drawn through ω, and through the straight line in which the plane section* (K) *cuts the umbilical directrix plane* (Δ).

(θ) Tous les plans tangens au cône qui a pour base un cercle de la sphère et pour sommet un point du diamètre perpendiculaire au plan de ce cercle sont également inclinés sur ce plan; donc

Un cône étant circonscrit à une surface de révolution, tous les plans menés par la droite d'intersection du plan de la courbe de contact et du plan directeur de la surface, couperont ce cône suivant des coniques qui, étant vues du foyer correspondant au plan directeur, sembleront être des cercles concentriques; le centre commun de ces cercles sera sur le rayon visuel mené au sommet du cône circonscrit.

Let (*d*) be the line in which the base of the cone (C) circumscribing (S) cuts the directrix plane (D). Let *c* be the vertex of this cone. The conjugate polar of this line (*d*), since it lies on the plane (D), will pass through the umbilical focus v; and as this line is also in the plane section (C) of the cone enveloping (S), it will pass through κ, the vertex of the cone (K) enveloping (Σ). Hence $v\kappa$ is the conjugate polar of the line (*d*); and as the secant planes of the cone (C) circumscribing (S) all pass through the line (*d*), their poles $\kappa_{\prime}$, $\kappa_{\prime\prime}$, &c. will be on the line κv; and as all the partial cones enveloping (S) have the same vertex but different bases, their reciprocals will have different vertices, κ, $\kappa_{,}$, $\kappa_{,,}$, &c., and the same base (K).

Again, let Q be a point on one of the plane sections of the cone (C) which passes through (*d*). Then as Q is a point in this plane, its polar plane (Π) will pass through κ; and as Q is a point in the side of the cone (C), its polar plane (Π) will pass through the tangent to the section of (Σ) in the base (K) whose vertex is κ; and as Q is a point on the tangent to the plane section of ($C_{,}$), its polar plane will pass through a side of the cone whose base is (K).

Hence (Π), the polar plane of Q, is a tangent to the cone whose vertex is $\kappa_{,}$ or $\kappa_{,,}$, and whose base is (K). Consequently the conjugate polar of FQ, the side of the cone whose vertex is at F, and whose side is FQ, is the straight line (δ) in which the tangent plane (Π) to the cone whose vertex is $\kappa_{,}$ and plane section (K) meets the umbilical directrix plane (Δ); and as the former is a right cone, so is the latter; and as the axis of the cone whose vertex is F and base (C) is the line Fc, so the plane passing through ω and the line in which (K) and (Δ) intersect is parallel to the circular section of the cone.

We may therefore conclude that if through v a straight line of indefinite length be drawn, and if in this line a point κ be assumed as the vertex of a cone, enveloping (Σ) along the plane of contact (K), and if any number of cones be described whose vertices range along the line $v\kappa$ having a common base (K), the sections of these cones by the umbilical directrix plane (Δ) will be the bases of right cones whose common vertex will be at the polar focus ω.

(λ) Let two cones (K) and ($K_{,}$) circumscribe (Σ), a surface with three unequal axes. Let their vertices κ and $\kappa_{,}$ be joined to v, one of the foci of the surface, and let these lines be produced to meet the corresponding umbilical directrix plane (Δ) in two points τ and $\tau_{,}$. The lines joining these points with the polar focus ω are the focals of a cone whose vertex is at ω, and whose base is the conic section in the umbilical directrix plane (Δ), which is the intersection with it of another cone whose vertex is at the focus of the surface, and whose base is the common plane of intersection of the two circumscribing cones.

(μ) When the surface is one of revolution, these two cones become identical, and we obtain the known theorem, that the focals of a cone whose vertex is the focus of a surface of revolution (S) and base the common intersection of two circumscribing cones, are the lines drawn from the focus to the vertices of the circumscribing cones.

When one of the cones becomes a tangent plane (Θ), we have the theorem that if a cone (K) circumscribes a surface (Σ), and is cut by a tangent plane (Θ) in a conic section, the cone whose vertex is a focus of the surface and base this section, is cut by the corresponding umbilical directrix plane (Δ) in a conic section (C), and the straight lines drawn from this focus to the point of contact of (Θ) and the vertex of (K) are met by the same directrix plane in two points τ and $\tau_{,}$. The straight lines joining the polar focus ω and the points $\tau, \tau_{,}$ are the focals of the cone whose vertex is ω and base (C).

CHAPTER XXVIII.

ON THE RECIPROCAL POLARS OF CONFOCAL SURFACES.

283.] The reciprocal polars of confocal surfaces are concyclic surfaces, or surfaces the planes of whose circular sections are parallel. We may apply the method developed in the foregoing pages to the derivation of the properties of one class of surfaces from those of the other, and proceed to show that if we take a series of confocal surfaces as the primitives (S), we shall make no change either in the position or inclination of the umbilical directrix planes (Δ) and ($\Delta_{,}$) to the plane of XY, and the distance of the polar focus ω from the polar directrix UY will continue the same.

Let A^2 and C^2, the squares of the semiaxes of the primitive surface (S), be changed, A^2 into $A_{,}^2 = A^2 \pm k^2$, and C^2 into $C_{,}^2 = C^2 \pm k^2$. This will introduce no change into the focal distance of ($S_{,}$) from the centre; for this distance in ($S_{,}$) is $\sqrt{C_{,}^2 - A_{,}^2} = \sqrt{C^2 - A^2}$, the same as in (S).

We shall assume the distance of ω the polar focus from O the centre of (S) to be constant, so also the polarizing sphere (Ω). Therefore D and R will be constant.

284.] The inclination of the umbilical directrix plane (Δ) to the plane of XY continues unchanged.

Let i be the angle which denotes this inclination. In sec. [240] it has been shown that $\cos^2 i = \frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)}$. Substituting in this expression the values of a, b, and c as given in (c), [237], we shall find

$$\cos^2 i = \frac{C^2 - A^2}{C^2 + D^2 - A^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

and the value of this expression is not altered by changing C^2 and A^2 into $C^2 \pm k^2$ and $A^2 \pm k^2$. The distance between ω, the polar focus, and UY, the polar directrix, is manifestly $\frac{b^2}{ae}$. Now $\frac{b^2}{a} = \frac{R^2}{A}$ and $e = \frac{D}{A}$; consequently $\frac{b^2}{ae} = \frac{R^2}{D}$, a constant quantity independent of C and A.

The length of the perpendicular let fall from the polar focus ω

on the umbilical directrix plane (Δ) is $\varpi = \frac{R^2}{\sqrt{D^2+C^2-A^2}}$. The value of this expression is not altered by changing C^2 and A^2 into $C^2 \pm k^2$, and $A^2 \pm k^2$.

Consequently, if we have a series of *confocal* primitive surfaces (S), ($S_{\prime}$), ($S_{\prime\prime}$), &c., their reciprocal surfaces (Σ), ($\Sigma_{\prime}$), ($\Sigma_{\prime\prime}$), &c. will be *concyclic*. They will have a common polar focus: they will have the same umbilical directrix planes; and the graphical properties of one set of surfaces may easily be transformed into their reciprocals on the polarized surfaces.

The portion of the vertical axis passing through the polar focus ω and cut off by the umbilical directrix plane (Δ) continues unchanged, when C^2 and A^2 are augmented by a constant quantity; for if ωz be this distance,

$$\omega z = \frac{bc}{a\eta}, \text{ but } \frac{bc}{a} = \frac{R^2}{C} \text{ and } \eta = \frac{\sqrt{C^2-A^2}}{C}.$$

Hence $\omega z = \frac{R^2}{\sqrt{C^2-A^2}}$, which is not affected by the constant added to C^2 and A^2.

285.] Every property of confocal surfaces of revolution (S), ($S_{\prime}$), &c. implies a corresponding theorem of concyclic surfaces (Σ), ($\Sigma_{\prime}$), &c., having the same polar focus ω, and the same umbilical directrix planes (Δ) and ($\Delta_{\prime}$).

Thus, for example, if from a point k a secant plane be drawn to a surface (S) cutting it in a section (C) and passing through the two foci F and $F_{\prime}$ of (S), and if from these foci focal chords f and $f_{\prime}$ are drawn to k, and tangents (t) and ($t_{\prime}$) from the point k to the section (C), the angles between f and (t), $f_{\prime}$ and ($t_{\prime}$) will be equal.

Let us take the polar of this theorem.

The polar planes of the points F, $F_{\prime}$, and k will be the umbilical directrix planes (Δ) and ($\Delta_{\prime}$), and also a secant plane (K) cutting (Σ), and the base of a cone circumscribing (Σ), whose vertex is κ, the pole of (C). Now, as f is a line joining F and the point k, its conjugate polar will be the line (δ) in which the planes (Δ) and (K) intersect. In the same way the planes ($\Delta_{\prime}$) and (K) will intersect in ($\delta_{\prime}$).

Again, as c, the point of contact of (t) with (C), is on the surface (S), its polar plane will be a tangent plane (Θ) to (Σ); and as it is a point on (C), its polar plane will pass through κ, the vertex of the cone circumscribing (Σ) along (K). Consequently the conjugate polar of the line Fk will be the line (τ) in which the planes (K) and (Θ) intersect; therefore the planes drawn through ω the polar focus and the straight lines (δ) and (τ), are inclined at the same angle as the planes $\omega(\delta_{\prime})$ and $\omega(\tau_{\prime})$.

We may consequently infer

That if a cone circumscribe (Σ), *a surface having three unequal axes, and if the plane of contact be produced to meet the umbilical planes in two straight lines and cut the tangent planes to the circumscribing cone, the planes through the polar focus and these straight lines in the directrix planes will be equally inclined to the planes through the polar focus and the straight lines, the intersections of the tangent planes with the secant plane.*

Now, if we take any number of surfaces (S), $(S_{,})$ whose semiaxes squared are C^2, A^2; C^2+k^2, A^2+k^2; $C^2+k_{,}^2$, $A^2+k_{,}^2$, &c., their reciprocal polars (Σ), $(\Sigma_{,})$, &c. will all have the *same* polar focus ω and the same umbilical planes (Δ) and $(\Delta_{,})$; and if we cut all these surfaces by a common secant plane, this plane will cut the umbilical planes in two lines (δ) and $(\delta_{,})$, and tangent planes to the circumscribing cones in two sets of tangents (τ), $(\tau_{,})$, $(\tau_{,,})$ and (τ'), (τ'), (τ''); the angles between one set of planes $(\delta)\omega(\tau_{,})$, $(\delta)\omega(\tau_{,,})$, $(\delta)\omega(\tau_{,,,})$, &c. will be equal to the angles $(\delta_{,})\omega(\tau')$, $(\delta_{,})\omega(\tau'')$, $(\delta_{,})\omega(\tau''')$, &c. between the corresponding set of planes.

CHAPTER XXIX.

ON METRICAL METHODS AS APPLIED TO THE THEORY OF RECIPROCAL POLARS.

286.] Throughout the investigations of the methods developed in the foregoing pages, lines and planes, surfaces and curves have been treated graphically, so to speak. Abstract numbers or their representatives have been rarely admitted. The distinction between graphical and metrical properties is sufficiently obvious. That the opposite sides of a hexagon inscribed in a conic section will meet in three points which range on a straight line is a graphical property; while the theorem that in any conic section the sum of the squares of any pair of conjugate diameters is constant, is evidently a metrical relation.

We shall find that, by the application of these methods, entirely new classes of properties of curves have been brought to light, the existence of which had hitherto been unsuspected and unknown. We give a simple illustration of this method.

In section [32] it has been shown that if perpendiculars be let fall from a series of fixed points A, $A_{,}$, &c. in a plane on a straight line in the same plane, and if the sum of the perpendiculars be constant, the line will envelop a circle.

We may take the reciprocal polar of this theorem and say, *That if a fixed point* O *be taken, and a number of fixed straight lines* BT, $B_{,}T_{,}$, *&c. be drawn, and another current point* S *be assumed from* *which and from the point* O *perpendiculars are let fall in pairs on these fixed lines, if the reciprocal of the vector* OS, *multiplied by the sum of the ratios of each pair of perpendiculars on the fixed lines, be constant, the point* S *will describe a conic section having its focus at* O. For the ratios of each pair of perpendiculars we may substitute the ratio of the segments OQ, QS of the line OS.

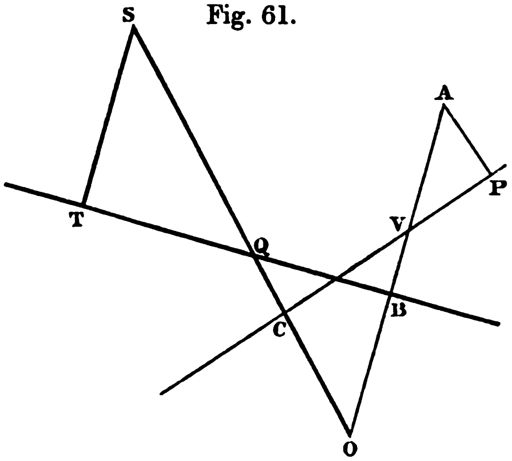


Fig. 61.

Let O be the fixed point, and let A be one of the fixed points referred to in the preceding theorem, from which perpendiculars AP=P are let fall on the line CP. In the preceding theorem, of which we require the polar, $P+P_{\prime}+P_{\prime\prime}+$ &c. equal to a constant, let R be the radius of the polarizing circle, and let the line BQ be the polar of the point A, and let S be the pole of CP. Let fall the perpendicular $ST=\Pi$ on this line. Now, as the point A and the line BQ are pole and polar, and also the point S and the line CP, we shall have $R^2=OA\times OB=OS\times OC$, or

$$\frac{OA}{OS}=\frac{OC}{OB}=\frac{OV}{OQ}=\frac{OA-OV}{OS-OQ}=\frac{AV}{SQ}=\frac{P}{\Pi},$$

or

$$\frac{OA}{OS}=\frac{P}{\Pi}, \text{ or } P=\frac{OA}{OS}\Pi=\frac{OA\times OB}{OS\times OB}\Pi=\frac{R^2\Pi}{OS.OB}.$$

Let $OS=r$, then $\frac{P}{R^2}=\frac{\Pi}{r.OB}$.

Now, the projective coordinates of the point S being x and y, and the tangential coordinates of the straight line BQ being ξ and υ, we shall have $\Pi=\frac{1-x\xi-y\upsilon}{\sqrt{\xi^2+\upsilon^2}}$; and the length of OB being $\frac{1}{\sqrt{\xi^2+\upsilon^2}}$, the resulting equation will become

$$\frac{\Pi}{OB}=1-x\xi-y\upsilon \text{ or } \frac{P}{R^2}=\frac{1-x\xi-y\upsilon}{r}. \quad . \quad . \quad . \quad (a)$$

Now, as the sum of the perpendiculars, divided by the square of the constant radius of the polarizing circle, is constant by supposition, we shall have $\frac{P+P_{\prime}+P_{\prime\prime}\ldots P_n}{R^2}=nC$, or substituting for P, $P_{\prime}$, $P_{\prime\prime}$, &c. their values, since $OS=r=\sqrt{x^2+y^2}$, the resulting expression will become

$$\frac{1}{\sqrt{x^2+y^2}}[1+\xi x+\upsilon y+1+\xi_{\prime}x+\upsilon_{\prime}y+1+\xi_{\prime\prime}x+\upsilon_{\prime\prime}y \text{ \&c.}]=nC.$$

Let the tangential coordinates ξ, υ, $\xi_{\prime}$, $\upsilon_{\prime}$, $\xi_{\prime\prime}$, $\upsilon_{\prime\prime}$ of the fixed lines BQ, $B_{\prime}Q_{\prime}$, $B_{\prime\prime}Q_{\prime\prime}$ &c. be added together, and let $\xi+\xi_{\prime}+\xi_{\prime\prime}$ &c. $=nV$, $\upsilon+\upsilon_{\prime}+\upsilon_{\prime\prime}$ &c. $=nU$, and the preceding equation will become

$$1+Vx+Uy=C\sqrt{x^2+y^2}, \quad \ldots\ldots \quad \text{(b)}$$

dividing out by n. Squaring this equation, and reducing,

$$(C^2-V^2)x^2+(C^2-U^2)y^2-2VUxy-2Vx-2Uy=1. \quad . \quad \text{(c)}$$

But it has been shown that if in the general equation of a conic section $Ax^2+A_{\prime}y^2+2Bxy+2Cx+2C_{\prime}y=1$, $A+C^2=A_{\prime}+C_{\prime}^2$, and $B+CC_{\prime}=0$, the origin is at a focus; and these conditions are satisfied in the preceding equation.

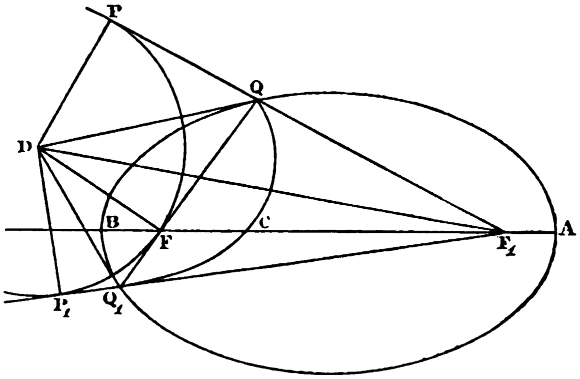
287.] Instead of defining a conic section as the curve of intersection of a right cone by a plane, the definition adopted by Apollonius, De la Hire, Hugh Hamilton, and others, we may define a conic section as a plane section of a surface of revolution of the second order, just as a circle may be defined as the curve of intersection of a sphere by a plane; and if, moreover, this plane be drawn through one of the foci of this surface, this point will also be one of the foci of the section. The directrix of this section will be the straight line in which its plane cuts the directrix plane of the surface.

It is needless to dwell longer on these trite and well-known properties of the focus in the major axis, and the directrix at right angles to it.

Of the theorem that if a plane be drawn through a focus of a surface of revolution, this point will also be a focus of the section, the following simple geometrical proof may be given.

Let F and $F_{\prime}$ be the foci of the surface of revolution (S), and let $FQCQ_{\prime}$ be a plane section of it passing through the focus F. Let $F_{\prime}$ be the vertex of the cone whose base is the section $FQCQ_{\prime}$. This will be a right cone, as we shall now proceed to show. Through the foci of the surface let any plane be drawn cutting the surface (S) in the conic section $AQBQ_{\prime}$ and the cone in the straight lines $F_{\prime}Q$ and $F_{\prime}Q_{\prime}$. Let a perpendicular FD to the plane $QCQ_{\prime}F$ be erected at F, meeting in D the tangent plane to the surface at Q. Now, as the focal lines FQ and $F_{\prime}Q$ make equal angles with the tangent plane at Q, the angle $DQP=DQF$; hence in the right-angled triangles DQP and DQF, DP=DF and $F_{,}P=AB$. In the same way $DP_{,}$ may be shown to be equal to DF. Hence a sphere described with D as centre and DF as radius will be touched by the two tangent planes to the cone through QP and $Q_{,}P_{,}$; but we know that when a sphere inscribed in a right cone touches a plane section of it, the point of contact will be a focus of this section. Hence F is a focus of the plane section.

Fig. 62.



It is easy to show that the centre of the sphere inscribed in the cone is on the directrix plane of the surface (S).

288.] If, now, we consider a conic section as a plane section of an oblate spheroid, and passing through the centre, this plane will cut the surface in a plane section, and the umbilical directrix planes (Δ) and $(\Delta_{,})$ in two parallel straight lines, since the surface is an oblate spheroid, which has its directrix planes parallel to the plane of XY, and therefore parallel to one another.

The sections of this oblate spheroid will have, with respect to these lines, properties not identical with, but analogous to, those which belong to the focus and directrix of the major axis. We may determine the position of these parallel directrices, which, as they are perpendicular to the minor axis of the figure, may be called the *minor directrices*, as those belonging to the major axis may be termed the major directrices.

We have shown in sec. [243], (d), that the length of a perpendicular P let fall from the polar focus ω on the corresponding directrix plane is $\frac{bc}{a\epsilon}$; but as in this case $b=a$, the expression becomes $\frac{c}{\epsilon}$ or $\frac{ac}{\sqrt{a^2-c^2}}$; and as $b=a$, $e=0$, or the polar focus ω coincides with the centre C of the surface (Σ).

We may write h for the distance between the minor directrix and the axis of X.

As the distance of the umbilical focus v from the plane of XY is $\frac{bc\eta}{a}$, see sec. [242], (b), and as $b=a$, and $\eta^2=\frac{b^2-c^2}{b^2}=\frac{a^2=c^2}{a^2}=\epsilon^2$, we shall have for the distance of v from the plane of XY $c\epsilon$; or as $h=\frac{c}{\epsilon}$, we shall have for that distance $h\epsilon^2$.

289.] In the hyperbola, as the minor axis is imaginary, the new directrices must be drawn in a somewhat different manner. Let 2χ be the angle between the asymptotes of the hyperbola, and on the transverse axis let two points be assumed at the distance $a\sin\chi$ from the centre; through these points let perpendiculars to the transverse axis be drawn, these lines are the *minor directrices* of the hyperbola; and if two other points be assumed at the distance $\frac{a}{\sin\chi}$ from the centre, these points are the minor foci. It is almost needless to mention that the distances of the common directrices and foci from the centre of the hyperbola are $a\cos\chi$ and $\frac{a}{\cos\chi}$ respectively; hence, in the equilateral hyperbola, where $\chi=\frac{\pi}{4}$, the ordinary and minor directrices coincide, as do also the common and minor foci; whence we may deduce the very general and remarkable conclusion that,

The common directrices and foci of the equilateral hyperbola *possess two distinct classes of properties—those which belong to them as being the common or ordinary directrices and foci, as also that other new and equally extensive class, to which they are in like manner related, as being the minor directrices and foci of a central conic section.*

In the circle the minor directrices are infinitely distant, and the minor foci coincide with the centre, as is also the case with the ordinary directrices and foci.

290.] We may, however, derive the properties of the minor directrices and their corresponding foci in a much simpler and less arbitrary way by the help of the method of reciprocal polars, from the well-known properties of the common focus and directrix.

Thus, let A and B be the semiaxes of the primitive ellipse or hyperbola. On 2A let a circle be described, and let this circle, whose radius is A, be taken as the polarizing circle (Ω) in the application of the method of reciprocal polars. Let (Σ), a new ellipse or hyperbola, be derived, the reciprocal polar of the former. Let a and b be the semiaxes of this derived section (Σ). Then manifestly $b=\mathrm{A}$, and $a=\frac{\mathrm{A}^2}{\mathrm{B}}$, since A is the radius of the polarizing circle (Ω). It is clear that the reciprocal polar (Σ) will have its minor semiaxis b or A coincident with the major axis of (S), while its major semiaxis will be coincident with the minor axis of (S).

The primitive and reciprocal sections (S) and (Σ) will be similar; for

$$\frac{A^2-B^2}{A^2}=\frac{b^2-\frac{b^4}{a^2}}{b^2}=\frac{a^2-b^2}{a^2}=e^2, \quad . \; . \; . \; . \; . \; . \quad (a)$$

or the eccentricity e is the same for (S) and (Σ). B is the semi-parameter of (Σ), since $B=\frac{b^2}{a}$. . . . . (b)

When the given section is an hyperbola, the reciprocal polar is also an hyperbola, having the same centre and transverse axis, and the angle between the asymptotes of the one equal to the supplement of the angle between the asymptotes of the other.

In the hyperbola the focal distance to the centre is $\sqrt{a^2+b^2}$, or $a \sec \chi$, while the distance of the directrix is $a \cos \chi$. The distance of the minor focus from the centre is $a \operatorname{cosec} \chi$, while the distance of the minor directrix is $a \sin \chi$.

When the hyperbola is equilateral, or $\chi=\frac{\pi}{4}$, the major and minor foci coincide, as do also the major and minor directrices; so also do the asymptotes.

The parabola has neither minor focus nor minor directrix.

291.] Let us assume the theorem established in lemma I., p. 228,

$$AB=\frac{R^2 r \sin\theta}{PP_{\prime}}, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (a)$$

a simple but most important formula due to Poncelet.

We shall now proceed to make some applications of this theorem.

In any central conic section the sum or difference of the distances of any point on the curve from the foci is constant, or

$$FC+F_{\prime}C=2A=2b,$$

since the major axis of the primitive (S) becomes the minor axis of (Σ) in this transformation.

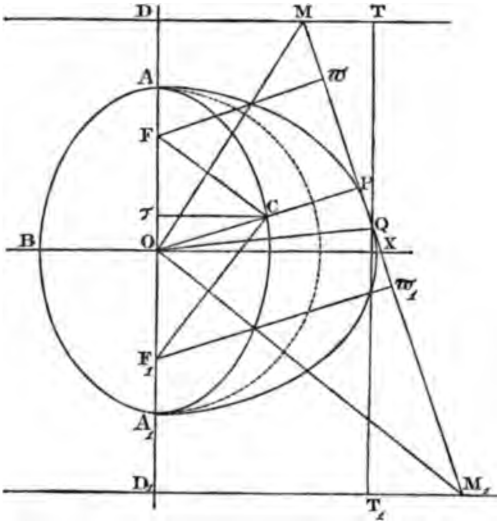
Now, by the preceding lemma, $FC=\frac{R^2 r\sin\theta}{PP_{\prime}}$; but $R=A=b$, $r=OM$, $OD=P=\frac{b}{e}$, and $P_{\prime}=OP$. Making these substitutions, $FC=be\sin\theta . \frac{OM}{OP}$. In like manner $F_{\prime}C=be\sin\theta . \frac{OM_{\prime}}{OP}$; adding these two expressions,

$$FC+F_{\prime}C=2b=\frac{(OM+OM_{\prime})}{OP}\, be\sin\theta.$$

Now $OP = OX \sin\theta$ and $2OX = DM + DM_{,}$; consequently

$$\frac{DM + D_{,}M_{,}}{OM + OM_{,}} = e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

Fig. 63.



Therefore, if a tangent be drawn to a conic section meeting the minor directrices in two points M *and* $M_{,}$, *the sum of the distances of these points from the minor axis is to the sum of their distances from the centre in a constant ratio e, the eccentricity of the section.* In the hyperbola the differences must be taken. It is to be observed that while F and $F_{,}$ are the major foci of the primitive conic (S), they are the minor foci of the reciprocal conic (Σ). The same is true also of the directrices; the major directrices of the one are the minor directrices of the other.

292.] *The product of the focal perpendiculars let fall from the minor foci* F *and* $F_{,}$ *on a tangent to the curve is to the square of the perpendicular from the centre on the same tangent as the square of the semidiameter* $a_{,}$ *passing through the point of contact* Q *is to the square of the semi- major axis a.*

Let ϖ and $\varpi_{,}$ be these perpendiculars. Then $\varpi = P - be\cos\theta$, $\varpi_{,} = P + be\cos\theta$ (see fig. 63). Therefore

$$\varpi\varpi_{,} = P^2 - b^2e^2\cos^2\theta = a^2\sin^2\theta + b^2\cos^2\theta - b^2e^2\cos^2\theta,$$

or $\varpi\varpi_{,} = \dfrac{a^4\sin^2\theta + b^4\cos^2\theta}{a^2}$, since θ is measured from the minor axis. But it has been shown in sec. [28] that

$$a^4\sin^2\theta + b^4\cos^2\theta = P^2(x^2 + y^2) = P^2a_{,}^2,$$

consequently

$$\frac{\varpi\varpi_{,}}{P^2} = \frac{a_{,}^2}{a^2}.$$

293.] From any point Q (see fig. 63) on the curve, perpendiculars QT, $QT_{,}$ are let fall on the minor directrices; the product of these perpendiculars is to the square of OQ, the semidiameter, in a constant ratio, or

$$\frac{\mathrm{QT}\times\mathrm{QT}_{,}}{\mathrm{QO}^2}=\frac{b^2}{a^2e^2}=\frac{h^2}{a^2}, \quad . \; . \; . \; . \; . \; . \; . \qquad \text{(a)}$$

$2h$ being the distance between the minor directrices.

This follows at once from the analogous theorem established in sec. [254] for surfaces of the second order.

The proof by the ordinary methods is very simple:

$$\left.\begin{aligned}&\mathrm{QT}=\frac{b}{e}-y_{,},\ \mathrm{QT}_{,}=\frac{b}{e}+y_{,},\ \text{or}\ \mathrm{QT}\times\mathrm{QT}_{,}=\frac{b^2-e^2y_{,}^2}{e^2}\\ &=\frac{a^2b^2-a^2e^2y_{,}^2}{a^2e^2}=\frac{a^2y_{,}^2+b^2x_{,}^2-a^2e^2y_{,}^2}{a^2e^2}=\frac{b^2(x^2+y^2)}{a^2e^2}=\frac{b^2a_{,}^2}{a^2e^2}.\end{aligned}\right\}. \qquad \text{(b)}$$

294.] In the ellipse or hyperbola the ratio of $\frac{\mathrm{F}c}{\mathrm{D}\tau}=e$, . . (a) see fig. 63. Now FC, as we have shown in sec. [290], is equal to $\frac{be}{\mathrm{P}}\sin\theta\,.\,\overline{\mathrm{OM}}$; and $\mathrm{D}\tau=\frac{b}{e}-\frac{b^2}{\mathrm{P}}\cos\theta=\frac{b}{\mathrm{P}e}(\mathrm{P}-be\cos\theta)$, or $\mathrm{D}\tau=\frac{b\varpi}{\mathrm{P}e}$. Consequently, by substitution in (a), $\varpi=e\sin\theta\,\overline{\mathrm{OM}}$. In like manner $\varpi_{,}=e\sin\theta\,.\,\mathrm{OM}_{,}$; consequently

$$\frac{\varpi}{\varpi_{,}}=\frac{\mathrm{OM}}{\mathrm{OM}_{,}}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad \text{(b)}$$

or, *the ratio of the focal perpendiculars on the tangent is the same as that of the distances from the centre of the points in which the tangent cuts the minor directrices.*

295.] Since, as we have shown in sec. [294],

$$\varpi=e\sin\theta\,.\,\mathrm{OM},\ \text{and}\ \varpi_{,}=e\sin\theta\,.\,\mathrm{OM}_{,},\ \varpi\varpi_{,}=e^2\sin^2\theta\,\mathrm{OM}\,.\,\mathrm{OM}_{,};$$

but $\varpi\varpi_{,}=\frac{\mathrm{P}^2a_{,}^2}{a^2}$, and $\sin\theta=\frac{\mathrm{QT}}{\mathrm{QM}}=\frac{\mathrm{QT}_{,}}{\mathrm{QM}_{,}}$, or $\sin^2\theta=\frac{\mathrm{QT}\times\mathrm{QT}_{,}}{\mathrm{QM}\times\mathrm{QM}_{,}}$.

Now putting for $\mathrm{QT}\times\mathrm{QT}_{,}$ its value as given in section [293], namely $\frac{b^2a_{,}^2}{a^2e^2}$, we shall have $\frac{\mathrm{OM}\times\mathrm{OM}_{,}}{\mathrm{QM}\times\mathrm{QM}_{,}}=\frac{\mathrm{P}^2}{b^2}$,

or, *if a tangent be drawn to a central conic section meeting the curve in the point* Q, *and the minor directrices in the points* M *and* M$_{,}$, *the rectangle under the distances of these points from the centre will be to the rectangle under the segments of this tangent as the square of the perpendicular from the centre upon it is to the square of the semi- minor axis.*

296.] Since $\frac{\varpi}{\varpi_{,}}=\frac{\mathrm{OM}}{\mathrm{OM}_{,}}=\frac{\mathrm{QM}}{\mathrm{QM}_{,}}=\frac{\mathrm{QT}}{\mathrm{QT}_{,}}$, ... (a)

it will follow that the ratios of *the focal perpendiculars on a tangent, of the distances to the centre* O *of the points* M *and* M$_{,}$ *in which* *this tangent meets the minor directrices, of the segments into which this tangent is divided between the point of contact* Q *and the minor directrices, as also of the perpendiculars from this point* Q *upon them, are the same for all.*

It will also follow that since $OM : OM_{\prime} :: QM : QM_{\prime}$, the angle $MOM_{\prime}$ is bisected by the line OQ.

297.] The sum of the reciprocals of the segments of any focal chord in a conic section is constant and equal to twice the reciprocal of the semiparameter, or

$$\frac{1}{FC}+\frac{1}{F_{\prime}C}=\frac{2A}{B^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

Now, if we refer to sec. [290], we shall find $FC=\frac{be\,.\,OM}{OX}$; in like manner $FC_{\prime}=\frac{be\,OM}{OX_{\prime}}$; and as $A=b$ and $B=\frac{b^2}{a}$, we shall have, by substitution,

$$\frac{OX+OX_{\prime}}{OM}=\frac{2a^2e}{b^2}=\frac{2ae}{\left(\frac{b^2}{a}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (b)$$

Consequently we may enunciate the following theorem :—

If from any point M *in the minor directrix of a conic section we draw two tangents to it meeting the transverse axis in the points* X *and* $X_{\prime}$, *the distance between these points will be to the distance of the point* M *from the centre as the distance between the foci is to the semiparameter.*

298.] It is a characteristic property of the focal distances of any point on a conic section that they may be expressed in rational functions of the projective coordinates of that point. A like property will be found to hold with respect to the distances from the centre of the points in which the minor directrices are cut by a tangent to the curve.

Let the equation of the tangent to the curve be

$$a^2y_{\prime}y+b^2x_{\prime}x=a^2b^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

$x_{\prime}$ and $y_{\prime}$ being the projective coordinates of the point of contact.

Let the coordinates of the point in which this tangent meets the minor directrix be $\frac{b}{e}$ and $\bar{x}$. Substituting these values in the equation (a) of the tangent, we shall have $\bar{x}$ or $DM=\frac{a^2(be-y_{\prime})}{bex_{\prime}}$.

In like manner $D_{\prime}M_{\prime}=\frac{a^2(be+y_{\prime})}{bex_{\prime}}$.

Now $\overline{OM}^2=\overline{DO}^2+\overline{DM}^2=\frac{b^2}{e^2}+\frac{a^4(be-y_{,})^2}{b^2e^2x_{,}^2}$; reducing, we shall find

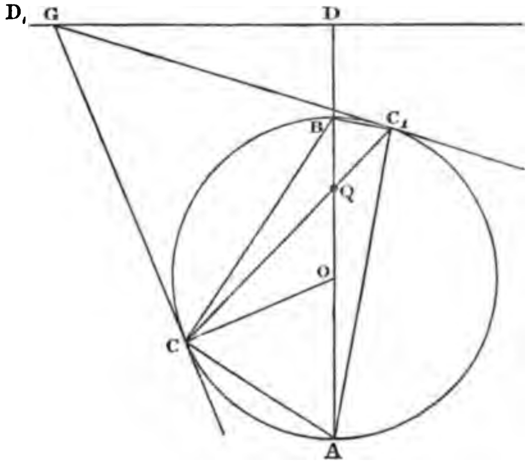
$$\overline{OM}^2=\frac{a^4(b-ey_{,})^2}{b^2e^2x_{,}^2}\text{, or }\overline{OM}=\frac{a^2(b-ey_{,})}{bex_{,}}.$$

In like manner $\overline{OM}_{,}=\frac{a^2(b+ey_{,})}{bex_{,}}$, or $\overline{OM}$ and $\overline{OM}_{,}$ may be expressed as rational functions of $x_{,}$ and $y_{,}$.

299.] The following investigation is deserving of attention, as it supplies an example of the method of deriving new theorems by means of a double reciprocation.

From any point G in the line $DD_{,}$ let two tangents, GC and $GC_{,}$,

Fig. 64.



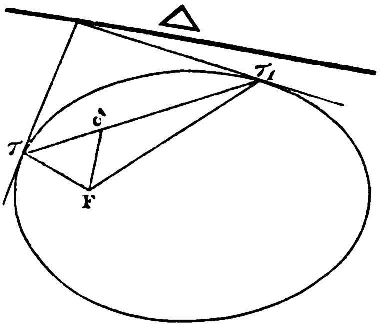
be drawn to a circle. The product of the tangents of half the angles which the lines GC and $GC_{,}$ make with the line $DD_{,}$ is constant.

Let Q be the pole of $DD_{,}$. Then as the angle $D_{,}GC$=COD, since the angles at D and C are right angles, the angle $D_{,}GC$=2CAB, and the angle $DGC_{,}$ is equal to $2C_{,}AB$; but the tangent of the angle CAB$=\frac{BC}{AC}$, and the tangent of the angle $C_{,}AB=\frac{BC_{,}}{AC_{,}}$.

Consequently the product of the tangents is equal to $\frac{BC\times BC_{,}}{AC\times AC_{,}}$. But $\frac{BC\times BC_{,}}{AC\times AC_{,}}$ is equal to the ratio of the perpendiculars let fall from B and A on the chord $CC_{,}$—that is, as $\frac{r-k}{r+k}$, putting r for the radius of the circle and k for OQ.

300.] Now, if we assume any point F in the plane of this circle, and take its reciprocal polar, which will be a conic section having its focus at F, the pole of the line $\mathrm{DD}_{\prime}$ will be a fixed point δ within the curve, and the poles of the tangents GC and $\mathrm{GC}_{\prime}$ will be points τ $\tau_{\prime}$ on the reciprocal polar; and as the three straight lines $\mathrm{DD}_{\prime}$, GC, and $\mathrm{GC}_{\prime}$ meet in the point G, the three poles δ, τ, and $\tau_{\prime}$ will range along the straight line $\tau\delta\tau_{\prime}$; and the angle between the lines $\mathrm{DD}_{\prime}$ and GC will be equal to the angle between the lines Fδ and Fτ. Hence we may infer *that if a chord be drawn through a fixed point δ in the plane of a conic section meeting the curve in the points τ and $\tau_{\prime}$, the product of the tangents of half the angles δFτ and δF$\tau_{\prime}$ will be constant.*

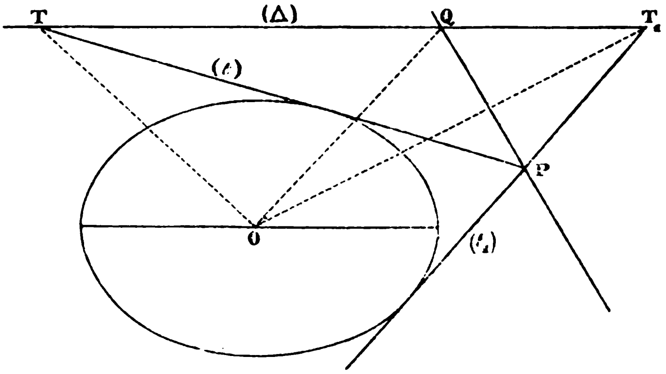
Fig. 65.



301.] If, now, we take the centre of this conic section as the centre of the polarizing circle, the polar of the focus F will be the minor directrix (Δ); the polars of the two points of contact τ, $\tau_{\prime}$ will be tangents (t), $(t_{\prime})$ to the reciprocal polar (Σ); and as the three points τ, $\tau_{\prime}$, and δ range on the same straight line, their polars (t), $(t_{\prime})$, and PQ will all meet on the same straight line. Hence we may derive this other theorem by a second polarization.

If a straight line PQ *be drawn in the plane of a conic section, and from any point* P *in it two tangents be drawn to the curve meeting the minor directrix in two points* T *and* $\mathrm{T}_{\prime}$, *while the given straight line meets it in* Q, *the product of the tangents of half the angles* TOQ *and* $\mathrm{T}_{\prime}$OQ *will be constant.*

Fig. 66.



We may diversify this theorem by taking other centres of polarization. Thus we may show that if the diameter of an ellipse be the base of a triangle whose vertex is at a focus, the product of the tangents of half the angles which the sides of the triangle make with the major axis is constant.

302.] If from any point E in the plane of a conic section two tangents be drawn, and the chord of contact produced to meet the major directrix in the point γ, the lines fE and $f\gamma$ drawn through the focus f are at right angles.

Fig. 67.

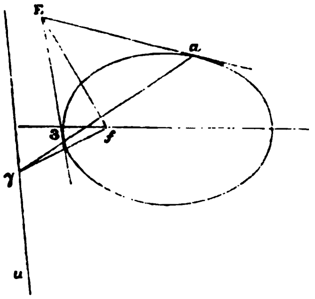
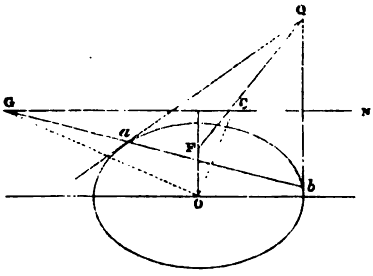


Fig. 68.



Hence, in the reciprocal polar, if a chord *ab* be drawn meeting the minor directrix in G, and a line be drawn from the minor focus F to meet in Q the intersection of the tangents drawn at *a* and *b*, and cutting the minor directrix in C, the lines OC and OG are at right angles.

For as Q, fig. 68, is the pole of $a\beta$, and F is the pole of γu, QF is the polar of the point γ; and as f is the pole of GN, therefore C is the pole of $f\gamma$. Again, as *ab* is the polar of E and GN the polar of f, G is the pole of fE; but fE and $f\gamma$ are at right angles; consequently GOC is a right angle.

303.] The locus of the feet of focal perpendiculars on a tangent to a conic section is the circle described on the major axis.

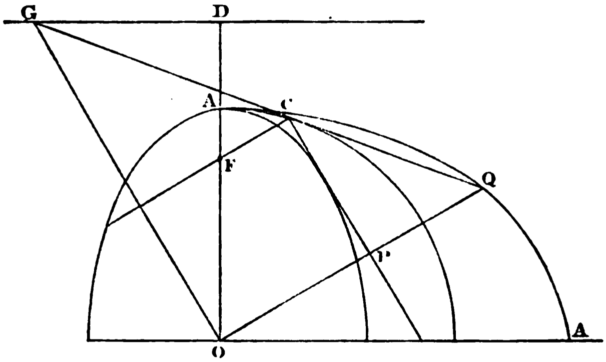
From any point C of the circle described on 2OA, the major axis of (S), draw a tangent CP to the curve (S) and the chord CF through the focus of (S); these lines we know are at right angles.

Taking the polar reciprocal of this theorem, the pole of the tangent CP will be a point Q on (Σ); and as the polar of F is the minor directrix DG, while the tangent to the circle through C is the polar of the point C, as the circle whose radius is OA is the polarizing circle, the point G where this tangent to C intersects the minor directrix will be the pole of FC; hence the points G and Q subtend a right angle at the centre; and as Q, G, C are the poles of the three lines CP, CF, and CG which all meet in the point C, these three points Q, C, and G will all range on the same straight line, the polar of C—that is, the tangent at C to the polarizing circle.

Hence the points Q, G are on the tangent which touches the circle at C; and as OQ is perpendicular to CP, Q is the pole of CP; but CP is at right angles to FC; therefore OQ is at right angles to OG.

Hence, if two diameters are drawn at right angles in a conic section, one meeting the curve in Q, *the other meeting the minor directrix in* G, *the line* QG *envelops the circle whose diameter is the minor axis.*

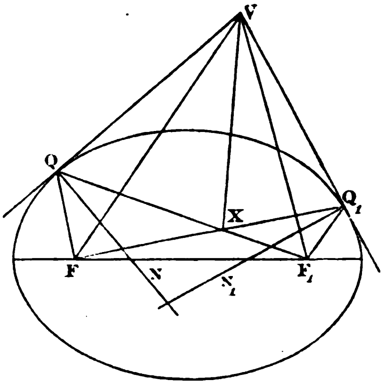
Fig. 69.



304.] (*a*) The lines drawn from the foci of a conic section to the point V, in which two tangents to the curve intersect, make equal angles with them.

Let the angle $FVF_{,}=\psi$, $FVQ=\theta$, $F_{,}VQ_{,}=\theta_{,}$; let f and $f_{,}$ be the

Fig. 70.



focal chords, P and p the perpendiculars from F on the tangents,

$P_{,}$ and $p_{,}$ the perpendiculars from $F_{,}$ on the same tangents. Then $P = f \sin\theta$, $P_{,} = f_{,}\sin(\psi+\theta)$, $p = f\sin(\psi+\theta_{,})$, $p_{,} = f_{,}\sin\theta_{,}$; but

$$\left.\begin{aligned} PP_{,} &= ff_{,}\sin\theta\sin(\psi+\theta) = b^2, \\ \text{and}\quad pp_{,} &= ff_{,}\sin\theta_{,}\sin(\psi+\theta_{,}) = b^2, \end{aligned}\right\} \quad \ldots\ldots \quad \text{(a)}$$

or

$$\sin\theta\sin(\psi+\theta) = \sin\theta_{,}\sin(\psi+\theta_{,}). \quad \ldots \quad \text{(b)}$$

But

$$\left.\begin{aligned} 2\sin\theta\sin(\psi+\theta) &= \cos\psi - \cos(\psi+2\theta) \\ \text{and}\quad 2\sin\theta_{,}\sin(\psi+\theta_{,}) &= \cos\psi - \cos(\psi+2\theta_{,}). \end{aligned}\right\} \quad \ldots \quad \text{(c)}$$

Consequently $\cos(\psi+2\theta) = \cos(\psi+2\theta_{,})$, or

$$\theta = \theta_{,}. \quad \ldots\ldots\ldots \quad \text{(d)}$$

(β) The line drawn from a focus F to the point V, the intersection of two tangents which touch the conic section in the points Q and $Q_{,}$, makes equal angles with the lines FQ, $FQ_{,}$.

This is evidently the reciprocal polar of the simple property of the circle, that any pair of tangents make equal angles with the chord of contact.

(γ) Twice the angle between the focal lines drawn to the point V in which two tangents meet, and touch the conic section at the points Q and $Q_{,}$, is equal to the sum of the angles which the foci subtend at the points Q and $Q_{,}$.

Let the normals to the curve at the points Q and $Q_{,}$ be QN and $QN_{,}$. Let the angle $FQN = F_{,}QN = \alpha$, and $F_{,}Q_{,}N_{,} = FQ_{,}N_{,} = \alpha_{,}$. Let $QFV = VFQ_{,} = \beta$, and $Q_{,}F_{,}V = VF_{,}Q = \beta_{,}$.

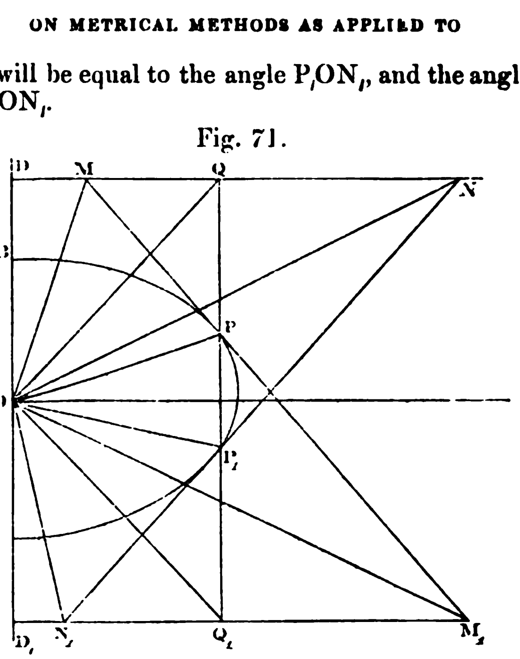
But the angle $FXF_{,} = 2\alpha + 2\beta$, and $FXF_{,} = 2\alpha_{,} + 2\beta_{,}$; consequently $FXF_{,} = \alpha + \alpha_{,} + \beta + \beta_{,}$.

But $FXF_{,} = FVF_{,} + \beta + \beta_{,}$; consequently $2FVF_{,} = 2\alpha + 2\alpha_{,}$, or

$$2FVF_{,} = FQF_{,} + FQ_{,}F_{,} \quad \ldots\ldots\ldots \quad \text{(e)}$$

305.] If we now take the reciprocal polars of these three theorems (α), (β), and (γ), we shall find, for (α), that as the two focal vectors FV and $F_{,}V$, and the two tangents QV and $Q_{,}V$ (fig. 70), all four meet in the same point V, the polar of V (namely $QQ_{,}$, fig. 71) will contain the four poles of these four lines; and as two of them pass through the foci (fig. 70), their poles Q, $Q_{,}$ will be on the minor directrices (fig. 71); and as two of them are tangents to the given curve, their poles P, $P_{,}$ will be on the polar curve; and as the angles FVQ and $F_{,}VQ_{,}$ are equal (fig. 70), the segments PQ and $PQ_{,}$ will subtend equal angles at the centre O (fig. 71); and as the segments of the tangent which passes through P, namely PM and $PM_{,}$, also subtend equal angles at the centre, the angle POM will be equal to the angle $POM_{,}$, and therefore the angle QOM will be equal to the angle $Q_{,}OM_{,}$. And as $NN_{,}$ is a tangent at $P_{,}$, the angle $P_{\prime}ON$ will be equal to the angle $P_{\prime}ON_{\prime}$, and the angle $QON =$ the angle $Q_{\prime}ON_{\prime}$.

Fig. 71.



306.] It has been shown in (e), in the preceding section, that

$$2FVF_{\prime} = FQF_{\prime} + FQ_{\prime}F_{\prime} \text{ (fig. 70).}$$

If we now take the reciprocal polar of this property, the poles of the lines FV and $F_{\prime}V$ in fig. 70 will be the points Q and $Q_{\prime}$ in fig. 71, and the poles of the lines FQ and $F_{\prime}Q$ in fig. 70 will be the points M and $M_{\prime}$ on the minor directrices, and the poles of the lines $FQ_{\prime}$ and $F_{\prime}Q_{\prime}$ will be the points N and $N_{\prime}$ on the minor directrices. Hence the following theorem:—

If a secant be drawn to a central conic section cutting the curve in the points P *and* $P_{\prime}$, *and the minor directrices in the points* Q *and* $Q_{\prime}$, *and if through the points* P *and* $P_{\prime}$ *tangents be drawn meeting the minor directrices in the points* $MM_{\prime}$ *and* $NN_{\prime}$, *the sum of the angles which the straight lines* $MM_{\prime}$ *and* $NN_{\prime}$ *subtend at the centre will be equal to twice the angle which* $QQ_{\prime}$ *subtends at the same centre.*

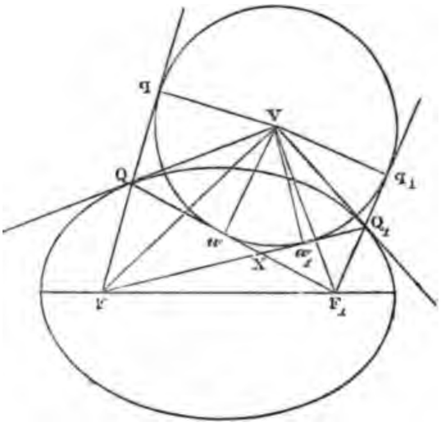
307.] If from any point V in the plane of a central conic section, of which the projective coordinates are p and q, lines $f, f_{\prime}$ be drawn to the foci $F, F_{\prime}$, and tangents to the curve touching it in the points Q and $Q_{\prime}$, the perpendiculars let fall from the point V on the four focal chords FQ, $F_{\prime}Q$, $FQ_{\prime}$, and $F_{\prime}Q_{\prime}$ are all equal.

This may briefly be shown by the method of tangential coordinates.

Let $\xi_{\prime}$ and $v_{\prime}$ be the tangential coordinates of the focal chord FQ, then the length of the perpendicular VP let fall upon it will be

$\frac{1-p\xi_{,}-qv_{,}}{\sqrt{\xi_{,}^2+v_{,}^2}}$, see sec. [5]. Now $\xi_{,}=\frac{1}{ae}$ and $v_{,}=\frac{a\xi-e}{b^2ev}$, ξ and v

Fig. 72.



being the tangential coordinates of the tangent VQ which passes through Q. Consequently

$$\sqrt{\xi_{,}^2+v_{,}^2}=\frac{ae\xi-1}{b^2ev}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

We must bear in mind that $\xi_{,}$ and $v_{,}$ are the tangential coordinates of the focal chord, while ξ and v are the tangential coordinates of the tangent to the curve passing through the point Q.

If now we substitute these values of $\xi_{,}$ and $v_{,}$ in the value of the perpendicular, we shall have

$$a\,.\,\mathrm{VP}=\frac{(ae-p)b^2v+a^2q\xi-aqe}{1-ae\xi}; \quad . \quad . \quad . \quad . \quad (b)$$

from this equation we must eliminate ξ and v.

The equation of the curve and the dual equation give

$$a^2\xi^2+b^2v^2=1, \text{ and } p\xi+qv=1. \quad . \quad . \quad . \quad . \quad (c)$$

Eliminating v,

$$\xi=\frac{b^2p+q\sqrt{\mathrm{M}}}{a^2q^2+b^2p^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

writing M for $a^2q^2+b^2p^2-a^2b^2$, we shall have also

$$v=\frac{a^2q-p\sqrt{\mathrm{M}}}{a^2q^2+b^2p^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (e)$$

Substituting these values of ξ and v in (b), we finally obtain

$$a^2\overline{\mathrm{VP}}^2=a^2q^2+b^2p^2-a^2b^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

Now, this expression for the perpendicular being a function of the projective coordinates p and q of the point V only and the constants of the tangential equation of the curve, it will hold for any one of the four focal chords. Consequently the four perpendiculars will be equal.

Hence we may obtain the following theorem (fig. 71):—

If through any two points P *and* $P_{\prime}$ *on a central conic section a secant be drawn, and two tangents through the points* P *and* $P_{\prime}$ *meeting the minor directrices in the four points* M, $M_{\prime}$, N, *and* $N_{\prime}$, *and perpendiculars be let fall from these four points on the secant, namely* MP, $M_{\prime}P_{\prime}$, $NP_{\prime\prime}$, $N_{\prime}P_{\prime\prime\prime}$, *the ratios*

$$\frac{MP}{MO}, \frac{M_{\prime}P_{\prime}}{M_{\prime}O}, \frac{NP_{\prime\prime}}{NO}, \frac{N_{\prime}P_{\prime\prime\prime}}{N_{\prime}O}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

will all be equal.

308.] A simple algebraical proof of this very elegant theorem may be given.

Let $x_{\prime}y_{\prime}$ and $x_{\prime\prime}y_{\prime\prime}$ be the coordinates of the points P and $P_{\prime}$. Then the equation of the straight line passing through these points will be

$$(y-y_{\prime})(x_{\prime}-x_{\prime\prime})-(y_{\prime}-y_{\prime\prime})(x-x_{\prime})=0; \quad . \quad . \quad . \quad \text{(a)}$$

or as $x_{\prime}y_{\prime}$ and $x_{\prime\prime}y_{\prime\prime}$ are points on the curve,

$$a^2y_{\prime}^2+b^2x_{\prime}^2=a^2b^2; \quad . \quad . \quad \text{(b)} \qquad a^2y_{\prime\prime}^2+b^2x_{\prime\prime}^2=a^2b^2, \quad . \quad . \quad \text{(c)}$$

or

$$a^2(y_{\prime}-y_{\prime\prime})(y_{\prime}+y_{\prime\prime})+b^2(x_{\prime}-x_{\prime\prime})(x_{\prime}+x_{\prime\prime})=0. \quad . \quad . \quad \text{(d)}$$

Consequently the equation of the secant becomes, by substitution,

$$a^2(y-y_{\prime})(y_{\prime}+y_{\prime\prime})+b^2(x-x_{\prime})(x_{\prime}+x_{\prime\prime})=0. \quad . \quad . \quad \text{(e)}$$

But this line meets the minor directrix in a point of which the coordinates are $\bar{x}$ and $\frac{b}{e}$.

Substituting these values in the preceding equation, we shall have

$$\bar{x}=\frac{a^2b^2e+e(a^2y_{\prime}y_{\prime\prime}+b^2x_{\prime}x_{\prime\prime})-a^2b(x_{\prime}+x_{\prime\prime})}{b^2e(x_{\prime}+x_{\prime\prime})}. \quad . \quad . \quad \text{(f)}$$

Now the tangent through P meets the minor directrix in a point of which the abscissa x^0 is given by the equation

$$x^0=\frac{a^2(be-y_{\prime})}{bex_{\prime}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(g)}$$

Subtracting x^0 from $\bar{x}$, we shall have

$$\bar{x}-x^0=\frac{a^2(y_{\prime\prime}x_{\prime}-y_{\prime}x_{\prime\prime})(b-ey_{\prime})}{b^2ex_{\prime}(x_{\prime}+x_{\prime\prime})}. \quad . \quad . \quad . \quad . \quad . \quad \text{(h)}$$

Here $\bar{x}-x^0$ denotes the distance between the points in which the secant and tangent cut the minor directrix, or $\bar{x}-x^0=\mathrm{QM}$, see fig. 71; but

$$\mathrm{OM}=\frac{a^2(b-ey_{\prime})}{bex_{\prime}},\text{ see sec. [298]; consequently}$$

$$\frac{\mathrm{QM}}{\mathrm{OM}}=\frac{(y_{\prime\prime}x_{\prime}-y_{\prime}x_{\prime\prime})}{b(x_{\prime}+x_{\prime\prime})}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}$$

But this is a symmetrical expression, independent of the particular position of any one of the four points M, $\mathrm{M}_{\prime\prime}$, N, $\mathrm{N}_{\prime}$ on the minor directrices.

Let MP be the perpendicular from the point M on the secant $\mathrm{QQ}_{\prime}$; let ϕ be the angle which this secant makes with the minor directrix. Then $\mathrm{MP}=\mathrm{QM}\sin\phi$. Now

$$\tan\phi=\frac{y_{\prime}-y_{\prime\prime}}{x_{\prime}-x^{\prime\prime}}=\frac{-b^2}{a^2}\,\frac{(x_{\prime}+x_{\prime\prime})}{(y_{\prime}+y_{\prime\prime})};$$

consequently

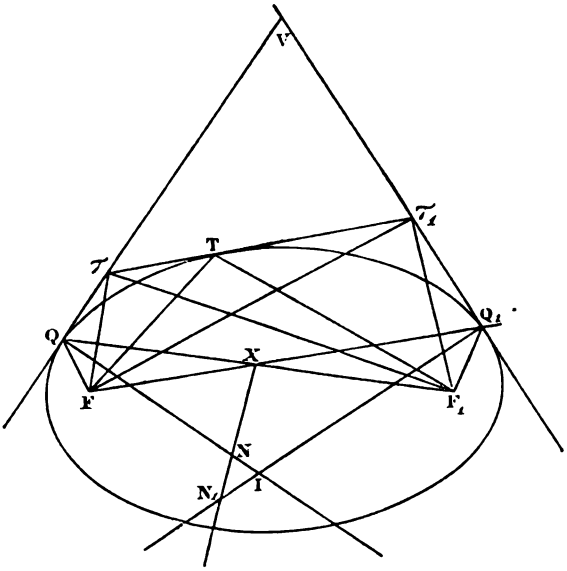
$$\sin\phi=\frac{b^2(x_{\prime}+x_{\prime\prime})}{[a^4(y_{\prime}+y_{\prime\prime})^2+b^4(x_{\prime}+x_{\prime\prime})^2]}.$$

Finally,

$$\frac{\mathrm{MP}}{\mathrm{MO}}=\frac{b(y_{\prime\prime}x_{\prime}-y_{\prime}x_{\prime\prime})}{[a^4(y_{\prime}+y_{\prime\prime})^2+b^4(x_{\prime}+x_{\prime\prime})^2]^{\frac{1}{2}}}, \quad . \quad . \quad . \quad \text{(j)}$$

a symmetrical expression in which $x_{\prime}$ and $y_{\prime}$, $x_{\prime\prime}$ and $y_{\prime\prime}$ are similarly involved.

Fig. 73.



309.] Let VQ, $\mathrm{VQ}_{\prime}$ be two fixed tangents drawn to a central conic section, and let $\tau\tau_{,}$ be a variable tangent touching the curve in the varying point T; the tangent $\tau\tau_{,}$ will subtend fixed angles at the foci, and the sum of these angles will be equal to the supplement of the angle $QVQ_{,}$.

Let the normals QN, $Q_{,}N_{,}$ be drawn to the curve at the points Q and $Q_{,}$, and let the angle $FXF_{,}$ be bisected by the line $XNN_{,}$, meeting the normals in the points N and $N_{,}$. Now let the angle

$$QF\tau = TF\tau = \alpha,\quad Q_{,}F\tau_{,} = TF\tau_{,} = \beta,$$

as also

$$Q_{,}F_{,}\tau_{,} = TF_{,}\tau_{,} = \alpha_{,},\quad QF_{,}\tau = TF_{,}\tau = \beta_{,};$$

let

$$FQN = NQX = \gamma, \text{ and } F_{,}Q_{,}N_{,} = N_{,}QX = \delta.$$

Now

$$FXF_{,} = 2\gamma + 2\alpha + 2\beta, \text{ or } \tfrac{1}{2}FXF_{,} = \alpha + \beta + \gamma;$$

but

$$\tfrac{1}{2}FXF_{,} \text{ or } F_{,}XN = \gamma + N, \text{ therefore } \alpha + \beta = N, \text{ or } \tau F\tau_{,} = QNX.$$

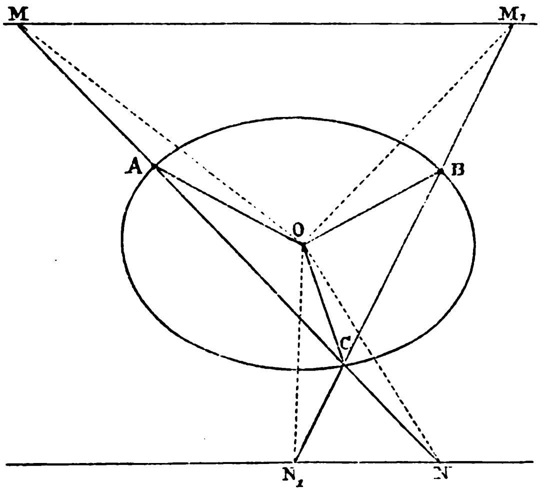
In like manner $\alpha_{,} + \beta_{,} = N_{,}$ or $\tau F_{,}\tau_{,} = Q_{,}N_{,}X$; consequently

$$\alpha + \beta + \alpha_{,} + \beta_{,} = N + N_{,} = QIQ = \text{supplement of } V,$$

since $VQQ_{,}I$ is a quadrilateral that may be inscribed in a circle.

Now applying to this theorem the method of reciprocal polars, we may infer, since the poles of the fixed tangents are two fixed points on the reciprocal (Σ), and the pole of the movable tangent a movable point on the same curve, and the polars of the three vertices V, τ, $\tau_{,}$ are three chords inscribed in (Σ), one fixed, the

Fig. 74.



others variable, and as the poles of the lines $F\tau$, $F\tau_{,}$ are the points in which these chords meet the minor directrix (Δ), and as the lines $F\tau$ and $F\tau_{,}$ contain a constant angle, and also $F_{,}\tau$ and $F_{,}\tau_{,,}$, we may infer the corresponding reciprocal theorem :—

If two fixed points A *and* B *be assumed on a conic section, and a third point* C *variable in position, and if the chords* CA, CB *be produced to meet the minor directrices* (Δ) *and* $(\Delta_{,})$ *in the points* $M, M_{,}$ *and* $N, N_{,,}$ *the line* $MM_{,}$ *will subtend a constant angle at the centre* O *; so also will the line* $NN_{,}$ *on the other directrix* $(\Delta_{,})$ *; and the sum of these angles will be constant and equal to the angle* AOB.

That the sum of the angles $MOM_{,}$ and $NON_{,}$ is equal to the angle AOB follows from the theorem established in sec. [305]; for the angle $AOM=CON$, and the angle $BOM_{,}$ is equal to $CON_{,}$; consequently the angle $AOB=MOM_{,}+NON_{,}$.

310.] When the tangents are drawn to the curve at the extremity of the major axis, the supplement of the angle between them is two right angles, since they are parallel. Hence the angle which the moving tangent subtends at one of the foci becomes a right angle, the poles of the two fixed tangents to (S) become the extremities of the minor axis of (Σ), and the preceding theorem becomes thus modified.

If a triangle be inscribed in a conic section whose base is the minor axis, and whose vertex is variable along the curve, the sides being produced will meet the minor directrix in two points $MM_{,,}$ *which will subtend a right angle at the centre.*

When the curve becomes a circle the minor directrix recedes to infinity, the lines OM, $OM_{,}$ coincide with the sides of the triangle; hence the angle in a semicircle is a right angle.

311.] Let a quadrilateral be circumscribed to a central conic section, and from one of the foci F let vectors FA, FB, FC, FD be drawn to the angles of the figure, and lines Fx, Fy, Fz, Fu to the points of contact; and as any two of these adjacent lines make equal angles with the vector line between them, let these angles be α, α; β, β; γ, γ; δ, δ; then $\alpha+\beta+\gamma+\delta=\pi$. Now the angle between the vectors FA and FB is $\alpha+\beta$, and the angle between the vectors FC and FD is $\gamma+\delta$.

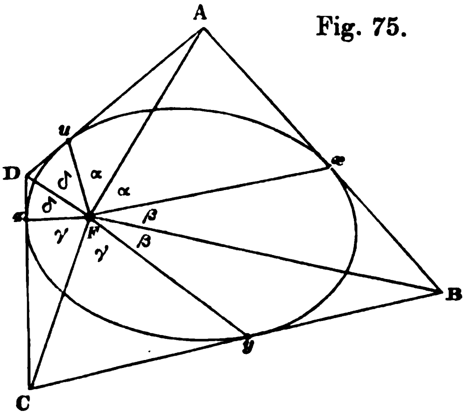


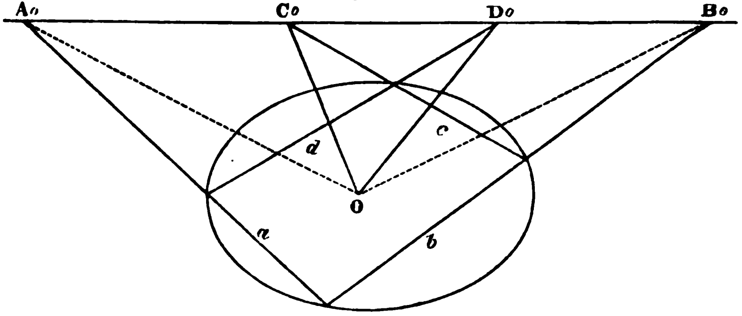
Fig. 75.

Let us now take the reciprocal polar of this theorem. The reciprocal polar of the quadrilateral *circumscribed* about the conic section will be a quadrilateral *inscribed* in a conic section; and the sides *a, b, c, d* of this inscribed conic will be the polars of the angles A, B, C, D of the circumscribed conic, and the pole of the vector FA will be the point A_0 on the minor directrix, in which the side *a* of the inscribed quadrilateral intersects it; the same may be said for the point B_0. Hence the line A_0B_0 on the minor directrix subtends at the centre the angle $\alpha+\beta$. In the same manner it may be shown that C_0 and D_0 subtend at the centre the angle $\gamma+\delta$. Consequently we shall have the following theorem:—

Let the sides of a quadrilateral inscribed in a conic section meet one of the minor directrices in the points A_0, B_0, C_0, D_0; *the sum of the angles which the segments* A_0B_0 *and* C_0D_0 *subtend at the centre will be equal to two right angles.*

When the curve is a circle the minor directrix recedes to infinity, and the lines OA_0, OB_0, OC_0, OD_0 become parallel to *a, b, c, d,* the sides of the inscribed figure. Hence we may infer that the sum of the opposite angles of a quadrilateral inscribed in a circle is equal to two right angles.

Fig. 76.



312.] Let ϕ be the angle between a pair of tangents to a conic section, $ff_{\prime}$ the focal radii through the point V (the intersection of the tangents), and P the perpendicular let fall from V on one of the focal chords (see fig. 72); we shall have

$$ff_{\prime}\sin\phi=2aP. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Let the perpendicular from the point V on the major axis divide the angle ϕ into the two angles ω and $\omega_{\prime}$, or let $\phi=\omega+\omega_{\prime}$; hence $\sin\phi=\sin\omega\cos\omega_{\prime}+\sin\omega_{\prime}\cos\omega$. Let the tangential coordinates of the tangent passing through V be $\xi_{\prime}\,v_{\prime}$ and $\xi_{\prime\prime}\,v_{\prime\prime}$, then

$$\sin\omega=\frac{\xi_{\prime}}{\sqrt{\xi_{\prime}^2+v_{\prime}^2}},\quad \cos\omega=\frac{v_{\prime}}{\sqrt{\xi_{\prime}^2+v_{\prime}^2}},$$

$$\sin\omega_{\prime}=\frac{\xi_{\prime\prime}}{\sqrt{\xi_{\prime\prime}^2+v_{\prime\prime}^2}},\quad \cos\omega_{\prime}=\frac{v_{\prime\prime}}{\sqrt{\xi_{\prime\prime}^2+v_{\prime\prime}^2}};$$

consequently

$$\sin\phi=\frac{\xi_{\prime}\upsilon_{\prime\prime}+\xi_{\prime\prime}\upsilon_{\prime}}{\sqrt{\xi_{\prime}^2+\upsilon_{\prime}^2}\sqrt{\xi_{\prime\prime}^2+\upsilon_{\prime\prime}^2}}. \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

We have now to calculate the value of this expression for $\sin\phi$. In sec. [307] it has been shown that

$$\xi_{\prime}=\frac{b^2p+q\sqrt{\text{M}}}{\text{L}},\quad \upsilon_{\prime}=\frac{a^2q-p\sqrt{\text{M}}}{\text{L}},\quad \xi_{\prime\prime}=\frac{b^2p-q\sqrt{\text{M}}}{\text{L}},$$

and $\upsilon_{\prime\prime}=\frac{a^2q+p\sqrt{\text{M}}}{\text{L}}$, writing L for $a^2q^2+b^2p^2$.

Consequently $\xi_{\prime}\upsilon_{\prime\prime}+\xi_{\prime\prime}\upsilon_{\prime}=\frac{2\sqrt{\text{M}}}{\text{L}}$; . . . . . . (c)

and as

$$(\xi_{\prime}^2+\upsilon_{\prime}^2)(\xi_{\prime\prime}^2+\upsilon_{\prime\prime}^2)=\xi_{\prime}^2\xi_{\prime\prime}^2+\xi_{\prime}^2\upsilon_{\prime\prime}^2+\xi_{\prime\prime}^2\upsilon_{\prime}^2+\upsilon_{\prime}^2\upsilon_{\prime\prime}^2,$$

we shall have, by substitution,

$$(\xi_{\prime}^2+\upsilon_{\prime}^2)(\xi_{\prime\prime}^2+\upsilon_{\prime\prime}^2)=\frac{[(p^2+q^2)^2+2(a^2-b^2)q^2-2(a^2-b^2)p^2+(a^2-b^2)^2]}{\text{L}^2}. \quad \text{(d)}$$

Now, as the focal lines f and $f_{\prime}$ are drawn from the point V,

$$f^2=q^2+(ae-p)^2,\ \text{and}\ f_{\prime}^2=q^2+(ae+p)^2; \quad . \quad . \quad . \quad \text{(e)}$$

or

$$f^2f_{\prime}^2=[(p+q)^2+2(a^2-b^2)q^2-2(a^2-b^2)p^2+(a^2-b^2)^2]. \quad \text{(f)}$$

Consequently $(\xi_{\prime}^2+\upsilon_{\prime}^2)(\xi_{\prime\prime}^2+\upsilon_{\prime\prime}^2)=\frac{f^2f_{\prime}^2}{\text{L}^2}$; and therefore

$$\sin\phi=\frac{\xi_{\prime}\upsilon_{\prime\prime}+\xi_{\prime\prime}\upsilon_{\prime}}{\sqrt{\xi_{\prime}^2+\upsilon_{\prime}^2}\sqrt{\xi_{\prime\prime}^2+\upsilon_{\prime\prime}^2}}=\frac{2\sqrt{\text{M}}}{ff_{\prime}}=\frac{2a\text{P}}{ff_{\prime}}.$$

Hence, finally, $ff_{\prime}\sin\phi=2a\text{P}$, ... (g)

where P is the perpendicular from V, the intersection of the tangents, and whose value we found in sec. [307] to be $a\text{P}=\sqrt{\text{M}}$.

313.] Hence we may readily obtain a very simple method of finding the curvature of a conic section.

Let AB be an arc of a conic section, on which two points, P, Q, indefinitely near to each other, are assumed. At these points let tangents to the curve be drawn meeting in s, and intersecting in the angle ϕ. Let $\text{P}s=\text{Q}s=c$; and let R, R, the radii of curvature at the points P and Q, meet in O.

Hence $\text{R}\sin\phi=\text{PT}=2c$ ultimately; for ultimately the two sides of the triangle PsQ coincide with the side PQ, which is ultimately equal to PT. Hence $\sin\phi=\frac{2c}{\text{R}}$.

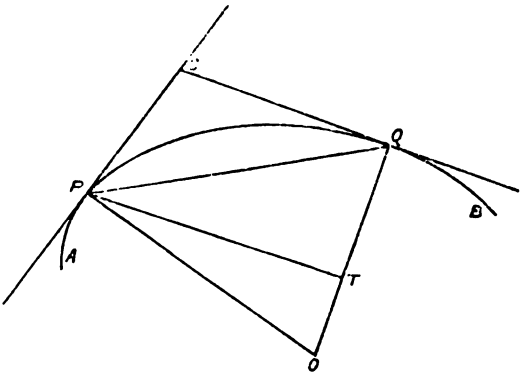
But in the preceding article it was shown that $\sin\phi=\frac{2a\text{P}}{ff_{\prime}}$.

Hence, eliminating sin ϕ,

$$R=\frac{cff_{\iota}}{aP}.$$

But when ultimately the point s coincides with the conic section, $ff_{\iota}=b_{\iota}^2$ and $\frac{c}{P}=\frac{b_{\iota}}{b}$.

Fig. 77.



Hence $R=\frac{b_{\iota}^3}{ab}$, a well-known expression for the radius of curvature of a conic section.

On groups of conic sections having the same minor directrices.

314.] We shall find that peculiar relations exist between conic sections having the same minor directrices. Let h be the distance between the common centre and one of the directrices; then the axes of the curve are connected by the relation

$$\frac{1}{b^2}=\frac{1}{a^2}+\frac{1}{h^2}. \quad \text{. . . . .} \quad \text{(a)}$$

A few examples of these properties are given.

Let a series of concentric conic sections, all having the same minor directrices, be cut by a transversal, the portions of this line intercepted by any pair of these curves will subtend equal angles at the centre; and if, through every pair of points in which this transversal intersects the sections, tangents are drawn intercepted both ways by the directrices, the sum of the angles which any pair of these tangents subtend at the centre is constant, being equal to twice the angle which the common transversal intercepted both ways by the directrices subtends at the centre.

Let $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } x\xi+y\upsilon=1 \quad \text{. . . . . .} \quad \text{(b)}$$

be the projective equation of the curve and the dual equation of the point (x, y). Let $y=mx$ be the equation of the diameter passing through the point (x, y). Eliminating x and y between these three equations, we shall have

$$a^2(1-b^2v^2)n^2-2a^2b^2\xi v\,.\,n+b^2(1-a^2\xi^2)=0. \quad . \quad . \quad (c)$$

Let $n_{,}$ and $n_{,,}$ be the roots of this quadratic equation, we shall have $n_{,}+n_{,,}=\frac{2a^2b^2\xi v}{a^2(1-b^2v^2)}$ and $n_{,}n_{,,}=\frac{b^2(1-a^2\xi^2)}{a^2(1-b^2v^2)}$. . . . . (d)

Let ϕ be the sum of the angles whose tangents are $n_{,}$ and $n_{,,}$. Then

$$\tan\phi=\frac{n_{,}+n_{,,}}{1-n_{,}n_{,,}}=\frac{2\xi v}{\frac{1}{b^2}-\frac{1}{a^2}+\xi^2-v^2}. \quad . \quad . \quad . \quad . \quad (e)$$

Now, as the sections are assumed to have the same minor directrices, $\frac{1}{b^2}-\frac{1}{a^2}=\frac{1}{h^2}$; consequently

$$\tan\phi=\frac{2h^2\xi v}{1+h^2(\xi^2-v^2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (f)$$

a value independent of a and b.

Let $n=\tan\omega$, then $n_{,}=\tan(2\pi-\omega_{,})=-\tan\omega_{,}$; therefore

$$\phi=\omega+2\pi-\omega_{,}=\omega-\omega_{,};$$

or ϕ is the geometrical difference of the angles ω and $\omega_{,}$; consequently

$$\tan\phi=\tan(\omega-\omega_{,})=\frac{2h^2\xi v}{1+h^2(\xi^2-v^2)}. \quad . \quad . \quad . \quad (g)$$

Now, when the difference between two variable quantities is constant, these variable quantities must receive equal increments; but these increments are the angles between each successive pair of diameters.

315.] If two diameters at right angles revolve round the centre of two conic sections having the same minor directrices, each diameter meeting one of the curves, the line joining these points will envelop a circle.

Assuming the equation (c) established in the preceding section we shall have

$$n_{,}n_{,,}=-1=\frac{b^2(1-a^2\xi^2)}{a^2(1-b^2v^2)}, \text{ or } \xi^2+v^2=\frac{1}{b^2}-\frac{1}{a^2}. \quad . \quad . \quad . \quad (a)$$

But when the curves have the same minor directrices

$$\frac{1}{b^2}-\frac{1}{a^2}=\frac{1}{h^2} \text{ or } h^2(\xi^2+v^2)=1,$$

the equation of a circle.

When the connected points are on different curves, let r and $r_{,}$ be the two semidiameters; then

$$\frac{\cos^2\phi}{a^2}+\frac{\sin^2\phi}{b^2}=\frac{1}{r^2}; \text{ but } \frac{1}{b^2}-\frac{1}{a^2}=\frac{1}{h^2};$$

consequently

$$\frac{1}{a^2}+\frac{\sin^2\phi}{h^2}=\frac{1}{r^2}. \quad \ldots\ldots\ldots \quad \text{(b)}$$

For the diameter at right angles

$$\frac{1}{a_{\prime}^2}+\frac{\cos^2\phi}{h^2}=\frac{1}{r_{\prime}^2}; \quad \ldots\ldots\ldots \quad \text{(c)}$$

adding,

$$\frac{1}{r^2}+\frac{1}{r_{\prime}^2}=\frac{1}{a^2}+\frac{1}{a_{\prime}^2}+\frac{1}{h^2}, \quad \ldots\ldots \quad \text{(d)}$$

or the sum of the squares of the reciprocals of the semidiameters, drawn one to each curve, is constant; and as

$$\frac{1}{\mathrm{P}^2}=\frac{1}{r^2}+\frac{1}{r_{\prime}^2}=\frac{1}{a^2}+\frac{1}{a_{\prime}^2}+\frac{1}{h^2},$$

P the perpendicular on the line joining the feet of r and $r_{\prime}$ is constant, and therefore the locus is a circle.

The difference of the squares of the reciprocals of any two coincident semidiameters of two conic sections having the same minor directrices is constant.

For $\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{a^2}+\frac{\sin^2\theta}{h^2}$, and $\frac{1}{r_{\prime}^2}=\frac{1}{a_{\prime}^2}+\frac{\sin^2\theta}{h^2}$; hence $\frac{1}{r^2}-\frac{1}{r_{\prime}^2}=\frac{1}{a^2}-\frac{1}{a_{\prime}^2}$.

316.] *Let a series of concentric conic sections, having the same minor directrices, be cut by a common diameter, the tangents drawn through the points where this diameter intersects the curves envelop a concentric conic section.*

The solution of this question is simply obtained by the method of tangential coordinates. Let

$$b^2v^2 \pm a^2\xi^2=1 \quad \ldots\ldots\ldots \quad \text{(a)}$$

be the tangential equation of one of the series of ellipses or hyperbolas, and, as they all have the same minor directrices,

$$\frac{1}{a^2}+\frac{1}{k^2}=\frac{1}{b^2}; \quad \ldots\ldots\ldots \quad \text{(b)}$$

let $y=nx$ be the projective equation of the diameter, then

$$\frac{b^2v}{a^2\xi}=n. \quad \ldots\ldots\ldots \quad \text{(c)}$$

Eliminating a and b between the equations (a), (b), and (c), we find

$$v^2 \mp \xi^2-\left(\frac{n^2-1}{n}\right)v\xi=\frac{1}{k^2}, \quad \ldots\ldots \quad \text{(d)}$$

the equation of a concentric equilateral hyperbola or ellipse.

On certain properties of the Equilateral Hyperbola.

317.] As the equilateral hyperbola is its own reciprocal, the centre of the curve being the centre of the polarizing circle, the major and minor foci will coincide; so also will the major and minor directrices, as shown in sec. [289]. Hence all the focal properties of this curve possess analogous central properties also. For example, we may instance the following:—

(α) *If a tangent to an equilateral hyperbola meet the directrices in the points* M *and* $M_{\prime}$, *the difference of the distances of these points from the transverse axis will be to the difference of the distances of the same points from the centre in the ratio of* $\sqrt{2} : 1$.

(β) *If a tangent be drawn to an equilateral hyperbola meeting the directrices in* M *and* $M_{\prime}$, *the ratio of the distances of these points from the centre is equal to the ratio of the focal perpendiculars on the tangent.*

(γ) *If two diameters at right angles be drawn, one meeting the equilateral hyperbola in the point* M, *while the other meets the directrix in* N, *the line* MN *envelops a circle described on the transverse axis as diameter.*

(δ) *If from any point two tangents are drawn to an equilateral hyperbola, the line joining this point with the focus, and meeting its directrix in* M, *while the chord of contact meets the same directrix in* N, *the portion of the directrix* MN *subtends a right angle at the centre.*

(ϵ) *If from a point in the directrix of an equilateral hyperbola two tangents be drawn to the curve, this point, and the point* C, *where the chord of contact meets the directrix, subtend a right angle at the centre.*

(ζ) *If from any point in the plane of an equilateral hyperbola two tangents be drawn meeting the directrix in two points* M *and* N, *the chord of contact meeting the directrix in* G, *the diameter* OG *bisects the angle* MON.

(η) *Two chords are drawn from the extremities of the transverse axis of an equilateral hyperbola meeting the curve in any point* C, *and produced to meet the directrix in the points* M *and* N. *These points will subtend a right angle at the centre.*

(θ) *If two fixed points be assumed on an equilateral hyperbola, and a third point variable, the chords drawn through this point and the two fixed points will meet the directrix in two points* M *and* N, *which will subtend at the centre a constant angle.*

(κ) *From any point* V *let two tangents be drawn to an equilateral hyperbola. Join* V *with the centre* O, *through* O *draw a perpendicular to* OV *meeting the chord of contact in* T. *The perpendiculars to the transverse axis through* T *and* V *with the two directrices are four harmonicals.*

(λ) *If two fixed points be assumed on an equilateral hyperbola, and a third point variable, the chords drawn through this point and the two fixed points will meet the directrices in four points* $\mathrm{M\,M}_{,}$ *and* $\mathrm{N\,N}_{,}$; *the sum of the angles at the centre subtended by* $\mathrm{MM}_{,}$ *and* $\mathrm{NN}_{,}$ *is equal to the angle subtended at the centre by the two fixed points.*

(μ) *If a straight line be drawn meeting the directrix of an equilateral hyperbola in* P, *and if from any point in this straight line two tangents be drawn meeting the directrix in* T *and* $\mathrm{T}_{,}$, *the product of the tangents of the semiangles* TOP *and* $\mathrm{T}_{,}\mathrm{OP}$ *is constant.*

There would be no difficulty in adding to these theorems others of a like kind; but enough perhaps has been given to show the exhaustless fertility of the method. The judicious student may easily augment the number; *primo avulso non deficit alter.*

CHAPTER XXX.

ON THE LOGOCYCLIC CURVE, THE TRIGONOMETRY OF THE PARABOLA, AND THE GEOMETRICAL ORIGIN OF LOGARITHMS*.

318.] It must doubtless have often appeared strange to geometers that no direct connexion could be traced between those arithmetical quantities called logarithms and certain other corresponding geometrical magnitudes. It is true that an imaginary relation between logarithms and circular arcs may be exhibited by means of the theorems of Cotes and De Moivre; but this imaginary relation would seem rather to indicate a real corresponding relation of duality with some other geometrical magnitude. As the principle of duality is of universal application in geometry, and as every property of circumscribed space—whatever be the nature of the bounding lines or surfaces—has its correlative or dual, it is a natural anticipation that this duality must exist somewhere for the trigonometry of the circle.

In the following pages an attempt is made to develop this principle of duality, and to show the connexion that exists between logarithms and parabolic trigonometry.

The obscurities which hitherto have hung over the geometrical theory of logarithms will be, it is hoped, now removed. It is possible to represent logarithms, as elliptic integrals usually have been represented, by curves devised to exhibit some special property only; and accordingly such curves, while they place before us the properties they have been devised to represent, fail generally to carry us any further. The close analogies which connect the theory of logarithms with the properties of the circle will no longer appear inexplicable.

---

* The substance of the following Chapters was embodied in a paper read before the Mathematical Section of the British Association at Cheltenham in 1856, and printed among the Reports of the Association for that year.

To devise a curve that shall represent one condition of a theory, or one truth of many, is easy enough. Thus, if we had first obtained by pure analysis all the properties of the circle without any previous conception of its form, and then proceeded to find a geometrical figure which should satisfy the conditions developed in the theory, we might hit upon several geometrical curves that would satisfy some of the established conditions, though not all. That all lines passing through a fixed point and terminated both ways by the curve shall be bisected in that point, would be satisfied as well by an ellipse or an hyperbola as by a circle. That all the lines passing through this point and terminated both ways by the curve shall be equal, would be satisfied as well by the cusp of a cardioid as by the centre of a circle; but no curve except the circle would fulfil all the analytical conditions of the theory of the circle.

In the same way, no curve but the parabola will satisfy all the conditions of the arithmetical theory of logarithms.

We shall now proceed to show that if the natural numbers ranging from zero to infinity be represented by the radii of a curve, called the Logocyclic curve, the logarithms of the numbers represented by these straight lines will be expressed by the *residual arcs** of a corresponding parabola described so as to have certain relations with the given Logocyclic curve.

This theory will decide a controversy long carried on between Leibnitz and J. Bernoulli on the subject of the logarithms of negative numbers. Leibnitz insisted they were imaginary, while Bernoulli argued they were real and the same as the logarithms of equal positive numbers. Euler espoused the side of the former, while D'Alembert coincided with the views of Bernoulli. Indeed, if we derive the theory of logarithms from the properties of the hyperbola (as geometers always have done), it will not be easy satisfactorily to answer the argument of Bernoulli—that as a hyperbolic area represents the logarithm of a positive number, denoted by the positive abscissa $+x$, so a negative number, according to conventional usage, being represented by the negative abscissa $-x$, the corresponding hyperbolic area should denote its logarithm also. And this is the more remarkable, because by Van Huraet's method the quadrature of the hyperbola itself depends on the rectification of the parabola, as is generally known. All this obscurity will be cleared up by the theory developed in the text, which completely establishes the correctness of the views of Leibnitz and Euler.

---

* By the *residual arc* of a curve is meant the difference between an arc of a curve, and that portion of the rectilinear tangent drawn at its extremity which is the *projection* of the radius vector upon it; or, in other words, the *residual arc* is the difference between this *arc* and its *protangent*.

It is somewhat remarkable in the history of mathematical science, that although the arithmetical properties of logarithms have been familiarly known to every geometer since the time of Napier, their inventor, or rather discoverer, no mathematician has hitherto divined their true geometrical origin. And this is the more singular, because the properties of the logarithms of imaginary numbers are intimately connected with those of the circle. No satisfactory reason has been shown why this should be so. The logarithmic curve, which has been devised to represent one well-known property of logarithms, is a transcendental curve, and has no connexion with the circle. Neither has any attempt been made to show how the Napierian base *e*, an abstract isolated incommensurable number, may be connected with our known geometrical knowledge. Had the circle never been made a geometrical conception, the same obscurity might probably have hung over the signification of π which has hitherto concealed from us the real interpretation of the Napierian base *e*.

This affords another instance, were any needed, to show how thin the veil may be which suffices to conceal from us the knowledge of apparently the simplest truths, the clue to whose discovery is even already in our hands. The geometrical origin of logarithms and the trigonometry of the parabola ought, in logical sequence, to have been developed by Napier, or by one of his immediate successors. They had many indications to direct them aright in their investigations. So true it is that men, in the contemplation of remote truths, often overlook those that are lying before their feet!

It will be shown in the following pages that the theory of logarithms is a result of the solution of the geometrical problem to find and compare the lengths of arcs of a parabola, just as plane trigonometry is nothing more than the development of the same problem for the circle. It has been shown, too, elsewhere*, that elliptic integrals of the three orders do in all cases represent the lengths of curves which are the symmetrical intersections of the surfaces of a sphere or a paraboloid by ruled surfaces. These functions divide themselves into two distinct groups, representing spherical and paraboloidal curves; and by no rational transformation can we pass from the one group to the other. The transition is always made by the help of imaginary transformations, as when we pass from the real logarithms of the parabola to the imaginary logarithms of the circle. When we take plane sections of those surfaces (that is to say, a circle and a parabola), the theory of elliptic integrals becomes simply common trigonometry, or parabolic trigonometry with the theory of logarithms.

---

* "Researches on the Geometrical Properties of Elliptic Integrals," Philosophical Transactions for 1852, p. 316.

These views will suggest the reflection, how very narrow is the field of that vast region, the Integral Calculus, which has hitherto been cultivated or even explored! When we find that the highest and most abstruse of known functions, not only circular functions and logarithms, but also elliptic integrals of the three orders, are exhausted, "used up," in representing the symmetrical intersections of surfaces of the second order, who shall exhibit and tabulate the integrals of those functions which represent the unsymmetrical sections of surfaces of the second order, or generally those curves of double curvature in which surfaces of the third and higher orders intersect? Considerations such as these but add fresh evidence to the truth, how small even in mathematics is the proportion which the known bears to the unknown!

319.] The properties of the Logocyclic curve (which we have so named from the similarity of many of its properties to those of the circle, and from its use in representing numbers with their logarithms) are very numerous, some are remarkable, and almost all of them may be easily investigated. We shall now proceed to develop a few of the leading properties of this curve.

On the Equations of the Logocyclic Curve.

Let r and θ be the polar coordinates of the curve, its equation is, a being a constant,

$$r = a(\sec\theta \pm \tan\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Since $\sec\theta = \frac{r}{x}$, $\tan\theta = \frac{y}{x}$, and $r^2 = x^2 + y^2$,

we get for the equation of the curve in rectangular coordinates

$$\left.\begin{aligned} &y^2(2a-x) = x(x-a)^2, \\ \text{or} \quad &(x^2+y^2)(2a-x) = a^2x. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

320.] Let a parabola be described whose focus is at the origin, and let its vertical focal distance be put $=a$. Assume the axis of this curve and the perpendicular to it through the focus as the axes of x and y. Let the vertical tangent OT and the directrix DS of the parabola be drawn, then we shall have the following properties of the Logocyclic curve.

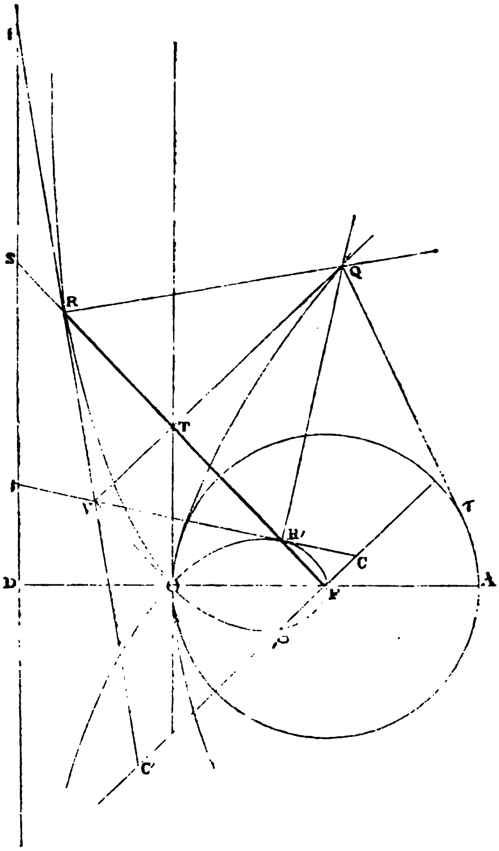
The directrix DS (fig. 78) of the parabola is an asymptote to the curve; for the perpendicular distance of the point R from this line is $a(1-\sin\theta)$, since $\text{RS} = a(\sec\theta - \tan\theta)$, and this value approximates to 0 as θ approaches to $\frac{\pi}{2}$.

Let ψ be the angle which the tangent to the curve makes with the radius vector (or vector, as we may more briefly call it), then

$$\tan\psi = \frac{rd\theta}{dr} = \cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (a)$$

When the vector coincides with the axis a, $\theta = 0$, hence $\tan\psi = 1$, or the curve intersects at right angles in the point O, making angles of 45° with the axis a.

Fig. 78.



The vector $r = a\,(\sec\theta \pm \tan\theta)$ drawn from the focus F of the parabola cuts the loop and one of the infinite branches in the points R and $R_{\prime}$, so that

$$FR = a\,(\sec\theta + \tan\theta), \quad FR_{\prime} = a\,(\sec\theta - \tan\theta)\,; \quad . \quad . \qquad (b)$$

the product of the segments of this secant, as in the circle, is constant and equal to a^2.

These points may be called *reciprocal points*.

Since $\mathrm{TO}=\mathrm{TR}=\mathrm{TR}_{,}=a\tan\theta$,

a circle described with T as centre touching the axis in O will pass through the points R and $\mathrm{R}_{,}$ on the vector FS. Hence the curve may be described from point to point with the use only of the rule and compass. Therefore $\mathrm{FR}_{,}$ is always equal to SR, and the angle $\mathrm{ROR}_{,}$ is always a right angle.

321.] Through T draw the straight line TQ at right angles to the line $\mathrm{RR}_{,}$, and touching the parabola in Q; join the points QR, $\mathrm{QR}_{,}$, and draw tangents to the logocyclic curve at the reciprocal points $\mathrm{RR}_{,}$ and let them meet in V. Now, as $\tan \mathrm{QR}_{,}\mathrm{T}=\dfrac{\mathrm{QT}}{\mathrm{TR}_{,}}$ and $\mathrm{QT}=a\sec\theta\tan\theta$, while $\mathrm{TR}_{,}=a\tan\theta$, $\therefore \tan\mathrm{QR}_{,}\mathrm{T}=\sec\theta$; but $\tan\mathrm{TR}_{,}\mathrm{V}$ or $\tan\psi=\cos\theta$; $\therefore$ ψ and the angle $\mathrm{QR}_{,}\mathrm{T}$ are complementary, or $\mathrm{QR}_{,}\mathrm{V}$ is a right angle. Hence, if normals to the logocyclic curve be drawn through any two reciprocal points, they will meet in the parabola conjugate to the logocyclic in a point Q, which point may be called the logarithmic point corresponding to the radius vector passing through $\mathrm{RR}_{,}$.

Since the tangents of the angles VRT, $\mathrm{VR}_{,}\mathrm{T}$ are each equal to $\cos\theta$, the lines VR and $\mathrm{VR}_{,}$ are equal and equally inclined to the chord $\mathrm{RR}_{,}$. This is also a property of the circle.

322.] The locus of V, the intersection of tangents to the logocyclic at any two reciprocal points R and $\mathrm{R}_{,}$, is a cissoid whose axis is a, whose cusp is at O, and whose asymptote is the directrix of the parabola. Join VO; then, since $\mathrm{VT}=\mathrm{TR}_{,}\tan\psi=a\sin\theta$, and as a perpendicular from O on the vector $\mathrm{RR}_{,}$ is also $=a\sin\theta$, VO is parallel to FR, and $\mathrm{VO}=a\sec\theta-a\cos\theta=\dfrac{a\sin^2\theta}{\cos\theta}$. But since VO is parallel to TF, the angle $\mathrm{VOD}=\mathrm{TFO}=\theta$; therefore VO and θ are the polar coordinates of the locus of V. Let this radius be ρ, then

$$\rho=\frac{a\sin^2\theta}{\cos\theta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(a)}$$

which is the polar equation of the cissoid.

In rectangular coordinates, since

$$\rho^2=\frac{a\rho^2\sin^2\theta}{\rho\cos\theta} \text{ and } \rho^2=x^2+y^2,\ x^2+y^2=\frac{ay^2}{x},$$

or
$$y^2(a-x)=x^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(b)}$$

Hence a straight line VQ at right angles to a vector VO drawn to the cusp of the curve through a point V on the cissoid envelops a parabola whose vertex is at the cusp, and whose vertical focal distance is equal to a the axis of the cissoid; or, in other words, the locus of the foot of a perpendicular let fall from the vertex of a parabola on a tangent to it is a cissoid.

This is easily shown by the method of tangential coordinates. Since the line VQ is at right angles to the line VO, $VO=\rho=\frac{a\sin^2\theta}{\cos\theta}$. Hence, as $\rho\xi=\cos\theta$ and $\rho v=\sin\theta$, ξ and v being the tangential coordinates of the line VQ,

$$\xi=av^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

which is the tangential equation of a parabola.

This may be shown by the ordinary method thus:

The equation of the right line VQ at right angles to the line VO is

$$y-y_\prime+\frac{x_\prime}{y_\prime}(x-x_\prime)=\mathrm{U}=0. \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Eliminating $y_\prime$ by the equation of the curve, and making $\frac{d\mathrm{U}}{dx_\prime}=0$, we should find the equation of the locus. The work is tedious, but involves no principle of any difficulty, it is therefore omitted.

323.] The sum of the ordinates of the reciprocal points is equal to the ordinate of the corresponding *logarithmic* point on the parabola;

for $\qquad y_\prime=r\sin\theta=a\,(1+\sin\theta)\tan\theta,$

and $\qquad y_{\prime\prime}=r_\prime\sin\theta=a\,(1-\sin\theta)\tan\theta;$

hence $\qquad y_\prime+y_{\prime\prime}=2a\tan\theta=$ the ordinate of Q. . . . (a)

In the same way it may be shown that

$$x_\prime+x_{\prime\prime}=2a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

We have also $\qquad y_\prime y_{\prime\prime}+x_\prime x_{\prime\prime}=a^2.$ . . . . (c)

324]. The distances of any point Q on a parabola from its focus and directrix are equal. We may generalize this well-known theorem, and say the distances are equal of any point on a parabola from its logocyclic curve measured along the two normals to this curve drawn through the point Q. This is evident, for the normals QR and $\mathrm{QR}_\prime$ make equal angles with the tangent QT.

Hence, if we conceive the surfaces of revolution and the paraboloid generated by the revolution of the logocyclic curve and the parabola round the axis DA, a luminous object of the form $\mathrm{OR}_\prime\mathrm{F}$ between the focus and vertex or the paraboloid would be reflected by the surface of the paraboloid in diverging rays from the conjugate surface OR, along the diverging lines RQ.

Through the two reciprocal points R, $\mathrm{R}_\prime$, on the logocyclic curve, the logarithmic point Q on the parabola and the intersection on the cissoid at V of the tangents drawn to the logocyclic through the reciprocal points R and $\mathrm{R}_{,}$ a circle may be described. This is evident from an inspection of the figure.

325]. The sum of the polar subtangents FC and $\mathrm{FC}_{,}$ belonging to the points R and $\mathrm{R}_{,}$ is constant and equal to $2a$;

for $\mathrm{FC}=\mathrm{R}_{,}\mathrm{F}\,.\,\tan \mathrm{CR}_{,}\mathrm{F}=r_{,}\tan\psi=a(\sec\theta-\tan\theta)\tan\psi$

$=a(\sec\theta-\tan\theta)\cos\theta$, ... (a) since $\tan\psi=\cos\theta$;

hence $\mathrm{FC}=a(1-\sin\theta)$. In like manner $\mathrm{FC}_{,}=a(1+\sin\theta)$;

therefore $$\mathrm{FC}+\mathrm{FC}_{,}=2a. \quad . \; . \; . \; . \; . \; . \; . \quad \text{(b)}$$

The sum of the reciprocals of the polar subnormals is constant and equal to $\frac{2}{a}$; for if ν be the subnormal to the point R,

$$\frac{1}{\nu}=\frac{\tan\psi}{r}=\frac{\cos\theta}{a(\sec\theta+\tan\theta)}=\frac{\cos^2\theta}{a(1+\sin\theta)}.$$

Similarly for the point $\mathrm{R}_{,}$,

$$\frac{1}{\nu}=\frac{\cos^2\theta}{a(1+\sin\theta)};\ \text{consequently}\ \frac{1}{\nu}+\frac{1}{\nu_{,}}=\frac{2}{a}. \quad . \; . \; . \quad \text{(c)}$$

Hence the locus of the extremities of the subtangents FC and $\mathrm{FC}_{,}$ belonging to the reciprocal points R and $\mathrm{R}_{,}$ is a cardioid whose diameter is $2a$, whose cusp is at F, and whose axis is the axis of the figure; while the locus of the extremities of the polar subnormals is a parabola confocal with the original parabola, and having its vertical focal distance equal to $\frac{a}{2}$.

326.] The lengths of the tangents to the curve between any two reciprocal points and the asymptote are equal to one another, or $\mathrm{R}t=\mathrm{R}_{,}t_{,}$.

For in the triangle $\mathrm{SR}_{,}t_{,}$,

$$\mathrm{R}_{,}t_{,} : \mathrm{R}_{,}\mathrm{S} :: \sin \mathrm{R}_{,}\mathrm{S}t_{,} : \sin \mathrm{S}t_{,}\mathrm{R}_{,},$$

or $$t : r :: \cos\theta : \sin\left(\frac{\pi}{2}-\theta+\psi\right);$$

hence $$t=\frac{r\cos\theta}{\cos\theta\cos\psi+\sin\theta\sin\psi};$$

but as $r\cos\theta=a(1+\sin\theta)$, and $\cos\theta\cos\psi=\sin\psi$,

we shall have $$t=\frac{a}{\sin\psi}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(a)}$$

We should find the same expression for the value of Rt.

Hence $$\mathrm{R}t=\mathrm{R}_{,}t_{,}\ \text{or}\ t=t_{,}=\frac{a}{\sin\psi}.$$

Hence also the perpendiculars $t'\mathrm{U}$ let fall from the points t and $t_{\prime}$ on the vector FS are equal to one another and to a.

Since $\quad \mathrm{S}t_{\prime}\cos\theta=a, \quad \mathrm{S}t_{\prime}=a\sec\theta.$

$$\mathrm{V}t_{\prime}=\mathrm{R}_{\prime}\mathrm{C}, \text{ and } \mathrm{V}t=\mathrm{RC}_{\prime}.$$

As the projection of $\mathrm{V}t_{\prime}$ on FS is equal to

$$\mathrm{ST}-\mathrm{SU}=a(\sec\theta-\tan\theta),$$

$$\mathrm{V}t_{\prime}\cos\psi=a(\sec\theta-\tan\theta)=\mathrm{FR}_{\prime}=\mathrm{R}_{\prime}\mathrm{C}\cos\psi;$$

therefore $\mathrm{V}t_{\prime}=\mathrm{R}_{\prime}\mathrm{C}$; hence $\mathrm{VC}=\mathrm{R}_{\prime}t_{\prime}$;

consequently $\quad \mathrm{VC}_{\prime}=\mathrm{R}t$. . . . . (b)

Therefore the point V is equidistant from the reciprocal points R and $\mathrm{R}_{\prime}$, and also from C and $\mathrm{C}_{\prime}$.

327.] Let p and $p_{\prime}$ be the perpendiculars let fall from the focus F on the tangents drawn through the reciprocal points R and $\mathrm{R}_{\prime}$, then

$$p=r\sin\psi; \text{ but } \sin\psi=\frac{\cos\theta}{\sqrt{(1+\cos^2\theta)}}, \text{ and } r=\frac{a(1+\sin\theta)}{\cos\theta};$$

hence $\quad p=\dfrac{a(1+\sin\theta)}{\sqrt{(1+\cos^2\theta)}}$. . . . . . . (a)

In like manner $\quad p_{\prime}=\dfrac{a(1-\sin\theta)}{\sqrt{(1+\cos^2\theta)}}$; . . . . (b)

hence $\quad pp_{\prime}=a^2\sin^2\psi.$

In [326] it has been shown that if t, $t_{\prime}$ denote the lengths of the tangents between the two reciprocal points and the asymptote,

$$tt_{\prime}=\frac{a^2}{\sin^2\psi}; \text{ hence } pp_{\prime}tt_{\prime}=a^4. \quad . \ . \ . \ . \ . \quad \text{(c)}$$

328.] Let u and $u_{\prime}$ denote the portions of the tangents intercepted betwen the points R, $\mathrm{R}_{\prime}$, and the feet of the perpendiculars let fall from the origin upon them.

Then $\quad u=r\cos\psi, \; u_{\prime}=r_{\prime}\cos\psi;$

therefore $\quad uu_{\prime}=a^2\cos^2\psi.$

Hence also $\quad uu_{\prime}+pp_{\prime}=a^2$. . . . . (a)

329.] Let λ and $\lambda_{\prime}$ be the angles which the perpendiculars p and $p_{\prime}$ make with the axis FO. Then an inspection of the figure will show that $\lambda=\theta+\psi-\frac{\pi}{2}$, and $\lambda_{\prime}=\theta-\psi+\frac{\pi}{2}$. Hence

$$\lambda+\lambda_{\prime}=2\theta, \quad . \ . \ . \ . \ . \ . \ . \ . \quad \text{(a)}$$

or the sum of the angles which the perpendiculars on the tangents through the reciprocal points make with the axis are equal to twice the angle which the vector makes with the same axis.

We may express the perpendiculars on the tangents through the reciprocal points in terms of ψ, the angle which these tangents make with the vector, since

$$\cos\theta=\tan\psi,\ \sin\theta=\sqrt{(1-\tan^2\psi)},$$
$$\sqrt{(1+\cos^2\theta)}=\sec\psi.$$

Making these substitutions in (a) and (b), sec. [327],

$$\left.\begin{aligned} p&=a\{\cos\psi+\sqrt{(\cos^2\psi-\sin^2\psi)}\},\\ p_{\prime}&=a\{\cos\psi-\sqrt{(\cos^2\psi-\sin^2\psi)}\}.\end{aligned}\right\} \quad .\ .\ .\ .\ \text{(b)}$$

Let $\sqrt{2}\sin\psi=\sin\phi,\ 2\cos^2\psi=2-\sin^2\phi,$

or $\cos\psi=\sqrt{(1-\tfrac{1}{2}\sin^2\phi)}.$

Hence
$$\left.\begin{aligned} p&=a\{\sqrt{(1-\tfrac{1}{2}\sin^2\phi)}+\cos\phi\},\\ p_{\prime}&=a\{\sqrt{(1-\tfrac{1}{2}\sin^2\phi)}-\cos\phi\}.\end{aligned}\right\} \quad .\ .\ .\ .\ \text{(c)}$$

These formulæ will be found useful in the rectification of the curve.

On the quadrature of the Logocyclic Curve.

330.] Since from sec. [319], (b), $y=\dfrac{(a-x)\sqrt{(x)}}{\sqrt{(2a-x)}}$,

$$y=\frac{ax}{\sqrt{(2ax-x^2)}}-\frac{x^2}{\sqrt{(2ax-x^2)}}.\ .\ .\ .\ .\ .\ \text{(a)}$$

Therefore $$\int y\mathrm{d}x=a\int\frac{x\mathrm{d}x}{\sqrt{(2ax-x^2)}}-\int\frac{x^2\mathrm{d}x}{\sqrt{(2ax-x^2)}}.\ .\ .\ .\ \text{(b)}$$

But $$a\int\frac{x\mathrm{d}x}{\sqrt{(2ax-x^2)}}=-a\sqrt{(2ax-x^2)}+a^2\,\text{versin}^{-1}\frac{x}{a},\ .\ .\ \text{(c)}$$

and $$-\int\frac{x^2\mathrm{d}x}{\sqrt{(2ax-x^2)}}=\sqrt{(2ax-x^2)}\left(\frac{x}{2}+\frac{3a}{2}\right)-\frac{3a^2}{2}\,\text{versin}^{-1}\frac{x}{a}.\quad \text{(d)}$$

Hence, adding these results together,

$$\int y\mathrm{d}x=\left(\frac{a+x}{2}\right)\sqrt{(2ax-x)^2}-\frac{a^2}{2}\,\text{versin}^{-1}\frac{x}{a};\quad .\ .\ \text{(e)}$$

no constant is added, for the area begins from $x=0$. Hence the area of the half loop is found by taking the integral between the limits $x=0$ and $x=a$.

Therefore $$\int_0^a y\mathrm{d}x=a^2-\frac{a^2\pi}{4}.\ .\ .\ .\ .\ .\ .\ .\ \text{(f)}$$

When we require the area between the infinite branch of the curve and the asymptote, as $x>a$, we shall have

$$\int y\mathrm{d}x=\frac{a^2}{2}\,\text{versin}^{-1}\frac{x}{a}-\left(\frac{a+x}{2}\right)\sqrt{(2ax-x^2)}+\mathrm{C}.$$

As the area of the infinite branch of the curve begins with $x=a$, we shall have

$$0=\frac{a^2\pi}{4}-a^2+\mathrm{C}, \text{ or } \mathrm{C}=a^2-\frac{a^2\pi}{4};$$

hence
$$\int_a^{2a} y\,\mathrm{d}x=\frac{a^2\pi}{2}+\mathrm{C}=a^2+\frac{a^2\pi}{4}; \quad . \quad . \quad . \quad . \quad . \quad (g)$$

consequently the whole area of the logocyclic curve, *i. e.* the area of the loop and of the curve between the infinite branches and the asymptote, is equal to $4a^2$, or to the square of the distance between the focus and the asymptote, while the difference between these areas is equal to $a^2\pi$. Thus, while the *sum* of the two areas is equal to the square of $2a$, the *difference* of the two areas is equal to the area of the circle inscribed in the same square.

331.] The area may be found very easily by a simple transformation. For the loop assume $a-x=a\cos\theta$. Then, substituting and reducing,

$$\int y\,\mathrm{d}x=a^2\cos\theta(1-\cos\theta)\mathrm{d}\theta.$$

Hence area of the loop $=a^2\left(\sin\theta+\sin\theta\cos\theta-\frac{\theta}{2}\right)$,

taking the integral between the limits $\theta=0$ and $\theta=\frac{\pi}{2}$.

The area of loop $=a^2-\frac{a^2\pi}{4}$.

When the area between the infinite branch and the asymptote is required, we must assume

$$x-a=a\sin\theta,$$

and
$$\int y\,\mathrm{d}x=-a^2\int\sin\theta(1+\sin\theta)\mathrm{d}\theta;$$

hence
$$\int y\,\mathrm{d}x=-a^2\cos\theta+\frac{a^2\theta}{2}-\frac{a^2}{2}\sin\theta\cos\theta+\mathrm{C}.$$

Since the area begins when $\theta=0$,

$$0=-a^2+\mathrm{C}, \text{ or } \mathrm{C}=a^2,$$

or
$$\int y\,\mathrm{d}x=a^2(1-\cos\theta)+\frac{a^2\theta}{2}-\frac{a^2}{2}\sin\theta\cos\theta;$$

therefore
$$\int_0^{\frac{\pi}{2}} y\,\mathrm{d}x=a^2+\frac{a^2\pi}{4}.$$

332.] If we take the cissoid whose cusp is at F, and whose asymptote is the line DS (see figure 78), its equation may be written $y=x\sqrt{\left(\frac{x}{2a-x}\right)}$; and if we take the curve (known as the witch or the curve of Agnesi) whose vertex is at F, and whose asymptote is also the line DS, its equation may be written, F being the origin, $y = 2a\sqrt{\left(\frac{x}{2a-x}\right)}$; and the equation of the logocyclic curve referred to the same axes being $y = (a-x)\sqrt{\left(\frac{x}{2a-x}\right)}$, we shall have, putting Y_c, Y_a, and Y_λ for any coincident ordinates of the cissoid, the curve of Agnesi, and the logocyclic

$$\tfrac{1}{2}Y_a - Y_c = \pm Y_\lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Hence the respective areas of these curves must be in the same ratio as these coincident ordinates; or if we draw a pair of ordinates common to the three curves, the area of the logocyclic curve between these parallel ordinates will be equal to the difference between the corresponding areas of the cissoid and half the area of the curve of Agnesi.

Hence, taking the whole of the areas between these three curves and their common asymptote DS, as the area of the cissoid is three times that of the base-circle, and half the area of the curve of Agnesi equal to twice that of the base-circle, the area of the logocyclic, which is the difference between the areas of these curves, must be equal to the area of the base-circle OAT, as is otherwise shown in sec. [319].

333]. Through any two reciprocal points of the logocyclic curve R and $R_{\prime\prime}$, let ordinates be drawn to the logocyclic, the cissoid, and the curve which bisects all the ordinates of the curve of Agnesi, and let these ordinates be Y_λ, Y_λ', Y_c, Y_c', and Y_a, Y_a', and as $x_{\prime} + x_{\prime\prime} = 2a$, $x_{\prime}$ and $x_{\prime\prime}$ being the ordinates of the reciprocal points as shown in [10], we shall have, omitting the traits,

$$Y_c = x\sqrt{\left(\frac{x}{2a-x}\right)}, \qquad Y_c' = (2a-x)\sqrt{\left(\frac{2a-x}{x}\right)},$$

$$Y_a = a\sqrt{\left(\frac{x}{2a-x}\right)}, \qquad Y_a' = a\sqrt{\left(\frac{2a-x}{x}\right)},$$

$$Y_\lambda = (a-x)\sqrt{\left(\frac{x}{2a-x}\right)}, \qquad Y_\lambda' = (a-x)\sqrt{\left(\frac{2a-x}{x}\right)}.$$

Hence obviously

$$Y_cY_c' = Y_aY_a' + Y\ Y_\lambda', \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

a curious relation between the six ordinates of the three curves drawn three by three through any two reciprocal points of the logocyclic curve.

334]. The logocyclic curve is "inverse to itself."

A curve is said to be "inverse to itself" when the product of any two coincident vectors drawn from the pole to two points, one on each branch, is constant.

In the case of the logocyclic this is evident; for, its vectors being the roots of the quadratic equation

$$r^2-2a\sec\theta\,.\,r+a^2=0, \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{a})^*$$

we shall have

$$\mathrm{R}=a(\sec\theta+\tan\theta) \text{ and } r=a(\sec\theta-\tan\theta), \text{ or } \mathrm{R}r=a^2.$$

This may be shown also in rectangular coordinates.

Let x, y and $x_{\prime}, y_{\prime}$ be the coordinates of any two *reciprocal points*. Then as $\frac{x}{r}=\frac{x_{\prime}}{\mathrm{R}}$, $x=\frac{r}{\mathrm{R}}x_{\prime}=\frac{a^2x_{\prime}}{\mathrm{R}^2}=\frac{a^2x_{\prime}}{x_{\prime}^2+y_{\prime}^2}$. In the same manner $y=\frac{a^2y_{\prime}}{x_{\prime}^2+y_{\prime}^2}$. Now, if we substitute these values of x and y in the equation of the logocyclic curve given in sec. [319], namely

$$(x^2+y^2)(2a-x)=a^2x,$$

we shall have

$$(x_{\prime}^2+y_{\prime}^2)\,(2a-x_{\prime})=a^2x_{\prime}; \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{b})$$

that is, we shall reproduce the original equation of the curve by this substitution. This property is peculiar to all curves that are inverse to themselves.

On the rectification of the Logocyclic Curve.

335.] It may be shown, if Σ and σ are the arcs of any two "inverse curves" terminating in a common vector which makes the angle θ with the axis, P and p being the perpendiculars let fall from the pole on the tangents drawn through the reciprocal points, that

$$\Sigma=k^2\int\frac{\mathrm{d}\theta}{p}, \text{ and } \sigma=k^2\int\frac{\mathrm{d}\theta}{\mathrm{P}}, \quad . \quad . \quad . \quad . \quad (\mathrm{a})\dagger$$

k being a constant. This is a new formula for rectification and one of remarkable simplicity. It is easily established.

Let $\mathrm{R}r=k^2$, and put $ru=1$. Then $\mathrm{R}=k^2u$, and

$$\Sigma=\int\sqrt{\left(\mathrm{R}^2+\frac{\mathrm{dR}^2}{\mathrm{d}\theta^2}\right)}\mathrm{d}\theta=k^2\int\sqrt{\left(u^2+\frac{\mathrm{d}u^2}{\mathrm{d}\theta^2}\right)}\mathrm{d}\theta. \quad . \quad (\mathrm{b})$$

---

* The polar equation of the circle is

$$r^2-2a\cos\theta\,.\,r+a^2=k^2,$$

k being the radius of the circle.

† This expression for the arc of an inverse curve suggests a very beautiful geometrical representation for the velocity of a body in any part of its orbit, subject to any law of central force whatever. For as the velocity in the orbit is inversely as the perpendicular from the centre of force on the tangent, or $v=\frac{h^2}{p}$, we shall have $\frac{\mathrm{d}\Sigma}{\mathrm{d}\theta}=\frac{k^2}{h^2}v$. Or the element of the curve inverse to the orbit between the vectors drawn through any two consecutive positions of the body will be a direct measure of the velocity with which the body describes that element of its orbit.

Now it is easily shown that $\sqrt{\left(u^2+\frac{du^2}{d\theta^2}\right)}$ is the reciprocal of the perpendicular p let fall from the pole on the tangent to the inverse curve drawn through the extremity of the vector, and which makes the angle θ with the axis. Hence

$$\Sigma=k^2\int\frac{d\theta}{p};\ \text{in like manner}\ \sigma=k^2\int\frac{d\theta}{P}.$$

336.] When the curve is the logocyclic, $k=a$, and

$$P=\frac{a\,(1+\sin\theta)}{\sqrt{(1+\cos^2\theta)}},\quad p=\frac{a\,(1-\sin\theta)}{\sqrt{(1+\cos^2\theta)}},$$

as shown in (a) and (b), sec. [327].

Hence

$$\frac{1}{P}=\frac{(1-\sin\theta)\,(1+\cos^2\theta)}{a\,.\cos^2\theta\sqrt{(1+\cos^2\theta)}},\quad \frac{1}{p}=\frac{(1+\sin\theta)(1+\cos^2\theta)}{a\,.\cos^2\theta\sqrt{(1+\cos^2\theta)}};\ .\ \text{(a)}$$

therefore
$$\Sigma=\frac{a}{\sqrt{(2)}}\int\frac{(1+\sin\theta)\,(1+\cos^2\theta)}{\cos^2\theta\sqrt{(1-\frac{1}{2}\sin^2\theta)}}\,.\,d\theta,\ .\ .\ \text{(b)}$$

and
$$\sigma=\frac{a}{\sqrt{(2)}}\int\frac{(1-\sin\theta)\,(1+\cos^2\theta)}{\cos^2\theta\sqrt{(1-\frac{1}{2}\sin^2\theta)}}.\ .\ .\ .\ .\ \text{(c)}$$

Adding (b) and (c), we shall have

$$\Sigma+\sigma=\sqrt{(2)}\,.a\int\frac{d\theta}{\cos^2\theta\sqrt{(1-\frac{1}{2}\sin^2\theta)}}+\sqrt{(2)}\,.a\int\frac{d\theta}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}}.\ \text{(d)}$$

Now $\sqrt{(2)}\,.\,a\int\frac{d\theta}{\cos^2\theta\sqrt{(1-\frac{1}{2}\sin^2\theta)}}$ is the expression for an arc of an equilateral hyperbola whose semi- transverse axis is $2a$, and whose central vector, drawn to the extremity of this arc, makes an angle ω with the axis, such that $\sin\theta=\sqrt{(2)}\,.\sin\omega$.

The second term is the expression for an arc of the lemniscate whose semi- transverse axis is $2a$, and whose vector is inclined by the angle ω to the axis.

337.] The arc of the logocyclic curve may be exhibited as a function of a hyperbolic arc, an elliptic arc, and a right line, as follows. In a paper published in the Philosophical Transactions * the two following expressions for the arc of an hyperbola have been given, which become when the hyperbola is equilateral (Υ being the arc)

$$\Upsilon=\frac{a}{\sqrt{(2)}}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-\frac{1}{2}\sin^2\theta)}}\ .\ .\ .\ .\ \text{(a)}$$

---

* "Researches on the Geometrical properties of Elliptic Integrals," Philosophical Transactions for 1852, p. 373, (292) (c) and (k).

and also

$$\Upsilon=a\sqrt{(2)}\tan\theta\sqrt{(1-\tfrac{1}{2}\sin^2\theta)}-a\sqrt{(2)}\int\sqrt{(1-\tfrac{1}{2}\sin^2\theta)}d\theta$$
$$+\frac{a}{\sqrt{(2)}}\int\frac{d\theta}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}}; \quad . \; . \; . \; . \; . \quad \text{(b)}$$

resuming (d), sec. [336],

$$\frac{\Sigma+\sigma}{2}=\frac{a}{\sqrt{(2)}}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-\frac{1}{2}\sin^2\theta)}}+\frac{a}{\sqrt{(2)}}\int\frac{d\theta}{\sqrt{(1-\frac{1}{2}\sin^2\theta)}},$$

and subtracting the two former equations from this latter,

$$\frac{\Sigma+\sigma}{2}-2\Upsilon=a\sqrt{(2)}\int\sqrt{(1-\tfrac{1}{2}\sin^2\theta)}d\theta$$
$$-a\sqrt{(2)}\tan\theta\sqrt{(1-\tfrac{1}{2}\sin^2\theta)}. \quad . \; . \; . \; . \quad \text{(c)}$$

Now the integral represents an arc of an ellipse whose semiaxes are $\sqrt{(2)}\,.\,a$ and a; and 2Υ is the arc of an equilateral hyperbola whose semi- transverse axis is $2a$. Hence the sum of two such arcs of a logocyclic curve may be represented as the sum of the arcs of an equilateral hyperbola and of an ellipse together with a right line.

338.] If we take the difference of the arcs, we shall find, subtracting (c) from (b), sec. [336], that

$$\frac{\Sigma-\sigma}{2}=a\int\frac{\sin\theta d\theta}{\cos^2\theta\sqrt{(1+\cos^2\theta)}}+a\int\frac{\sin\theta d\theta}{\sqrt{(1+\cos^2\theta)}}. \quad . \quad \text{(a)}$$

Let $\cos\theta=\tan\psi$, ψ being, see (a), sec. [320], the angle between the vector and the tangent to the curve at one of the reciprocal points. Then, introducing the necessary transformations,

$$\frac{\Sigma-\sigma}{2}=\frac{a}{\sin\psi}-a\int\frac{d\psi}{\cos\psi}. \quad . \; . \; . \; . \; . \; . \; . \quad \text{(b)}$$

Hence, integrating,

$$\frac{\Sigma-\sigma}{2}=\frac{a}{\sin\psi}-a\log(\sec\psi+\tan\psi)+C; \quad . \; . \quad \text{(c)}$$

$\frac{a}{\sin\psi}=t$ is the distance between the extremity of the arc and the asymptote, and measured along the tangent to the arc at this point, as has been already shown in (a), sec. [326].

To determine the constant; when $\theta=0$, $\psi=\frac{1}{4}\pi$, $\Sigma=0$, $\sigma=0$. Consequently

$$0=a\sqrt{(2)}-a\log\{\sqrt{(2)}+1\}+C,$$

or

$$C=a\log\{\sqrt{(2)}+1\}-a\sqrt{(2)}. \quad . \; . \; . \; . \quad \text{(d)}$$

Hence

$$\frac{\Sigma-\sigma}{2}=t-a\sqrt{(2)}+a\log\left\{\frac{\sqrt{(2)}+1}{\sec\psi+\tan\psi}\right\}. \quad . \; . \; . \; . \quad \text{(e)}$$

This may be put in another form, since

$$\frac{\sqrt{(2)}+1}{\sec\psi+\tan\psi}=(\sec\tfrac{1}{4}\pi+\tan\tfrac{1}{4}\pi)(\sec\psi-\tan\psi)$$

$$=\sec\tfrac{1}{4}\pi\sec\psi-\tan\tfrac{1}{4}\pi\tan\psi+\sec\psi\tan\tfrac{1}{4}\pi-\sec\tfrac{1}{4}\pi\tan\psi,$$

or

$$\{\sec\tfrac{1}{4}\pi\sec\psi-\tan\tfrac{1}{4}\pi\tan\psi\}-\{\sec\tfrac{1}{4}\pi\tan\psi-\sec\psi\tan\tfrac{1}{4}\pi\};$$

and as this may be written

$$\sec(\tfrac{1}{4}\pi\top\psi)-\tan(\tfrac{1}{4}\pi\top\psi), \quad .\ .\ .\ .\ . \quad \text{(f)}$$

we shall have

$$\frac{\Sigma-\sigma}{2}=t-a\sqrt{(2)}+a\log[\sec(\tfrac{1}{4}\pi\top\psi)-\tan(\tfrac{1}{4}\pi\top\psi)]. \quad .\ . \quad \text{(g)}$$

Now this last expression represents the residual arc of the parabola, or the arc diminished by its *protangent*, which corresponds to the vector drawn to the loop of the logocyclic, and which is represented by the expression

$$r=a[\sec(\tfrac{1}{4}\pi\top\psi)-\tan(\tfrac{1}{4}\pi\top\psi)]. \quad .\ .\ . \quad \text{(h)}$$

(See the Trigonometry of the Parabola, next Chapter.)

339.] We shall find a remarkable relation to exist between the arc of the logocyclic curve and the arc of the cissoid which is the locus of V, the point of intersection of two tangents touching the logocyclic curve in any two reciprocal points such as R and $R_{\prime}$ (see figure 78).

Let S be an arc of the cissoid whose equation is

$$\rho=a\sin\theta\tan\theta. \quad .\ .\ .\ .\ .\ .\ .\ . \quad \text{(a)}$$

We shall find by the common methods,

$$\mathrm{S}=a\int\frac{\sin\theta\sqrt{(1+3\cos^2\theta)}}{\cos^2\theta}\,d\theta. \quad .\ .\ .\ .\ .\ . \quad \text{(b)}$$

Make $\qquad 3\cos^2\theta=\tan^2\phi,$

and we shall have

$$\frac{\mathrm{S}}{\sqrt{(3)}}=\frac{a}{\sin\phi}-a\int\frac{d\phi}{\cos\phi}+\mathrm{C}. \quad .\ .\ .\ .\ .\ . \quad \text{(c)}$$

This is identically the expression that was found in (b), sec. [338], for the difference of two logocyclic arcs terminating in two reciprocal points on the same vector, or *coradial*, if one might use such a term.

The constant may be found from the consideration that the arc $\mathrm{S}=0$, when $\theta=0$, or $\phi=60$; hence

$$0=\frac{2a}{\sqrt{(3)}}-a\log\{2+\sqrt{(3)}\}+\mathrm{C},$$

and therefore

$$\frac{S}{\sqrt{(3)}}=a\left\{\frac{1}{\sin\phi}-\frac{2}{\sqrt{(3)}}\right\}+a\log\left\{\frac{2+\sqrt{(3)}}{\sec\phi+\tan\phi}\right\}. \quad . \quad \text{(d)}$$

In (e), sec. [338], a similar expression has been given for the logocyclic residual—namely,

$$\frac{\Sigma-\sigma}{2}=a\left\{\frac{1}{\sin\psi}-\sqrt{(2)}\right\}+a\log\left\{\frac{\sqrt{(2)}+1}{\sec\psi+\tan\psi}\right\}.$$

Take the lengths of these curves between the limits

$$\theta=0 \text{ and } \cos^{-1}\theta=\frac{1}{\sqrt{(3)}},$$

or ϕ between the limits 60 and 45, and ψ between the limits 45 and 30, then

$$\frac{S}{\sqrt{(3)}}=a\left\{\sqrt{(2)}-\frac{2}{\sqrt{(3)}}\right\}+a\log\left\{\frac{2+\sqrt{(3)}}{\sqrt{(2)}+1}\right\}$$

and $\frac{\Sigma-\sigma}{2}=a\{2-\sqrt{(2)}\}+a\log\sqrt{(3)}-a\log\left\{\frac{2+\sqrt{(3)}}{\sqrt{(2)}+1}\right\}.$

Hence, adding these equations together,

$$\frac{\Sigma-\sigma}{2}+\frac{S}{\sqrt{(3)}}=2a\left\{1-\frac{1}{\sqrt{(3)}}\right\}+a\log\sqrt{(3)}. \quad . \quad . \quad . \quad \text{(e)}$$

This is the relation which subsists between the arc of the cissoid and the residual arc of the logocyclic between the limits $\theta=0$ and $\cos^{-1}\theta=\frac{1}{\sqrt{(3)}}$.

340.] The logocyclic curve is the envelope of all the circles whose centres range along the parabola, and whose radii are successively equal to $\sqrt{(f^2-a^2)}$, f being the distance of the centre Q of the circle from the focus of the parabola. This follows from an inspection of the figure; but it may easily be proved as an independent theorem as follows:—

Let the equation of one of the circles be

$$(x-\alpha)^2+(y-\beta)^2=f^2-a^2.$$

Now $\qquad \alpha=a-a\tan^2\theta, \quad \beta=2a\tan\theta, \quad f=a\sec^2\theta.$

Making these substitutions in the preceding equation, and reducing,

$$x^2+y^2+2a(\tan^2\theta-1)x-4a\tan\theta y+a^2=0=V; \quad . \quad . \quad \text{(a)}$$

taking the differential of this expression with respect to θ, and reducing,

$$\frac{dV}{d\theta}=\frac{4ax\tan\theta}{\cos^2\theta}-\frac{4ay}{\cos^2\theta}=0.$$

or

$$\tan\theta=\frac{y}{x}.$$

Introducing this value of $\tan\theta$ in (a), we shall find

$$(x^2+y^2)(2a-x)=a^2x,$$

the equation of the logocyclic curve.

The vertical tangent OT bisects all the chords of the logocyclic curve passing through F.

The angles VRT or $VR_{,}T=\psi$ and θ or OFR are so connected that $\cos\theta=\tan\psi$. Hence the maximum ordinate of the loop is found by making $\psi=\theta$, or $\tan\theta=\cos\theta$, or

$$\sin\theta=\frac{\sqrt{(5)}-1}{2}.$$

If any point Q be taken on the parabola as centre, and through the two reciprocal points on the logocyclic curve a circle be drawn, it will always cut at right angles the fixed circle whose centre is F and radius $=a$.

This is evident; for the radius of this circle is equal to $(FQ)^2-a^2=(QT)^2$, QT being the tangent drawn from Q to the fixed circle.

341.] It is not difficult to show that if we put Δ and $\Delta_{,}$ for the diameters of curvature of the logocyclic curve at any two reciprocal points of which the vectors are r and R, we shall have

$$\frac{r}{\Delta}+\frac{R}{\Delta_{,}}=\sin\psi. \quad \ldots\ldots\ldots \quad (a)$$

If we put C and $C_{,}$ for the chords of curvature of the two reciprocal points and passing through the pole, we shall have

$$\frac{r}{C}+\frac{R}{C_{,}}=1. \quad \ldots\ldots\ldots \quad (b)$$

These simple and remarkable expressions for the curvature of the logocylic curve at any two reciprocal points, are true of all inverse curves whatsoever, at any two reciprocal points. Thus, in the simple case of a circle, let a and b be the segments of any chord $a+b$, then evidently $\frac{a}{a+b}+\frac{b}{a+b}=1$.

This very general theorem may be proved as follows:—

Let $\frac{1}{2}\Delta$ be the radius of curvature of a curve at any point, r the vector from the pole, and p the perpendicular from the pole on the tangent to the curve through this point. Then in most elementary works it is shown that $\Delta=2r\frac{dr}{dp}$; and if C be the chord of curvature through the pole, $C=\Delta\frac{p}{r}=2p\frac{dr}{dp}$, and therefore $\frac{2r}{C}=\frac{\frac{dp}{p}}{\frac{dr}{r}}$, and, for any other curve, $\frac{2R}{C_{,}}=\frac{\frac{dP}{P}}{\frac{dR}{R}}$; hence

$$\frac{r}{C}+\frac{R}{C_{,}}=\frac{1}{2}\left\{\frac{\frac{dp}{p}}{\frac{dr}{r}}+\frac{\frac{dP}{P}}{\frac{dR}{R}}\right\} \quad \ldots\ldots\ldots \quad (c)$$

Now, in all "inverse curves," as the tangents at any two reciprocal points are equally inclined to the common radius vector, if P, R, and $C_{,}$ are the corresponding quantities for the inverse curve,

$$p:P::r:R, \text{ or } p=\frac{rP}{R}, \quad \ldots\ldots \quad (d)$$

and

$$Rr=k^2. \quad \ldots\ldots\ldots\ldots \quad (e)$$

Differentiating (d) and (e),

$$dp=dP\frac{r}{R}+P\frac{dr}{R}-rP\frac{dR}{R^2}, \text{ and } Rdr+rdR=0;$$

hence

$$\frac{dp}{p}=\frac{r}{Rp}dP+\frac{P}{Rp}dr-\frac{Pr}{R^2p}dR.$$

Now $\frac{Rp}{r}=P$, $\frac{Rp}{P}=r$, $\frac{Rp}{Pr}=1$; therefore $\frac{dp}{p}=\frac{dP}{P}+\frac{dr}{r}-\frac{dR}{R}$; (f)

and therefore as

$$\frac{\frac{dp}{p}}{\frac{dr}{r}}=\frac{\frac{dP}{P}}{\frac{dr}{r}}+1-\frac{\frac{dR}{R}}{\frac{dr}{r}}; \text{ and as } \frac{dr}{r}=-\frac{dR}{R},$$

we shall have finally

$$\frac{\frac{dp}{p}}{\frac{dr}{r}}+\frac{\frac{dP}{P}}{\frac{dR}{R}}=2, \quad \ldots\ldots\ldots \quad (g)$$

or

$$\frac{r}{C}+\frac{R}{C_{,}}=1. \quad \ldots\ldots\ldots\ldots \quad (h)$$

342]. The preceding formula will enable us, by a simple transformation, to express the relation between the central forces, by which any curve and its inverse may be described, having the same centre of force. For the formula

$$\tfrac{1}{2}\frac{dp}{p}\,\frac{r}{dr}+\tfrac{1}{2}\frac{dP}{P}\,\frac{R}{dR}=1,$$

may be written

$$rp^2\left(\frac{dp}{2p^3dr}\right)+RP^2\left(\frac{dP}{2P^3dR}\right)=1\,;$$

and we have also

$$p=r\sin\psi,\text{ and }P=R\sin\psi\,;$$

hence

$$r^3\left(\frac{dp}{2p^3dr}\right)+R^3\left(\frac{dP}{2P^3dR}\right)=\frac{1}{\sin^2\psi}.\quad\ldots\ldots\quad(a)$$

But in every elementary treatise on central forces it is shown that the expression for the centripetal force in any given orbit is as $\left(\frac{dp}{2p^3dr}\right)$. Putting F for this force, and Φ for the force at the reciprocal point of the inverse curve, we shall have this general expression for the relation which connects the laws of the central forces in the two orbits,

$$[r^3F+R^3\Phi]\sin^2\psi=1.\quad\ldots\ldots\quad(b)$$

Thus, let one of the curves be the focal parabola, the inverse curve will be the cardioid, and $\sin^2\psi=\frac{a}{r}$, while $F=\frac{1}{4a\,.\,r^2}$. Hence, making these substitutions in (b) and putting $Rr=k^2$, we shall have

$$\Phi=\tfrac{3}{4}\,\frac{k^2}{aR^4},\quad\ldots\ldots\ldots\quad(c)$$

or in the cardioid the force is inversely as the fourth power of the distance, the cusp being the centre of force.

The polar equation of the curve which is inverse to the focal ellipse is

$$R=a(1+e\cos\theta),\text{ if }Rr=b^2.$$

Now, in the focal ellipse, $F=\frac{a}{2b^2r^2}$, and $\sin^2\psi=\frac{b^2}{r(2a-r)}$. Hence

$$\Phi=\tfrac{3}{2}\,\frac{a}{R^4}-\frac{b^2}{R^5},\quad\ldots\ldots\ldots\quad(d)$$

or such an orbit might be described by a body attracted by a force varying inversely as the fourth power of the distance, and repelled by a force varying inversely as the fifth power of the distance.

When the centre of force is the centre of an equilateral hyperbola, the inverse curve is the lemniscate, and

$$\mathrm{F}=\frac{r}{2a^4},\quad \sin\psi=\frac{a^2}{r^2};$$

hence
$$\mathrm{R}^7\Phi=\frac{k^8}{2a^4} \text{ or } \Phi=\frac{k^8}{2a^4}\times\frac{1}{\mathrm{R}^7}; \quad \ldots\ldots \quad \text{(e)}$$

or the law of force in the lemniscate is inversely as the seventh power of the distance, the force being attractive to the centre.

In the same way we might show that as the spiral of Archimedes is the inverse curve to the hyperbolic spiral, and as the law of central force in this latter is inversely as the cube of the vector, and ω being the vector angle in the former spiral whose equation is

$$\mathrm{R}=a\omega, \text{ we shall have } \tan\psi=\omega,$$

or
$$\sin^2\psi=\frac{\omega^2}{1+\omega^2}.$$

Now, in the hyperbolic spiral whose equation is $r\omega=a$, $r^3\mathrm{F}=\frac{1}{2}$; hence, reducing,

$$\Phi=\tfrac{1}{2}\times\frac{1}{\mathrm{R}^3}+\frac{a^2}{\mathrm{R}^5}, \quad \ldots\ldots\ldots \quad \text{(f)}$$

or the force in the spiral of Archimedes is inversely as the third and fifth powers of the distance.

In the same way we might proceed to discover the law of force in other inverse curves.

A direct expression for the radius of curvature of the inverse curve, in terms of u, the reciprocal of the radius vector of the original curve, may easily be found by putting $\mathrm{R}=k^2u$, and therefore

$$\tfrac{1}{2}\Delta_{,}=\frac{k^2\left(u^2+\frac{du^2}{d\theta^2}\right)^{\frac{3}{2}}}{u^2+2\frac{du^2}{d\theta^2}-u\frac{d^2u}{d\theta^2}}. \quad \ldots\ldots \quad \text{(g)}$$

Let ρ and $\rho_{,}$ be the reciprocals of the radii of curvature of the logocyclic curve at the points R and $\mathrm{R}_{,}$; and let $\rho_{,,}$ be the reciprocal of the radius of curvature of the parabola at a point through which the normal makes the angle ψ with the axis, then

$$\rho+\rho_{,}=8\rho_{,,}; \quad \ldots\ldots\ldots \quad \text{(h)}$$

ψ and θ are connected by the condition $\tan\psi=\cos\theta$*.

---

* So far as we have been able to learn, the first notice of "Inverse Curves" will be found in a short paper published by Mr. Stubbs, of Trinity College, Dublin, in the Philosophical Magazine for 1843. Mr. Ferrers has also given a paper on the "Inversion of Curves," in which he combines the principle of "inversion" with that of "reciprocal polars." It will be found at p. 32, vol. i. of the Quarterly Journal of Pure and Applied Mathematics. Other notices of this method will doubtless have appeared elsewhere; but the properties given in the text would not seem to have been published before.

CHAPTER XXXI.

ON THE TRIGONOMETRY OF THE PARABOLA.

343.] Let the angles ω, ϕ, and χ, which we shall call *conjugate amplitudes*, be connected by the equation

$$\tan\omega = \tan\phi\sec\chi + \tan\chi\sec\phi. \quad . \quad . \quad . \quad . \quad \text{(a)}$$

Hence ω is such a function of ϕ and χ as will render

$$\tan[\phi,\chi] = \tan\phi\sec\chi + \tan\chi\sec\phi.$$

We must adopt some appropriate notation to represent this function. Let the function $[\phi,\chi]$ be written $\phi \perp \chi$, so that

$$\tan(\phi\perp\chi) = \tan\phi\sec\chi + \tan\chi\sec\phi.$$

This equation must be taken as the *definition* of the function $\phi\perp\chi$.

In like manner we may represent by $\tan(\phi\top\chi)$ the expression

$$\tan\phi\sec\chi - \tan\chi\sec\phi.$$

From (a) we obtain

$$\sec\omega = \sec(\phi\perp\chi) = \sec\phi\sec\chi + \tan\phi\tan\chi. \quad . \quad . \quad . \quad \text{(b)}$$

If we now differentiate the equation

$$\tan\omega = \tan\phi\sec\chi + \tan\chi\sec\phi,$$

we shall have

$$\left.\begin{aligned}\frac{d\omega}{\cos\omega}\,.\,\sec\omega = \frac{d\phi}{\cos\phi}\,.\,\sec\phi\sec\chi + \frac{d\chi}{\cos\chi}\tan\phi\tan\chi \\ + \frac{d\phi}{\cos\phi}\tan\phi\tan\chi + \frac{d\chi}{\cos\chi}\sec\phi\sec\chi.\end{aligned}\right\} \quad . \quad \text{(c)}$$

Adding these expressions together, and introducing the relation established in (b), we shall find

$$\frac{d\omega}{\cos\omega} = \frac{d\phi}{\cos\phi} + \frac{d\chi}{\cos\chi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

This is the differential equation which connects the amplitudes ω, ϕ, and χ.

As ω, ϕ, and χ are supposed to vanish together, we shall have by integration

$$\int\frac{d\omega}{\cos\omega} = \int\frac{d\phi}{\cos\phi} + \int\frac{d\chi}{\cos\chi}, \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

or, in the more compact notation,

$$\int\sec\omega\,d\omega = \int\sec\phi\,d\phi + \int\sec\chi\,d\chi. \quad . \quad . \quad . \quad \text{(f)*}$$

---

* The relation between the conjugate amplitudes ω, ϕ, and χ was originally obtained in this way. In the theory of elliptic integrals, any three conjugate amplitudes are connected by the equation

$$\cos\omega = \cos\phi\cos\chi - \sin\phi\sin\chi\sqrt{1 - i^2\sin^2\omega} \ . \ . \ . \ . \ . \ ;$$

Hence, if ω, ϕ, and χ are connected by the relation assumed in (a), we shall have the simple relation between the integrals expressed in (e).

If in (a) we make the following imaginary substitutions—that is to say, put $\sqrt{-1}\sin\alpha$ for $\tan\phi$, $\sqrt{-1}\sin\beta$ for $\tan\chi$, $\sqrt{-1}\sin\gamma$ for $\tan\omega$, $\cos\alpha$ for $\sec\phi$, $\cos\beta$ for $\sec\chi$, $\cos\gamma$ for $\sec\omega$, and change $\perp$ into $+$ and $\top$ into $-$, we shall have

$$\sin\gamma=\sin(\alpha+\beta)=\sin\alpha\cos\beta+\sin\beta\cos\alpha,$$

the well-known expression for the sine of the sum of two arcs of a circle.

We shall show presently that an arc of a parabola measured from the vertex may be expressed by the integral $\int\sec\theta\,d\theta$, θ being the angle which the normal to the arc at its other extremity makes with the axis, or the angle between the normals drawn to the arc at its extremities.

$\perp$ and $\top$ may be called logarithmic plus and minus. As examples of the analogy which exists between the trigonometry of the parabola and that of the circle, the following expressions in parallel columns are given—premising that the formulæ marked by corresponding letters may be derived singly, one from the other, by the help of the preceding imaginary transformations.

In applying the imaginary transformations, or while $\tan\phi$ is changed into $\sqrt{-1}\sin\phi$, $\sec\phi$ into $\cos\phi$, $\cot\phi$ into $-\sqrt{-1}\operatorname{cosec}\phi$, $\perp$ must be changed into $+$, $\top$ into $-$, and $\int\sec\phi\,d\phi$ into $\phi\sqrt{-1}$*.

The reader who has not proceeded beyond the elements of trigonometry may assume the fundamental formula as proved. He will find little else that requires more than a knowledge of plane trigonometry.

---

i is called the modulus. When we make $i=0$, we get

$$\cos\omega=\cos\phi\cos\chi-\sin\phi\sin\chi \text{ or } \omega=\phi+\chi$$

in the trigonometry of the circle. When we take the complement of 0, or make $i=1$, we get

$$\sec\omega=\sec\phi\sec\chi+\tan\phi\tan\chi \text{ or } \omega=\phi\perp\chi$$

in the trigonometry of the parabola. Whence, as above,

$$\tan\omega=\tan\phi\sec\chi+\tan\chi\sec\phi.$$

* The advanced reader hardly needs to be reminded that this is the imaginary transformation by which we are enabled, in elliptic functions of the third order, to pass from the *circular* form to the *logarithmic* form, or to pass from the properties of a curve described on the surface of a sphere to its analogue described on the surface of a paraboloid of revolution. See the author's paper "On the Geometrical Properties of Elliptic Integrals," in the Philosophical Transactions for 1852, pp. 362, 368, and for 1854, p. 53.

Trigonometry of the Parabola.

344.]

$\tan(\phi \perp \chi) = \tan\phi \sec\chi + \tan\chi \sec\phi$. . . . . . . ($\alpha$)

$\tan(\phi \top \chi) = \tan\phi \sec\chi - \tan\chi \sec\phi$. . . . . . . ($\beta$)

$\sec(\phi \underset{\top}{\perp} \chi) = \sec\phi \sec\chi \pm \tan\phi \tan\chi$. . . . . . . ($\gamma$)

$\sin(\phi \perp \chi) = \dfrac{\sin\phi + \sin\chi}{1 + \sin\phi \sin\chi}$. . . . . ($\delta$)

$\sin(\phi \top \chi) = \dfrac{\sin\phi - \sin\chi}{1 - \sin\phi \sin\chi}$. . . . . ($\epsilon$)

Let $\phi = \chi$.

$\tan(\phi \perp \phi) = 2 \tan\phi \sec\phi$. . . . . ($\eta$)

$\sec(\phi \perp \phi) = \sec^2\phi + \tan^2\phi$. . . . . ($\theta$)

$\sin(\phi \perp \phi) = \dfrac{2 \sin\phi}{1 + \sin^2\phi}$. . . . . ($\iota$)

$\sec\phi = \dfrac{e^{\int \sec\phi\, d\phi} + e^{-\int \sec\phi\, d\phi}}{2}$, $\tan\phi = \dfrac{e^{\int \sec\phi\, d\phi} - e^{-\int \sec\phi\, d\phi}}{2}$. ($\kappa$)

$1 + \sqrt{-1} \tan\phi \perp \phi) = \{\sec\phi + \sqrt{-1} \tan\phi\}^2$. . . ($\lambda$)

$$\left.\begin{aligned} 2\tan^2\phi &= \sec(\phi \perp \phi) - 1 \\ 2\sec^2\phi &= \sec(\phi \perp \phi) + 1 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (\mu)$$

Let the amplitudes be $\phi \perp \chi$ and $\phi \top \chi$.

$\tan(\phi \perp \chi) \tan(\phi \top \chi) = \tan^2\phi - \tan^2\chi$. . . . . ($\nu$)

Trigonometry of the Circle.

$\sin(\phi + \chi) = \sin\phi \cos\chi + \sin\chi \cos\phi$. . . . (a)

$\sin(\phi - \chi) = \sin\phi \cos\chi - \sin\chi \cos\phi$. . . . (b)

$\cos(\phi \pm \chi) = \cos\phi \cos\chi \mp \sin\phi \sin\chi$. . . . (c)

$\tan(\phi + \chi) = \dfrac{\tan\phi + \tan\chi}{1 - \tan\phi \tan\chi}$. . . . . (d)

$\tan(\phi - \chi) = \dfrac{\tan\phi - \tan\chi}{1 + \tan\phi \tan\chi}$. . . . . (e)

Let $\phi = \chi$.

$\sin 2\phi = 2 \sin\phi \cos\phi$. . . . . (ē)

$\cos 2\phi = \cos^2\phi - \sin^2\phi$. . . . . (th)

$\tan 2\phi = \dfrac{2 \tan\phi}{1 - \tan^2\phi}$. . . . . (i)

$\cos\phi = \dfrac{e^{\phi\sqrt{-1}} + e^{-\phi\sqrt{-1}}}{2}$, $\sin\phi = \dfrac{e^{\phi\sqrt{-1}} - e^{-\phi\sqrt{-1}}}{2\sqrt{(-1)}}$. (k)

$1 + \sin 2\phi = (\cos\phi + \sin\phi)^2$. . . . . (l)

$$\left.\begin{aligned} 2\sin^2\phi &= 1 - \cos 2\phi \\ 2\cos^2\phi &= 1 + \cos 2\phi \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (m)$$

Let the amplitudes be $\phi + \chi$ and $\phi - \chi$.

$\sin(\phi + \chi) \sin(\phi - \chi) = \sin^2\phi - \sin^2\chi$. . . . (n)

Since

$$\sec(\phi \perp \phi) = \sec^2\phi + \tan^2\phi, \text{ and } \tan(\phi \perp \phi) = 2\tan\phi\sec\phi,$$

$$\sec(\phi \perp \phi) + \tan(\phi \perp \phi) = (\sec\phi + \tan\phi)^2.$$

Again, as

$$\sec(\phi \perp \phi \perp \phi) = \sec(\phi \perp \phi)\sec\phi + \tan(\phi \perp \phi)\tan\phi,$$

and

$$\tan(\phi \perp \phi \perp \phi) = \tan(\phi \perp \phi)\sec\phi + \sec(\phi \perp \phi)\tan\phi,$$

it follows that

$$\sec(\phi \perp \phi \perp \phi) + \tan(\phi \perp \phi \perp \phi) = (\sec\phi + \tan\phi)^3,$$

and so on to any number of angles. Hence

$$\sec(\phi \perp \phi \perp \phi \ldots \text{to } n\phi) + \tan(\phi \perp \phi \perp \phi \ldots \text{to } n\phi) = (\sec\phi + \tan\phi)^n. \tag{ξ}$$

Introduce into the last expression the imaginary transformation

$$\tan\phi = \sqrt{-1}\sin\phi,$$

and we get Demoivre's imaginary theorem for the circle,

$$\cos n\phi + \sqrt{-1}\sin n\phi = \{\cos\phi + \sqrt{-1}\sin\phi\}^n.$$

This is a particular case of the more general theorem

$$\left.\begin{array}{l}\sec(\alpha \perp \beta \perp \gamma \perp \delta \perp \&c.) + \tan(\alpha \perp \beta \perp \gamma \perp \delta \perp \&c.) \\ = (\sec\alpha + \tan\alpha)(\sec\beta + \tan\beta)(\sec\gamma + \tan\gamma)(\sec\delta + \tan\delta)\ \&c.^*\end{array}\right\} \tag{ζ}$$

In the circle,

$$\frac{1+\tan\phi}{1-\tan\phi} = \sqrt{\frac{1+\sin 2\phi}{1-\sin 2\phi}}; \quad . \quad . \quad . \quad . \quad . \tag{aa}$$

accordingly, in the parabola,

$$\frac{1+\sqrt{-1}\sin\phi}{1-\sqrt{-1}\sin\phi} = \sqrt{\frac{1+\sqrt{-1}\tan(\phi \perp \phi)}{1-\sqrt{-1}\tan(\phi \perp \phi)}}. \quad . \tag{$\alpha\alpha$}$$

In the circle,

$$\tan^2\phi = \frac{2\sin 2\phi - \sin 4\phi}{2\sin 2\phi + \sin 4\phi}; \quad . \quad . \quad . \quad . \quad . \tag{bb}$$

hence, in the parabola,

$$\sin^2\phi = \frac{2\tan(\phi \perp \phi) - \tan(\phi \perp \phi \perp \phi \perp \phi)}{2\tan(\phi \perp \phi) + \tan(\phi \perp \phi \perp \phi \perp \phi)}. \quad . \quad . \tag{$\beta\beta$}$$

In the circle,

$$\cos 2\phi = \cos^4\phi - \sin^4\phi; \quad . \quad . \quad . \quad . \quad . \quad . \tag{cc}$$

hence, in the parabola,

$$\sec(\phi \perp \phi) = \sec^4\phi - \tan^4\phi. \quad . \quad . \quad . \quad . \tag{$\gamma\gamma$}$$

---

[*] Hence

$$\cos(\alpha+\beta+\gamma+\delta+\&c.) + \sqrt{-1}\sin(\alpha+\beta+\gamma+\delta+\&c.) = (\cos\alpha + \sqrt{-1}\sin\alpha)$$
$$(\cos\beta + \sqrt{-1}\sin\beta)(\cos\gamma + \sqrt{-1}\sin\gamma)(\cos\delta + \sqrt{-1}\sin\delta)\ \&c.$$

In the circle,

$$\tan^2\phi-\tan^2\chi=\frac{\sin(\phi+\chi)\sin(\phi-\chi)}{\cos^2\phi\cos^2\chi}; \quad . \quad . \quad . \quad \text{(dd)}$$

therefore, in the parabola,

$$\sin^2\phi-\sin^2\chi=\frac{\tan(\phi\perp\chi)\tan(\phi\top\chi)}{\sec^2\phi\sec^2\chi}. \quad . \quad . \quad . \quad (\delta\delta)$$

In the circle,

$$\tan\phi=\sqrt{\frac{1-\cos 2\phi}{1+\cos 2\phi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ee)}$$

Accordingly, in the trigonometry of the parabola,

$$\sin\phi=\sqrt{\frac{\sec(\phi\perp\phi)-1}{\sec(\phi\perp\phi)+1}}. \quad . \quad . \quad . \quad . \quad . \quad (\epsilon\epsilon)$$

If

$$\frac{\sin\phi}{\sin\psi}=\frac{\sin(\phi-\chi)}{\sin(\chi-\psi)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(kk)}$$

it is easily shown that $\tan\phi$, $\tan\chi$, and $\tan\psi$ are in harmonic progression.

Hence it follows in parabolic trigonometry, that if

$$\frac{\tan\phi}{\tan\psi}=\frac{\tan(\phi\top\chi)}{\tan(\chi\top\psi)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\kappa\kappa)$$

$\sin\phi$, $\sin\chi$, and $\sin\psi$ are in harmonic progression.

Let ϖ be conjugate to ψ and ω, while ω, as before, is conjugate to ϕ and χ. Then we shall have

$$\tan\varpi=\tan(\phi\perp\chi\perp\psi),$$

or

$$\begin{aligned}\tan(\phi\perp\chi\perp\psi)&=\tan\phi\sec\chi\sec\psi+\tan\chi\sec\psi\sec\phi\\&\quad+\tan\psi\sec\phi\sec\chi+\tan\phi\tan\chi\tan\psi, \quad . \quad . \quad . \quad . \quad (\varpi)\end{aligned}$$

$$\begin{aligned}\sec(\phi\perp\chi\perp\psi)&=\sec\phi\sec\chi\sec\psi+\sec\phi\tan\chi\tan\psi\\&\quad+\sec\chi\tan\psi\tan\phi+\sec\psi\tan\phi\tan\chi, \quad . \quad . \quad . \quad . \quad (\rho)\end{aligned}$$

and

$$\sin(\phi\perp\chi\perp\psi)=\frac{\sin\phi+\sin\chi+\sin\psi+\sin\phi\sin\chi\sin\psi}{1+\sin\chi\sin\psi+\sin\psi\sin\phi+\sin\phi\sin\chi}; \quad (\sigma)$$

whence, in the trigonometry of the circle,

$$\begin{aligned}\sin(\phi+\chi+\psi)&=\sin\phi\cos\chi\cos\psi+\sin\chi\cos\psi\cos\phi\\&\quad+\sin\psi\cos\phi\cos\chi-\sin\phi\sin\chi\sin\psi, \quad . \quad . \quad . \quad . \quad \text{(p)}\end{aligned}$$

$$\begin{aligned}\cos(\phi+\chi+\psi)&=\cos\phi\cos\chi\cos\psi-\cos\phi\sin\chi\sin\psi\\&\quad-\cos\chi\sin\psi\sin\phi-\cos\psi\sin\phi\sin\chi, \quad . \quad . \quad . \quad . \quad \text{(r)}\end{aligned}$$

$$\tan(\phi+\chi+\psi)=\frac{\tan\phi+\tan\chi+\tan\psi-\tan\phi\tan\chi\tan\psi}{1-\tan\chi\tan\psi-\tan\psi\tan\phi-\tan\phi\tan\chi}. \quad \text{(s)}$$

We have here a remarkable illustration of that fertile principle of *duality* which may be developed to such an extent in every department of pure mathematical science.

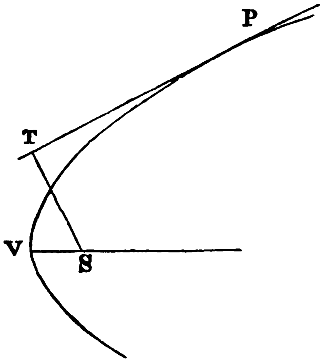
The angle $\phi \perp \phi$ may be called the *duplicate* of the angle ϕ, the angle $\phi \perp \phi \perp \phi$ the *triplicate*, and the angle ($\phi \perp \phi$ to n terms) the *n-plicate* of the angle ϕ.

The reader will observe that the signs $\perp$ and $\top$ connect the angular magnitudes only of the parabola, while numerical quantities are connected by $+$ and $-$. Thus, in the circle, we have $\phi+\chi$ and $a+b$ indifferently, while in the parabola we must use the notation $\phi \perp \chi$ or $\phi \top \chi$ for angles, but $a+b$ or $a-b$ for lines, as in the circle.

345.] An expression for the length of a curve in terms of a perpendicular p let fall from a fixed point on a tangent to it, and making the angle θ with a line passing through the given point or pole, namely $s=\int p d\theta+t$, has been established in sec. [201]. In the following figure,

$$p=\mathrm{ST}, \quad \theta=\mathrm{VST}, \quad t=\mathrm{PT}.$$

Fig. 79.



Let $\Pi(m.\theta)$ denote the length of the arc of a parabola, whose parameter is $4m$, measured from the vertex to a point at which the tangent to the arc is inclined to the ordinate of that point by the angle θ. When $m=1$, the symbol becomes $\Pi(\theta)$.

In the parabola whose equation is $y^2=4mx$, the focus S is taken as the pole, and therefore $p=m \sec \theta$, while PT or $t=m \sec\theta \tan\theta$.

The arc of a parabola, measured from the vertex, may therefore be expressed by the formula

$$\Pi(m.\theta)=m \sec\theta \tan\theta+m\int \sec\theta \, d\theta. \quad . \quad . \quad . \quad \text{(a)}$$

The difference between the arc and its subtangent t has been named the *residual arc*.

For brevity, and for a reason which will presently be shown, the distance between the focus and the vertex of a parabola will be called its *modulus*. Hence the parameter of a parabola is equal to four times its modulus.

Let $\Pi(m.\omega)$, $\Pi(m.\phi)$, $\Pi(m.\chi)$ denote three parabolic arcs VD, VB, VC, measured from the vertex V of the parabola. Let, moreover, ω, ϕ, and χ be *conjugate amplitudes*. Then

$$\left.\begin{aligned}\Pi(m\,.\,\omega)&=m\tan\omega\sec\omega+m\int\sec\omega\,d\omega,\\ \Pi(m\,.\,\phi)&=m\tan\phi\sec\phi+m\int\sec\phi\,d\phi,\\ \Pi(m\,.\,\chi)&=m\tan\chi\sec\chi+m\int\sec\chi\,d\chi.\end{aligned}\right\}\;.\;\;.\;\;.\quad(b)$$

Whence, since $\int\sec\omega\,d\omega=\int\sec\phi\,d\phi+\int\sec\chi\,d\chi$, because ω, ϕ, and χ are conjugate amplitudes, we get, after some reductions,

$$\Pi(m\,.\,\omega)-\Pi(m\,.\,\phi)-\Pi(m\,.\,\chi)=2m\tan\omega\tan\phi\tan\chi.\quad(c)$$

It is not difficult to show that

$$\tan\omega\sec\omega-\tan\phi\sec\phi-\tan\chi\sec\chi=2\tan\omega\tan\phi\tan\chi.\quad(d)$$

Put for $\tan\omega$, $\sec\omega$, their values given in (a) and (b), sec. [343]. Write $(\sec^2\phi-\tan^2\phi)$ and $(\sec^2\chi-\tan^2\chi)$ for 1, the coefficient of $\tan\phi\sec\phi$ and $\tan\chi\sec\chi$ in the preceding expression, and we shall obtain the foregoing result.

346.] Let y, $y_{\prime}$, $y_{\prime\prime}$ be the ordinates, to the axis of the parabola, of the extremities of the arcs $\Pi(m\,.\,\omega)$, $\Pi(m\,.\,\phi)$, and $\Pi(m\,.\,\chi)$. Then $y=2m\tan\omega$, $y_{\prime}=2m\tan\phi$, $y_{\prime\prime}=2m\tan\chi$. Therefore

$$2m\tan\omega\tan\phi\tan\chi=\frac{yy_{\prime}y_{\prime\prime}}{4m^2}.\quad.\;\;.\;\;.\;\;.\;\;.\quad(a)$$

We have therefore the following theorem:—

The algebraic sum of the three conjugate arcs of a parabola, measured from the vertex, is equal to the product of the ordinates of their extremities divided by the square of the semiparameter.

To exemplify the preceding theorem. Let

$$\tan\omega=2,\qquad\tan\phi=\frac{1}{2},\qquad\tan\chi=\frac{\sqrt{5}}{2},$$

then

$$\sec\omega=\sqrt{5},\quad\sec\phi=\frac{\sqrt{5}}{2},\quad\sec\chi=\frac{3}{2};$$

and these values satisfy the fundamental equation of condition,

$$\tan\omega=\tan\phi\sec\chi+\tan\chi\sec\phi.$$

Now

$$\Pi(m.\omega)=m2\sqrt{5}+m\log(2+\sqrt{5}),$$

$$\Pi(m.\phi)=m\frac{\sqrt{5}}{4}+m\log\left(\frac{1+\sqrt{5}}{2}\right),$$

$$\Pi(m.\chi)=m\frac{3\sqrt{5}}{4}+m\log\left(\frac{3+\sqrt{5}}{2}\right).$$

Hence, since $\log(2+\sqrt{5})=\log\left(\frac{1+\sqrt{5}}{2}\right)+\log\left(\frac{3+\sqrt{5}}{2}\right)$, we shall

have $\Pi(m.\omega)-\Pi(m.\phi)-\Pi(m.\chi)=m\sqrt{5}$; . . . (b)
and $m\sqrt{5}=2m\tan\omega\tan\phi\tan\chi$.

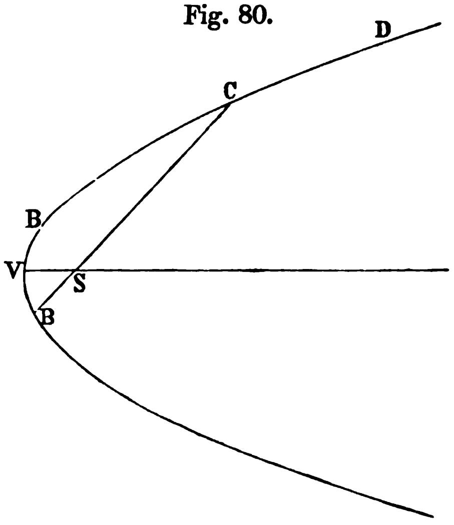
347.] If we call an arc measured from the vertex of a parabola an *aspidal* arc, to distinguish it from an arc taken anywhere along the parabola, the preceding theorem will enable us to express an arc of a parabola, taken anywhere along the curve, as the sum or difference of an apsidal arc and a right line.

Thus, let VCD be a parabola, S its focus, and V its vertex (fig. 80). Let $VB=\Pi(m.\phi)$, $VC=\Pi(m.\chi)$, $VD=\Pi(m.\omega)$, and let $\frac{yy_{\prime}y_{\prime\prime}}{4m^2}=h$. Then (c), sec. [345] shows that the parabolic arc $(VC+VB)$ $=$ arc $VD-h$, and the parabolic arc $VD-VB=BD=VC+h$.

When the arcs $\Pi(m.\phi)$ and $\Pi(m.\chi)$ together constitute a focal arc, or an arc whose chord passes through the focus, $\phi+\chi=\frac{\pi}{2}$, and h is the ordinate of the arc VD. Accordingly we derive the following theorem:—

Any focal arc of a parabola is equal to the difference between the conjugate apsidal arc and its ordinate.

Fig. 80.



The relation between the amplitudes $\phi=\left(\frac{\pi}{2}-\chi\right)$ and ω in this case is given by the equation $\sin 2\phi=\frac{2\cos\omega}{1-\cos\omega}$. Thus, when the focal chord makes an angle of 30° with the axis, we get $\cos\omega=\frac{1}{5}$, or $y=10m$. Here, therefore, the ordinate of the conjugate arc is ten times the modulus.

When $\phi=\chi$, (c), sec. [345], is changed into

$$\Pi(m\,.\,\omega)-2\Pi(m\,.\,\phi)=2m\tan\omega\tan^2\phi\,;\quad .\;\;.\;\;.\quad (a)$$

or as $\tan\omega=2\tan\phi\sec\phi$, see ($\eta$) of [344],

$$\Pi(m\,.\,\omega)-2\Pi(m\,.\,\phi)=4m\tan^3\phi\sec\phi.\quad .\;\;.\;\;.\quad (b)$$

Let $\phi=45$, then $\Pi\left(m.\frac{\pi}{4}\right)$ is the arc of the parabola intercepted between the vertex and the focal ordinate; and as

$$\sec\omega=\sec(\phi\perp\phi)=\sec^2\phi+\tan^2\phi,$$

we shall have, since $\tan\phi=1$ and $\sec\phi=\sqrt{2}$, $\sec\omega=3$; therefore

$$\Pi(m\,.\sec^{-1}.\,3)-2\Pi\left(m.\frac{\pi}{4}\right)=4m\sqrt{2}.\quad .\;\;.\;\;.\quad (c)$$

Now, as $\sec\omega=3$, $\tan\omega=2\sqrt{2}$, and the ordinate $Y=4m\sqrt{2}$, we may therefore conclude that *the parabolic arc whose ordinate is* $4m\sqrt{2}$, *diminished by this ordinate, is equal to the sum of the arcs of the parabola between the focal ordinate produced both ways, and the vertex.*

It is easy to give an independent proof of this particular case without the help of the preceding theory.

The length of the parabolic arc whose amplitude is 45° will be found by the usual formula to be

$$\Pi\left(m.\frac{\pi}{4}\right)=m\sqrt{2}+m\log(1+\sqrt{2})\,;$$

and twice this arc is

$$2\Pi\left(m.\frac{\pi}{4}\right)=m2\sqrt{2}+m\log(3+2\sqrt{2}),\ \text{since}\ (1+\sqrt{2})^2=3+2\sqrt{2}.$$

The parabolic arc whose amplitude is $\sec^{-1}3$, is found in like manner to be

$$\Pi(m\,.\sec^{-1}3)=m3\,.\,2\sqrt{2}+m\log(3+2\sqrt{2}).$$

Subtracting the former equation from the latter,

$$\Pi(m\,.\sec^{-1}3)-2\Pi\left(m.\frac{\pi}{2}\right)=4m\sqrt{2}.$$

Now the ordinate Y of the parabolic arc whose amplitude is $\sec^{-1}3$ is equal to

$$2m\,.\,2\sqrt{2}=4m\sqrt{2}\,;$$

therefore

$$\Pi(m\,.\sec^{-1}3)-2\Pi\left(m.\frac{\pi}{4}\right)=Y.\quad .\;\;.\;\;.\quad (d)$$

It is easily shown that $4m\sqrt{2}$ is the radius of curvature of the extremity of the arc whose amplitude is 45°.

To find a parabolic arc which shall differ from twice another parabolic arc by an algebraical quantity, may be thus exemplified.

Let $\tan\phi=2$, $\tan\omega=4\sqrt{5}$, $\sec\phi=\sqrt{5}$, $\sec\omega=9$; substituting these values in (a), sec. [345], we shall have

$$\Pi(m\,.\,\sec^{-1}9)=m36\sqrt{5}+m\log(9+4\sqrt{5}),$$

and $$2\Pi(m\tan^{-1}2)=2m\,2\sqrt{5}+m\log(2+\sqrt{5})^2.$$

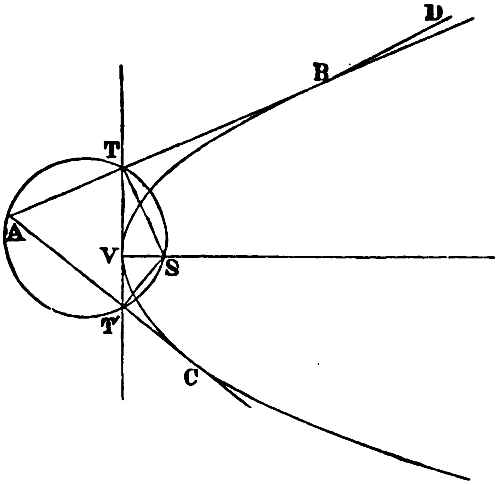
Consequently, since $(2+\sqrt{5})^2=9+4\sqrt{5}$,

$$\Pi(m\,.\,\sec^{-1}9)-2\Pi(m\,.\,\tan^{-1}2)=m32\sqrt{5}=2m\tan\omega\tan^2\phi. \quad \text{(e)}$$

348.] We may in all cases represent by a simple geometrical construction the ordinates of the conjugate parabolic arcs whose amplitudes are ϕ, χ, and ω.

Let BC be a parabola whose focus is S and whose vertex is V.

Fig. 81.



Let $VS=m$; moreover let VB be the arc whose amplitude is ϕ, and VC the arc whose amplitude is χ. At the points V, B, C draw tangents to the parabola; they will form a triangle circumscribing the parabola, whose sides represent half the ordinates of the conjugate arcs VB, VC, VD.

We know that the circle circumscribing this triangle passes through the focus of the parabola. Now

$VT=m\tan\phi$, $VT_{,}=m\tan\chi$, $T_{,}A=m\tan\phi\sec\chi$, $TA=m\tan\chi\sec\phi$; hence

$$T_{,}A+TA=m(\tan\phi\sec\chi+\tan\chi\sec\phi)=m\tan\omega;$$

therefore VT, VT$^{\prime}$, and TA + AT$^{\prime}$ represent half the ordinates of the arcs whose amplitudes are ϕ, χ, and ω.

When VB, VC together constitute a focal arc, the angle TAT$_{\prime}$ is a right angle.

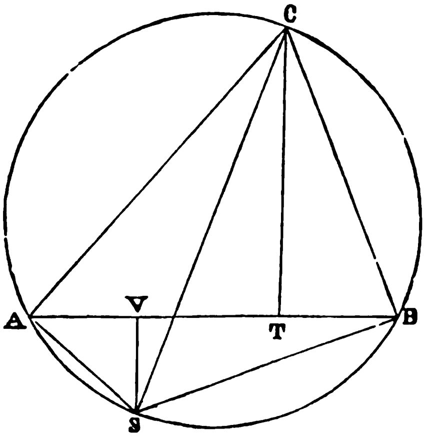
The diameter of this circle is $m \sec\phi \sec\chi$.

The demonstration of these properties follows obviously from the figure.

349.] It may be convenient, by a simple geometrical illustration, to show the magnitude of the functions $\sec(\phi \perp \chi)$ and $\tan(\phi \perp \chi)$.

Let SV$=m$, ASV$=\chi$, BSV$=\phi$, the line AB being at right angles to SV. Through the three points ABS describe a circle. Draw the diameter SC, and join the point C with A and B. Let fall the perpendicular CT.

Fig. 82.



Then $m \sec(\phi \perp \chi) = \text{SC} + \text{CT}$, and $m \tan(\phi \perp \chi) = \text{AC} + \text{CB}$.

Moreover also it follows, since

$$\sec(\phi \perp \chi) + \tan(\phi \perp \chi) = (\sec\phi + \tan\phi)(\sec\chi + \tan\chi),$$

as has been established in (ζ) of sec. [344], that

$$m(\text{SC} + \text{CT} + \text{AC} + \text{CB}) = (\text{SB} + \text{BV})(\text{AS} + \text{AV}). \quad . \quad \text{(a)}$$

Of this theorem it is easy to give an independent geometrical demonstration.

We have manifestly also

$$\text{CT}(\text{SC} + m + \text{SA} + \text{SB}) = (\text{AC} + \text{AT})(\text{BC} + \text{BT}). \quad . \quad . \quad \text{(b)}$$

350.] Let $\overline{\omega}$ be the conjugate amplitude of ω and ψ, while ω is the conjugate amplitude, as before, of ϕ and χ. Then, as

$$\int \sec\overline{\omega}\, d\overline{\omega} = \int \sec\omega\, d\omega + \int \sec\psi\, d\psi,$$

and

$$\int \sec\omega\, d\omega = \int \sec\phi\, d\phi + \int \sec\chi\, d\chi,$$

we shall have

$$\int \sec\bar{\omega}\, d\bar{\omega} = \int \sec\phi\, d\phi + \int \sec\chi\, d\chi + \int \sec\psi\, d\psi\,; \quad . \quad \text{(a)}$$

and if $\Pi(m.\bar{\omega})$, $\Pi(m.\phi)$, $\Pi(m.\chi)$, and $\Pi(m.\psi)$ denote four conjugate parabolic arcs,

$$\Pi(m.\bar{\omega}) - \Pi(m.\phi) - \Pi(m.\chi) - \Pi(m.\psi) =$$
$$2m \tan(\phi \perp \chi) \tan(\phi \perp \psi) \tan(\chi \perp \psi), \quad . \quad . \quad \text{(b)}$$

which gives a simple relation between four conjugate parabolic arcs*.

To exemplify the foregoing formula. Let us assume the following arithmetical values for the angles $\bar{\omega}$, ϕ, χ, ψ:—

$$\left.\begin{array}{llll} \tan\bar{\omega} = \dfrac{10+4\sqrt{5}}{3}, & \tan\phi = \dfrac{1}{2}, & \tan\chi = \dfrac{\sqrt{5}}{2}, & \tan\psi = \dfrac{4}{3}, \\ \sec\bar{\omega} = \dfrac{8+5\sqrt{5}}{3}, & \sec\phi = \dfrac{\sqrt{5}}{2}, & \sec\chi = \dfrac{3}{2}, & \sec\psi = \dfrac{5}{3}. \end{array}\right\} \text{(c)}$$

Hence

$$\left.\begin{array}{l} \Pi\left(m.\tan^{-1}\left[\dfrac{10+4\sqrt{5}}{3}\right]\right) = m(20+9\sqrt{5}) + m\dfrac{\sqrt{5}}{9} + m\log(6+3\sqrt{5}), \\ \Pi\left(m.\tan^{-1}\dfrac{1}{2}\right) = m\dfrac{\sqrt{5}}{4} + m\log\left(\dfrac{1+\sqrt{5}}{2}\right), \\ \Pi\left(m.\tan^{-1}\dfrac{\sqrt{5}}{2}\right) = m\dfrac{3\sqrt{5}}{4} + m\log\left(\dfrac{3+\sqrt{5}}{2}\right), \\ \Pi\left(m.\tan^{-1}\dfrac{4}{3}\right) = m\dfrac{20}{9} + m\log 3. \end{array}\right\} \text{(d)}$$

[*] This latter theorem may be proved as follows:—Since $\bar{\omega}$ is conjugate to ω and ψ, we shall have, by (c), sec. [345],

$$\Pi(m.\bar{\omega}) - \Pi(m.\omega) - \Pi(m.\psi) = 2m \tan\bar{\omega} \tan\omega \tan\psi\,;$$

and since ω is conjugate to ϕ and χ,

$$\Pi(m.\omega) - \Pi(m.\phi) - \Pi(m.\chi) = 2m \tan\omega \tan\phi \tan\chi.$$

Hence, adding these equations, $\Pi(m.\omega)$ will disappear, and

$$\Pi(m.\bar{\omega}) - \Pi(m.\phi) - \Pi(m.\chi) - \Pi(m.\psi) = 2m \tan\omega\,[\tan\bar{\omega}\tan\psi + \tan\phi\tan\chi].$$

Now $\quad \tan\bar{\omega} = \tan(\omega \perp \psi).$

Therefore $\quad \tan\bar{\omega} = \tan\omega \sec\psi + \tan\psi \sec\omega.$

But $\quad \tan\omega = \tan\phi \sec\chi + \tan\chi \sec\phi.$

Substituting this value in the preceding equation, and multiplying by $\tan\psi$,

$$\tan\bar{\omega}\tan\psi = \tan\phi\sec\chi\sec\psi\tan\psi + \tan\chi\sec\phi\sec\psi\tan\psi$$
$$+ \sec\phi\sec\chi\tan^2\psi + \tan\phi\tan\chi\tan^2\psi,$$

and $\quad \tan\phi\tan\chi = \sec^2\psi\tan\phi\tan\chi - \tan^2\psi\tan\phi\tan\chi.$

Consequently

$$\tan\bar{\omega}\tan\psi + \tan\phi\tan\chi = (\sec\psi\tan\phi + \sec\phi\tan\psi)(\sec\chi\tan\psi + \sec\psi\tan\chi)$$
$$= \tan(\phi \perp \psi)\tan(\chi \perp \psi), \text{ and } \omega = \phi \perp \chi.$$

Adding the latter three equations together, and subtracting the sum from the former, the logarithms will disappear; for

$$\log\left(\frac{1+\sqrt{5}}{2}\right)+\log\left(\frac{3+\sqrt{5}}{2}\right)+\log 3=\log\left[3.\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)\right]$$
$$=\log(6+3\sqrt{5})\,; \quad \text{. . . . .} \quad \text{(e)}$$

consequently

$$\Pi(m.\bar{\omega})-\Pi(m.\phi)-\Pi(m.\chi)-\Pi(m.\psi)$$
$$=m\left(\frac{160+73\sqrt{5}}{9}\right)=2m.2.\left(\frac{5+4\sqrt{5}}{6}\right)\left(\frac{12+5\sqrt{5}}{6}\right), \quad . \quad \text{(f)}$$

since

$$\tan(\phi\perp\chi)=2,\ \tan(\phi\perp\psi)=\frac{5+4\sqrt{5}}{6},\ \text{and}\ \tan(\chi\perp\psi)=\frac{12+5\sqrt{5}}{6}.$$

351.] Let, in the preceding formula (b), $\phi=\chi=\psi$, and we shall have

$$\Pi(m.\bar{\omega})-3\Pi(m.\phi)=2m\tan^3(\phi\perp\chi)=16m\tan^3\phi\sec^3\phi.$$

We are thus enabled to assign the difference between an arc of a parabola whose amplitude is $\bar{\omega}=(\phi\perp\phi\perp\phi)$ and three times another arc.

If in (ϖ), [344], we make $\phi=\chi=\psi$,

$$\tan\bar{\omega}=4\tan^3\phi+3\tan\phi. \quad \text{. . . . . .} \quad \text{(a)}$$

Introduce into this expression the imaginary transformation

$$\tan\phi=\sqrt{-1}\sin\theta,\ \text{change}\ \perp\ \text{into}\ +,$$

and we shall get $\sin 3\theta=-4\sin^3\theta+3\sin\theta$, which is the known formula for the trisection of a circular arc. (a) may therefore be taken as the formula which gives the trisection of an arc of a parabola.

The following illustration of the triplication of the arc of a parabola may be given:—

Take the arcs whose ordinates Y and y are $4m$ and m respectively. Let ω and ϕ be the amplitudes which correspond to these ordinates; then, as

$$Y=2m\tan\bar{\omega}=4m, \quad \tan\bar{\omega}=2, \quad \sec\bar{\omega}=\sqrt{5}\,;$$

and as

$$y=2m\tan\phi=m, \quad \tan\phi=\frac{1}{2}, \quad \sec\phi=\frac{\sqrt{5}}{2}.$$

Now these values of $\tan\bar{\omega}$ and $\tan\phi$ satisfy the equation of condition (a), namely

$$4\tan^3\phi+3\tan\phi=\tan\bar{\omega}.$$

But

$$\Pi(m.\tan^{-1}2)=m2\sqrt{5}+m\log(2+\sqrt{5}),$$

and

$$\Pi\left(m.\tan^{-1}\frac{1}{2}\right)=m\frac{1}{2}\frac{\sqrt{5}}{2}+m.\log\left(\frac{1+\sqrt{5}}{2}\right);$$

and three times this arc is

$$3\Pi\left(m.\tan^{-1}\frac{1}{2}\right)=m\frac{3}{4}\sqrt{5}+m\log(2+\sqrt{5}),$$

since

$$\left(\frac{1+\sqrt{5}}{2}\right)^3=2+\sqrt{5}.$$

Subtracting this latter equation from the former, the logarithms disappear, and we get

$$\Pi(m.\tan^{-1}2)-3\Pi\left(m.\tan^{-1}\frac{1}{2}\right)=\frac{m5\sqrt{5}}{4}=16m\tan^3\phi\sec^3\phi. \quad \text{(b)}$$

Now, as the radius of curvature R is equal to the cube of the normal divided by the square of the semiparameter, $R=\frac{m5\sqrt{5}}{4}$, since $N=2m\sec\bar{\omega}$. We have therefore the following theorem:—

The arc of the parabola whose ordinate is equal to 4m, or to the abscissa, diminished by the radius of curvature of its extremity, is equal to three times the arc whose ordinate is m, or one fourth that of the former arc.

It is evident that the chord of the greater arc is inclined by an angle of 45° to the axis, or the ordinate is equal to the abscissa, while in the lesser arc the ordinate is four times the abscissa.

This is the point on the parabola up to which the ordinate is greater than the abscissa; beyond this point it is less than the abscissa.

Another example of the triplication of the arc of a parabola, or of finding an arc which, diminished by an algebraical quantity, shall be equal to three times another arc, may be given.

Let

$$\tan\phi=\frac{3}{2}, \qquad \tan\bar{\omega}=18,$$

$$\sec\phi=\frac{\sqrt{13}}{2}, \quad \sec\bar{\omega}=5\sqrt{13}.$$

These values satisfy the equation of condition,

$$4\tan^3\phi+3\tan\phi=\tan\bar{\omega}.$$

Hence

$$\Pi(m.\tan^{-1}.18)=m90.\sqrt{13}+m\log(18+5\sqrt{13}),$$

$$\Pi\left(m.\tan^{-1}\frac{3}{2}\right)=m\frac{3\sqrt{13}}{4}+m\log\left(\frac{3+\sqrt{13}}{2}\right);$$

and three times this arc is

$$3\Pi\left(m.\tan^{-1}\frac{3}{2}\right)=\frac{m9\sqrt{13}}{4}+m\log(18+5\sqrt{13}),$$

since

$$\left(\frac{3+\sqrt{13}}{2}\right)^3=18+5\sqrt{13}.$$

Therefore, subtracting the latter equation from the former,

$$\Pi(m.\tan^{-1}18)-3\Pi\left(m.\tan^{-1}\frac{3}{2}\right)=m\frac{351\sqrt{13}}{4}=16m\left(\frac{3}{2}\right)^3\left(\frac{\sqrt{13}}{2}\right)^3. \quad \text{(c)}$$

When there are five parabolic arcs, whose amplitudes ϕ, χ, ψ, υ, Ω are related as above, namely

$$\omega=\phi\perp\chi,\ \overline{\omega}=\omega\perp\psi=\phi\perp\chi\perp\psi,\ \Omega=\phi\perp\chi\perp\psi\perp\upsilon,$$

we may proceed to obtain in like manner a formula which will connect five parabolic arcs whose amplitudes are connected by the given law.

352.] To find the arc of a parabola which shall differ from n times a given arc by an algebraical quantity, may be thus investigated.

Let ϕ be the amplitude of the given arc, then

$$\Pi(m.\phi)=m\sec\phi\tan\phi+m\log(\sec\phi+\tan\phi),$$

and n times this arc is

$$n\Pi(m.\phi)=nm\sec\phi\tan\phi+m\log(\sec\phi+\tan\phi)^n.$$

Let $\phi\perp\phi\perp\phi\perp\phi$ to n terms $=\Phi$, then

$$\Pi(m.\Phi)=m\sec\Phi\tan\Phi+m\log(\sec\Phi+\tan\Phi), \text{ sec. [344]}.$$

Now $\sec\Phi+\tan\Phi=(\sec\phi+\tan\phi)^n$, as shown in (ξ).

Hence $\quad \Pi(m.\Phi)-n\Pi(m.\phi)=m[\sec\Phi\tan\Phi-n\sec\phi\tan\phi]$.

Let $\sec\phi+\tan\phi=\lambda$, then $\sec\Phi+\tan\Phi=\lambda^n$, and

$$\sec\phi=\frac{\lambda+\lambda^{-1}}{2},\quad \tan\phi=\frac{\lambda-\lambda^{-1}}{2}.$$

We have also $\sec\Phi=\dfrac{\lambda^n+\lambda^{-n}}{2}$, $\tan\Phi=\dfrac{\lambda^n-\lambda^{-n}}{2}$. Hence

$$\Pi(m.\Phi)-n\Pi(m.\phi)=m\left[\frac{(\lambda^{2n}-\lambda^{-2n})-n(\lambda^2-\lambda^{-2})}{4}\right].$$

Let $n=3$, $\tan\phi=\frac{3}{4}$, $\sec\phi=\frac{5}{4}$, $\lambda=2$. Then

$$\Pi(m.\Phi)-3\Pi(m.\phi)=\frac{m}{4}\left(\frac{3.5}{4}\right)^3.$$

When $n=4$,

$$\Pi(m.\Phi)-4\Pi(m.\phi)=m\frac{5.3^3.457}{2^{10}};$$

and so may n be taken as any other integral number.

353.] The equation (a), sec. [351], affords a very simple mode of expressing the real root of a cubic equation.

Let the cubic equation under the ordinary form be $x^3+px=q$.

Let the parabolic equation $\tan^3\omega+\frac{3}{4}\tan\omega=\frac{\tan\Omega}{4}$ be written

$$\tan^3\omega+\frac{3m^2}{4}\tan\omega=\frac{m^3}{4}\tan\Omega \quad . \quad . \quad . \quad . \quad . \quad (a)$$

by introducing the modulus m; hence

$$p=\frac{3}{4}m^2, \quad q=\frac{m^3}{4}\tan\Omega.$$

Now, since the value of x, found by the ordinary methods, is

$$x=\sqrt[3]{\frac{q}{2}+\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}}+\sqrt[3]{\frac{q}{2}-\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}},$$

we shall have

$$2x=m\sqrt[3]{\sec\Omega+\tan\Omega}-m\sqrt[3]{\sec\Omega-\tan\Omega}, \quad . \quad . \quad (b)$$

while

$$m=2\sqrt{\frac{p}{3}}, \quad \tan\Omega=\frac{3q}{2p}\sqrt{\frac{3}{p}}.$$

When the sign of p is negative, the solution must be sought in the trigonometry of the circle.

CHAPTER XXXII.

ON THE GEOMETRICAL ORIGIN OF LOGARITHMS.

354.] In the trigonometry of the circle we find the formula

$$\vartheta=\tan\vartheta-\frac{\tan^3\vartheta}{3}+\frac{\tan^5\vartheta}{5}-\frac{\tan^7\vartheta}{7}+\&c. \quad . \quad . \quad . \quad (a)$$

And if we develop by common division the expression

$$\frac{1}{\cos\theta}=\frac{\cos\theta}{1-\sin^2\theta}=\cos\theta(1+\sin^2\theta+\sin^4\theta+\sin^6\theta+\&c.),$$

and integrate,

$$\int \frac{d\theta}{\cos\theta} = \int \sec\theta \, d\theta = \sin\theta + \frac{\sin^3\theta}{3} + \frac{\sin^5\theta}{5} + \frac{\sin^7\theta}{7} + \&c. \qquad \text{(b)}$$

If we now inquire what, in the circle, is the magnitude of the *trigonometrical tangent* of the arc which differs from its subtangent by the distance between the vertex and its focus; or, as the subtangent is 0 in the circle, and the focus is the centre, the question may be changed into this other, what is the trigonometrical tangent of the arc of a circle which is equal in length to the radius? This question would be answered by putting 1 for ϑ in (a), and reverting the series

$$1 = \tan(1) - \frac{\tan^3(1)}{3} + \frac{\tan^5(1)}{5} - \frac{\tan^7(1)}{7} + \&c. \quad . \quad . \quad . \qquad \text{(c)}$$

By this process we should get, in functions of the numbers of Bernoulli, the value of $\tan(1)$, as is shown in most treatises on trigonometry.

Let us now make a like inquiry in the case of the parabola, and ask what is the value of the *subtangent of the amplitude* which will give the difference between the arc of the parabola and this subtangent equal to the distance between the focus and the vertex of the parabola. Now, if θ be this angle, we must have

$$\Pi(m.\theta) - m \sec\theta \tan\theta = m.$$

But in general, as shown in sec. [345],

$$\Pi(m.\theta) - m \sec\theta \tan\theta = m \int \sec\theta \, d\theta.$$

We must therefore have, in this case, $\int \sec\theta \, d\theta = 1$. If we now revert the series (b), putting 1 for $\int \sec\theta \, d\theta$, we shall get from this particular value of the series, namely

$$1 = \sin\theta + \frac{\sin^3\theta}{3} + \frac{\sin^5\theta}{5} + \frac{\sin^7\theta}{7} + \&c., \quad . \quad . \quad . \qquad \text{(d)}$$

an arithmetical value for $\sin\theta$*. This we shall find to be $\sin\theta = \frac{e^1 - e^{-1}}{e^1 + e^{-1}}$, e being the number called the base of the Napierian logarithms. Hence $\sec\theta + \tan\theta = e$; or if we write ϵ for this particular value of θ to distinguish it from every other,

$$\sec\epsilon + \tan\epsilon = e = 2{\cdot}718281828, \&c. \quad . \quad . \quad . \quad . \qquad \text{(e)}$$

---

* As

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} \ \&c.\right),$$

let $x = \sin\theta$, then

$$\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = 2\left(\sin\theta + \frac{\sin^3\theta}{3} + \frac{\sin^5\theta}{5} + \frac{\sin^7\theta}{7} + \frac{\sin^9\theta}{9} \ \&c.\right) = 2,$$

or

$$\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = e^2, \text{ or} \left(\frac{1+\sin\theta}{\cos\theta}\right) = e. \text{ Hence } \sec\theta + \tan\theta = e.$$

We are thus (for the first time, it is believed) put in possession of the geometrical origin of that quantity so familiarly known to mathematicians—the Napierian base. From the above equations we may derive

$$\sec\epsilon=\frac{e^1+e^{-1}}{2},\quad \tan\epsilon=\frac{e^1-e^{-1}}{2},\quad \ldots\ldots \qquad (\mathrm{f})$$

or

$$\tan\epsilon=1{\cdot}175201195,\ \text{whence}\ \epsilon={\cdot}8657695,\ \ldots \qquad (\mathrm{g})$$

or

$$\epsilon=49^\circ\ 36'\ 18''*.$$

The corresponding arc of the parabola will be given by the following series :—

$$\Pi(m.\epsilon)=2m\left[1+\frac{2^1}{123}+\frac{2^3}{12345}+\frac{2^5}{1234567}\ \&\text{c.}\right],$$

---

* Mr. Merrifield, F.R.S., has been good enough to calculate the exact magnitude of the angle ϵ, the Napierian base angle. No relation has hitherto been discovered between this angle ϵ and the angle which subtends the arc of a circle equal to unity.

To find

ϵ from $\sec\epsilon+\tan\epsilon=\mathrm{e}$, or from $\log_{10}\tan(\frac{1}{4}\pi+\frac{1}{2}\epsilon)=10{\cdot}43429\ldots.$

The best working formula is :

$$\text{if } \log\tan(x\pm h)=\log\tan x\pm r,$$

$$\log h=\log(\tfrac{1}{2}\mathrm{M}r\sin 2x)\pm\tfrac{1}{2}r\cos 2x \text{ nearly.}$$

$$\begin{array}{lrl}
\log\tan(x+h) & =10{\cdot}43429\ 44819\ 0325 & \\
\log\tan x & =10{\cdot}43423\ 67350\ 3220 & =\log\tan 69^\circ\ 48' \\
\hline
r & =\qquad 5\ 77468\ 7105 &
\end{array}$$

r is positive, but $\cos 2x$ is negative $=-\cos 40^\circ\ 24'$; the correction of $\log h$ is therefore subtractive, and the work is as follows :—

Correction of log h.

$$\begin{array}{rr}
\log r= & 5{\cdot}7615285 \\
\log\frac{1}{2}= & 9{\cdot}6989700 \\
\log\cos 2x= & 9{\cdot}8816918 \\
\hline
 & 5{\cdot}3421903 \\
\hline
\text{Correction}= & 2{,}19882
\end{array}$$

$$\begin{array}{rr}
\log r= & 5{\cdot}76152\ 84574 \\
\log\frac{1}{2}= & 9{\cdot}69897\ 00043 \\
\log \mathrm{M}= & 0{\cdot}36221\ 56887 \\
\log\sin 2x= & 9{\cdot}81165\ 53569 \\
\text{ar. co. }\log 1''= & 5{\cdot}31442\ 51332 \\
\hline
 & 0{\cdot}94879\ 46405 \\
\text{Correction}= & -2\ 19882 \\
\hline
\end{array}$$

$$h=8{\cdot}88735\ 75414\ \ldots\ldots\ \log h=0{\cdot}94877\ 26523$$

$$\therefore\ 45^\circ+\tfrac{1}{2}\epsilon=69^\circ\ 48'\ 8''{\cdot}88735\ 75414$$

$$\epsilon=49^\circ\ 36'\ 17''{\cdot}77471\ 5083$$

The last two figures are of doubtful accuracy.

since the subtangent in this case is equal to

$$m \sec \epsilon \tan \epsilon = \frac{m}{4}\left(e^2 - e^{-2}\right).$$

355.] If we now extend this inquiry, and ask what is the magnitude of the amplitude of the arc of the parabola which shall render the difference between this parabolic arc and its subtangent equal to n times the distance between the focus and the vertex, we shall have, as before, by the terms of the question,

$$\Pi(m.\theta) - m \sec \theta \tan \theta = nm.$$

But, in general,

$$\Pi(m.\theta) - m \sec \theta \tan \theta = m \int \sec \theta \, d\theta ;$$

hence we must have

$$n = \int \sec \theta \, d\theta = \log(\sec \theta + \tan \theta), \text{ or } \sec \theta + \tan \theta = e^n.$$

Now we may solve this equation in two ways:—either by making n a given number, and then determine the value of $\sec\theta + \tan\theta$, which may be called the *base*; or we may assign an arbitrary value to $\sec\theta + \tan\theta$, and then derive the value of n. Taking the latter course, let, for example,

$$\sec \theta + \tan \theta = 10, \text{ then } n = \log 10 ;$$

or putting δ for this angle,

$$\sec \delta + \tan \delta = 10.$$

To represent Numbers and their Logarithms by the Logocyclic Curve and its Conjugate Parabola.

356.] A parabola whose focal vertical distance is a or 1 being drawn, and also its logocyclic curve, let a vector be drawn to the latter equal to the given number N. Then (see fig. 83)

$$\text{N} = \sec \theta + \tan \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Let this line meet the vertical tangent in T, the parabolic arc OQ−QT is the logarithm of N.

It is clear that the infinite branch of the curve from $+\infty$ to 0 will give vectors of every magnitude from ∞ to 1, and parabolic arcs from ∞ to 0; hence, while the vectors or numbers range from ∞ to 1, the parabolic arcs or logarithms range from ∞ to 0. When the number lies between 1 and 0, the vector representing it is drawn *below* the axis; its extremity will be found on the loop, and the corresponding arc of the parabola will be negative: hence the logarithm of a positive number is equal to the logarithm of its reciprocal, with the sign changed; for the magnitude of the parabolic arc depends on θ, and θ is the same in $\sec \theta + \tan \theta$, as in its reciprocal $\sec \theta - \tan \theta$.

Hence, while the infinite branch of the logocyclic curve from

$+\infty$ through R, O, ρ to F, may by its vectors represent all positive numbers from $+\infty$ to $+0$, the *two* infinite branches of the parabola will be used up in representing the logarithms of positive numbers from $+\infty$ to $+0$; that is, the upper or positive branch of the parabola will be expended in representing the logarithms of positive numbers from $+\infty$ to $+1$, and the lower or negative branch of the parabola in representing the logarithms of positive fractional numbers from $+1$ to $+0$. There is, therefore, no construction by which we can represent negative numbers or their logarithms; consequently such numbers can have no logarithms.

Let vectors be drawn from F to the logocyclic curve equal to $e, e^2, e^3, e^4 \dots e^n$, e being the Napierian base; then these lines will meet the tangent to the vertex of the parabola in the points $T, T_{\prime}, T_{\prime\prime} \dots T_n$; and tangents being drawn from these points, touching the parabola in $Q, Q_{\prime}, Q_{\prime\prime}, Q_{\prime\prime\prime}, Q_n$, the logarithms of these numbers will be

$$OQ-QT=1, \quad OQ_{\prime}-Q_{\prime}T_{\prime}=2, \quad OQ_{\prime\prime}-Q_{\prime\prime}T_{\prime\prime}=3, \dots$$
$$OQ_n-Q_nT_n=(n+1)\,; \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

hence the logarithms of e, e^2, e^3, e^n are $1, 2, 3, \dots n$.

In like manner we should find the logs of $e, e^{\frac{1}{2}}, e^{\frac{1}{3}}, e^{\frac{1}{4}} \dots e^{\frac{1}{n}}$ to be $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots \frac{1}{n}$.

357.] Let a series of vectors be drawn from the point F to the logocyclic curve in geometrical progression, and let them be

$$(\sec\theta+\tan\theta),\ (\sec\theta+\tan\theta)^2,\ (\sec\theta+\tan\theta)^3 \dots (\sec\theta+\tan\theta)^n,$$

meeting the vertical tangent to the parabola in the points $T_{\prime}, T_{\prime\prime}, T_{\prime\prime\prime} \dots T_n$, and let the tangents drawn from the points $T_{\prime}, T_{\prime\prime}$, &c. touch the parabola in the points $Q_{\prime}, Q_{\prime\prime}, Q_{\prime\prime\prime} \dots Q_n$; let the difference between the first parabolic arc and its *protangent* be δ, or let δ be the residual arc, then we shall have

$$OQ_{\prime}-Q_{\prime}T_{\prime}=\delta, \quad OQ_{\prime\prime}-Q_{\prime\prime}T_{\prime\prime}=2\delta, \quad OQ_{\prime\prime\prime}-Q_{\prime\prime\prime}T_{\prime\prime\prime}=3\delta,$$
$$OQ^n-Q_nT_n=n\delta.$$

Or while numbers increase in geometrical progression, their logarithms increase in arithmetical progression.

Assuming the common methods of logarithmic differentiation as known, it becomes evident that the residual arc of the parabola is the logarithm of the corresponding number; for if the given number r be put under the form

$$r=a\,(\sec\theta+\tan\theta), \quad \frac{dr}{r}=\frac{d\theta}{\cos\theta},$$

integrating, $\log r=\int\frac{d\theta}{\cos\theta}$.

Now the parabolic residual arc$=\mathrm{OQ}-\mathrm{QT}=a\int\frac{d\theta}{\cos\theta}$.

Hence $$\log r=\frac{\mathrm{OQ}-\mathrm{QT}}{a}. \qquad \text{(a)}$$

As every number whose logarithm is to be exhibited must be put under the form $\sec\theta+\tan\theta$, which is of the form $1+x$, since the limiting value of $\sec\theta$ is 1, we discover the reason why in developing the logarithm of a number, the number itself must be put under the form $1+x$, or some derivative from it, and not simply under that of x.

If we equate $\sec\theta+\tan\theta$ with $1+x$, we shall find

$$x=\frac{2\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}. \text{ Let } u=\tan\tfrac{1}{2}\theta, \qquad \text{(b)}$$

then $\mathrm{N}=\sec\theta+\tan\theta=1+x=\frac{1+u}{1-u}$, which is another familiar form under which a number is put whose logarithm is to be developed in a series. In such a form, u represents the tangent of half the angle that the vector of the logocyclic (which represents the number) makes with the axis.

358.] Let us assume the relations established in sec. [344]:

$$\left.\begin{aligned}\tan(\alpha\perp\beta)&=\tan\alpha\sec\beta+\tan\beta\sec\alpha,\\ \sec(\alpha\perp\beta)&=\sec\alpha\sec\beta+\tan\alpha\tan\beta.\end{aligned}\right\} \qquad \text{(a)}$$

Therefore

$$\sec(\alpha\perp\beta)+\tan(\alpha\perp\beta)=(\sec\alpha+\tan\alpha)(\sec\beta+\tan\beta).$$

In like manner it may be shown that

$$\begin{aligned}&\sec(\alpha\perp\beta\perp\gamma)+\tan(\alpha\perp\beta\perp\gamma)\\ &\quad=(\sec\alpha+\tan\alpha)(\sec\beta+\tan\beta)(\sec\gamma+\tan\gamma), \qquad \text{(b)}\end{aligned}$$

and so on for any number of angles.

Hence, if we draw vectors from F to the logocyclic, making the angles $\alpha,\beta,\gamma,\delta$, &c. with the axis respectively, and another making the angle $(\alpha\perp\beta\perp\gamma\perp\delta\perp\&c.)$ with the same axis, this latter vector will be equal to the continued product of the former, or

$$r_\alpha . r_\beta . r_\gamma . r_\delta=\mathrm{R}_{(\alpha\perp\beta\perp\&c.)}, \qquad \text{(c)}$$

and the sum of the *residual* arcs of the parabola corresponding to the former will be equal to the *residual* arc of the parabola corresponding to the latter.

If in (b) we put $\alpha=\beta=\gamma=\delta$,

$$\begin{aligned}&\sec(\alpha\perp\alpha\perp\alpha\perp\text{to } n \text{ places})+\tan(\alpha\perp\alpha\perp\alpha\perp\text{to } n \text{ places})\\ &\qquad=(\sec\alpha+\tan\alpha)^n. \qquad \text{(d)}\end{aligned}$$

Change sec into cos, tan into $\sqrt{(-1)}$ sin, and $\perp$ into $+$, then

$$\{\cos n\alpha+\sqrt{(-1)}\sin n\alpha\}=\{\cos\alpha+\sqrt{(-1)}\sin\alpha\}^n.$$

The values of the expressions $\sec(a \perp a \perp \&c.)$ and $\tan(a \perp a \perp \&c.)$ are very easily found.

Let $$\sec a + \tan a = \text{N},$$

then
$$\left.\begin{array}{ll}
\sec a = \dfrac{\text{N}+\text{N}^{-1}}{2}, & \tan a = \dfrac{\text{N}-\text{N}^{-1}}{2}, \\
\sec(a \perp a) = \dfrac{\text{N}^2+\text{N}^{-2}}{2}, & \tan(a \perp a) = \dfrac{\text{N}^2-\text{N}^{-2}}{2}, \\
\sec(a \perp a \perp a) = \dfrac{\text{N}^3+\text{N}^{-3}}{2}, & \tan(a \perp a \perp a) = \dfrac{\text{N}^3-\text{N}^{-3}}{2}, \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
\sec(a \perp a \perp a \perp \text{to } i \text{ terms}) = \dfrac{\text{N}^i+\text{N}^{-i}}{2}, & \tan(a \perp a \perp a \perp \text{to } i \text{ terms}) = \dfrac{\text{N}^i-\text{N}^{-i}}{2}.
\end{array}\right\} \text{(e)}$$

This gives a very beautiful and simple law for the magnitudes of the secants and tangents of multiple logocyclic or parabolic angles.

But this theorem may be extended; indeed it follows from (b). Let

$$\sec a + \tan a = n, \quad \sec \beta + \tan \beta = m, \quad \sec \gamma + \tan \gamma = p,$$
$$\sec \delta + \tan \delta = q, \ \&c.$$

$$\left.\begin{array}{ll}
\text{Then} & \sec(a \perp \beta \perp \gamma \perp \delta) = \dfrac{mnpq + m^{-1}n^{-1}p^{-1}q^{-1}}{2}, \\
\text{and} & \tan(a \perp \beta \perp \gamma \perp \delta) = \dfrac{mnpq - m^{-1}n^{-1}p^{-1}q^{-1}}{2}.
\end{array}\right\} \ \cdot\ \cdot\ \cdot\ \text{(f)}$$

Now, in circular trigonometry, let

$$\cos a + \sqrt{(-1)}\sin a = m, \quad \cos\beta + \sqrt{(-1)}\sin\beta = n,$$
$$\cos\gamma + \sqrt{(-1)}\sin\gamma = p, \quad \cos\delta + \sqrt{(-1)}\sin\delta = q, \ \&c.,$$

$$\left.\begin{array}{ll}
\text{then} & \cos(a+\beta+\gamma+\delta) = \dfrac{mnpq + m^{-1}n^{-1}p^{-1}q^{-1}}{2}, \\
\text{and} & \sin(a+\beta+\gamma+\delta) = \dfrac{mnpq - m^{-1}n^{-1}p^{-1}q^{-1}}{2\sqrt{(-1)}}.
\end{array}\right\} \ \cdot\ \cdot\ \cdot\ \text{(g)}$$

These relations place in a striking point of view the analogies between circular and parabolic trigonometry. It is not a little curious that these imaginary expressions for the cosines and sines of multiple arcs in the circle should have been discovered so long before their real analogues were found for the secants and tangents of multiple angles in parabolic trigonometry.

359.] Since we have shown that negative numbers have no logarithms, at least no real ones, and imaginary ones can only be educed by the transformation so often referred to, this leads us to seek them among the properties of the circle. For as θ always lies between

0 and a right angle, or between 0 and the half of $\pm\pi$, $\sec\theta\pm\tan\theta$ is *always* positive; therefore *negative* numbers can have no real or *parabolic* logarithms, but they may have imaginary or *circular* logarithms; for if in the expression

$$\log\{\cos\vartheta+\sqrt{-1}\sin\vartheta\}=\vartheta\sqrt{-1},$$

we make $\vartheta=(2n+1)\pi$, we shall get $\log(-1)=(2n+1)\pi\sqrt{-1}$.

Hence also, as the length of the parabolic arc TP, without reference to the sign, depends solely on the amplitude θ, it follows that the logarithm of $\sec\theta-\tan\theta$ is equal to the logarithm of $\sec\theta+\tan\theta$. We may accordingly infer that the logarithm of any number is equal to the logarithm of its reciprocal, with the sign changed, since

$$(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=1.$$

When θ is very large, $\sec\theta+\tan\theta=2\tan\theta$ nearly. It follows, therefore, if we represent a large number by an ordinate of a parabola whose focal distance to the vertex is 1, the difference between the corresponding arc and its subtangent will represent its logarithm.

360.] Let $\int\sec\phi\,d\phi=p$, $\int\sec\chi\,d\chi=q$; then as

$$\int\sec\omega\,d\omega=\int\sec\phi\,d\phi+\int\sec\chi\,d\chi,$$

$$\int\sec\omega\,d\omega=p+q,\ \text{and}\ \omega=\phi\perp\chi.$$

Hence if ϕ be the amplitude which gives the residual arc$=p$, and χ the amplitude which gives the residual arc$=q$, $\phi\perp\chi$ is the amplitude which will give the residual arc$=p+q$. In the same way we might show that, if ψ be the angle which gives this residual arc$=r$, $(\phi\perp\chi\perp\psi)$ is the angle which will give this residual arc $=p+q+r$.

Let α be the amplitude of the number A, and p its logarithm; β the amplitude of the number B, and q its logarithm; γ the amplitude of the number C, and r its logarithm. Then

$$A=\sec\alpha+\tan\alpha,\quad B=\sec\beta+\tan\beta,\quad C=\sec\gamma+\tan\gamma,$$

and $\log A=p$, $\log B=q$, $\log C=r$, or

$$p+q+r=\log A+\log B+\log C.$$

We have also

$$\begin{aligned}ABC&=(\sec\alpha+\tan\alpha)(\sec\beta+\tan\beta)(\sec\gamma+\tan\gamma)\\&=\sec(\alpha\perp\beta\perp\gamma)+\tan(\alpha\perp\beta\perp\gamma).\end{aligned}$$

Now, as p is the logarithm of $\sec\alpha+\tan\alpha$, q the logarithm of $\sec\beta+\tan\beta$, r the logarithm of $\sec\gamma+\tan\gamma$,

$p+q+r$ is the log of $\sec(\alpha\perp\beta\perp\gamma)+\tan(\alpha\perp\beta\perp\gamma)$, or of A B C, as shown above. We may therefore conclude that

$$\log(ABC)=\log A+\log B+\log C.$$

361.] If ϵ be the angle which gives the difference between the parabolic arc and its subtangent equal to m, $(\epsilon \perp \epsilon)$ is the angle which will give this difference equal to $2m$, $(\epsilon \perp \epsilon \perp \epsilon)$ is the angle which will give this difference equal to $3m$, and so on to any number of angles. Hence, in the circle, if ϑ be the angle which gives the circular arc equal to the radius, 2ϑ is the angle which will give an arc equal to twice the radius, and so on for any number of angles. This is of course self-evident in the case of the circle; but it is instructive to point out the complete analogy which holds in the trigonometries of the circle and of the parabola.

Hence the amplitude which gives the difference between the parabolic arc and its subtangent equal to the semiparameter is given by the simple equation

$$\sec \epsilon_{,} + \tan \epsilon_{,} = e^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

And more generally, if ϵ^{ν} be the amplitude which gives the difference between the parabolic arc and its subtangent equal to n times the modulus, we shall have

$$\sec \epsilon^{\nu} + \tan \epsilon^{\nu} = e^n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

In the same way it may be shown that if ϵ_{ν} be the angle which gives the difference between the parabolic arc and its subtangent equal to $\frac{1}{n}$-th part the modulus, we shall have

$$\sec \epsilon_{\nu} + \tan \epsilon_{\nu} = e^{\frac{1}{n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Let the difference be equal to one half the modulus, then $n = 2$, and $\sec \epsilon_{,} + \tan \epsilon_{,} = e^{\frac{1}{2}}$.

This is easily shown.

Let $\epsilon_{,} \perp \epsilon_{,} = \epsilon$. Then $\sec (\epsilon_{,} \perp \epsilon_{,}) = \sec \epsilon = \sec^2 \epsilon_{,} + \tan^2 \epsilon_{,}$, and

$$\tan (\epsilon_{,} \perp \epsilon_{,}) = \tan \epsilon = 2 \sec \epsilon_{,} \tan \epsilon_{,}.$$

Therefore $\sec (\epsilon_{,} \perp \epsilon_{,}) + \tan (\epsilon_{,} \perp \epsilon_{,}) = \sec \epsilon + \tan \epsilon = e$

$$= \sec^2 \epsilon_{,} + \tan^2 \epsilon_{,} + 2 \sec \epsilon_{,} \tan \epsilon_{,} = (\sec \epsilon_{,} + \tan \epsilon_{,})^2.$$

Hence $\sec \epsilon_{,} + \tan \epsilon_{,} = \sqrt{e}$. . . . . (d)

Since $\tan \epsilon = \dfrac{e^1 - e^{-1}}{2}$, $\sec \epsilon = \dfrac{e^1 + e^{-1}}{2}$;

$$\tan (\epsilon \perp \epsilon) = \frac{e^2 - e^{-2}}{2}, \quad \sec (\epsilon \perp \epsilon) = \frac{e^2 + e^{-2}}{2};$$

$$\tan (\epsilon \perp \epsilon \perp \epsilon) = \frac{e^3 - e^{-3}}{2}, \quad \sec (\epsilon \perp \epsilon \perp \epsilon) = \frac{e^3 + e^{-3}}{2};$$

$$\tan (\epsilon \perp \epsilon \perp \text{to } n \text{ terms}) = \frac{e^n - e^{-n}}{2}, \quad \sec (\epsilon \perp \epsilon \text{ to } n \text{ terms}) = \frac{e^n + e^{-n}}{2}.$$

Therefore $2 \sec \epsilon \tan \epsilon = \tan (\epsilon + \epsilon)$

$$2 \sec (\epsilon \perp \epsilon) \tan (\epsilon \perp \epsilon) = \tan (\epsilon \perp \epsilon \perp \epsilon \perp \epsilon),$$

and generally

$$2\sec(\epsilon\perp\epsilon\perp\text{to } n \text{ terms})\tan(\epsilon\perp\epsilon\perp\text{to } n \text{ terms}) = \tan(\epsilon\perp\epsilon\perp\epsilon\perp\epsilon\perp\text{to } 2n \text{ terms}).$$

Now 2 sec ($\epsilon\perp\epsilon\perp$ to n terms) tan ($\epsilon\perp\epsilon\perp$ to n terms) is the portion of the tangent to the curve intercepted between the axis of the parabola and the point of contact whose amplitude, or the angle it makes with the ordinate, is ($\epsilon\perp\epsilon\perp$ to n terms), while tan ($\epsilon\perp\epsilon\perp\epsilon\perp\epsilon\perp$ to $2n$ terms) is half the ordinate of that point of the curve whose amplitude is ($\epsilon\perp\epsilon\perp\epsilon\perp\epsilon$ to $2n$ terms). Hence we derive this very general theorem:—

That if two points be taken on a parabola such that the intercept of the tangent to the one between the point of contact and the axis shall be equal to one half the ordinate to the other, the amplitudes of the two points will be

($\epsilon\perp\epsilon\perp$ to n terms) *and* ($\epsilon\perp\epsilon\perp\epsilon\perp\epsilon$ to $2n$ terms) *respectively.*

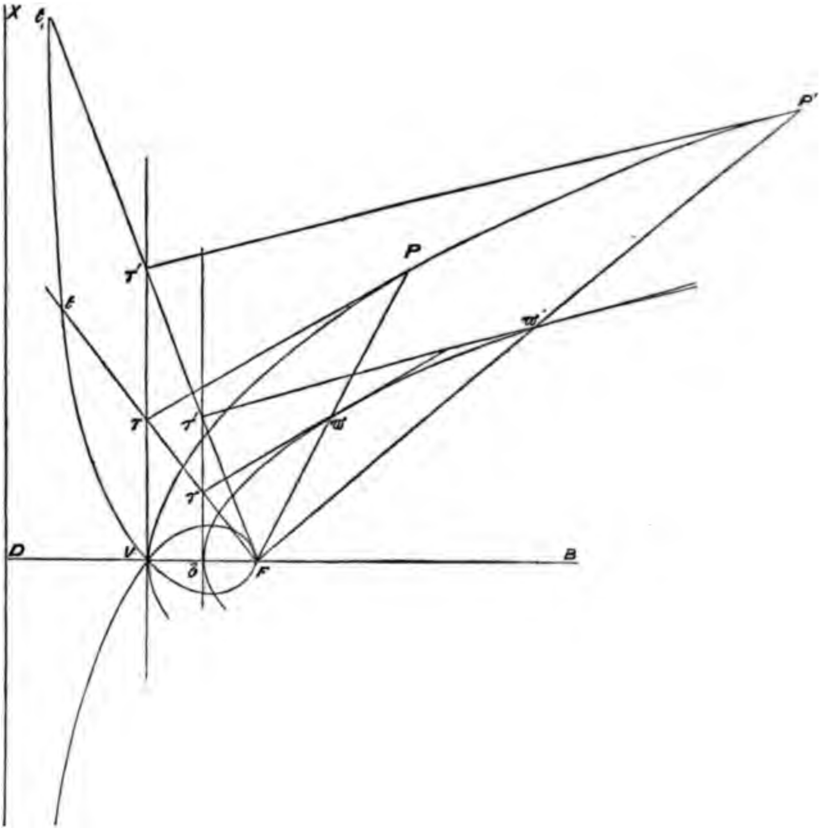
This theorem suggests a simple method of graphically finding a parabolic arc whose amplitude shall be the *duplicate* of the amplitude of a given arc. Let P be the point on the parabola whose amplitude is given. Draw the tangent PQ meeting the axis in Q. Erect VT at the vertex=PQ. Through T draw the tangent $\mathrm{TP}_{\prime}$; the amplitude of the arc $\mathrm{VP}_{\prime}$ will be the duplicate of the amplitude of the arc VP, or ($\theta\perp\theta\perp$ to n terms) and ($\theta\perp\theta\perp$ to $2n$ terms) will be the amplitudes of VP and $\mathrm{VP}_{\prime}$ respectively. We may therefore conclude that in the circle

$$2\cos(\theta+\theta+\text{to } n \text{ terms})\sin(\theta+\theta+\text{to } n \text{ terms}) = \sin(\theta+\theta+\theta+\theta \text{ to } 2n \text{ terms}).$$

362.] To represent the decimal or any other system of logarithms by a corresponding parabola.

The parabola which is to give the Napierian system of logarithms being drawn, whose vertical focal distance m is assumed as the *arithmetical unit,* let another *confocal* parabola be described having its axis coincident with that of the former, and such that its vertical focal distance shall be $m_{\prime}$. The numbers being represented as before by the vectors of the logarithmic curve whose asymptote coincides with the directrix of the parabola whose parameter is $4m$, the differences between the similar parabolic arcs and their subtangents in the two parabolas will give the logarithms, in the two systems, of the *same* number represented by the vector of the logocyclic curve; for as all parabolas, like circles, are similar figures, and these are confocal and similarly placed, any line drawn through their common focus will cut the curves in the same angle, and cut off proportional segments. Hence the two triangles FPT and F$\varpi\tau$ are similar, and the tangential differences PV − PT and $\varpi v - \varpi\tau$ are proportional to $4m$ and $4m_{\prime}$, the parameters of the parabolas.

Fig. 83.



Let log denote the Napierian logarithm, and Log the decimal logarithm of the same number.

Draw the line FT, making the angle ϵ with the axis such that $\sec\epsilon+\tan\epsilon=e$. Then as $PV-PT : \varpi v-\varpi\tau :: m : m_{,}$, and $PV-PT=m=1$, since e is the base of the Napierian system, and $\varpi v-\varpi\tau=\text{Log}\, e$ on the decimal parabola, therefore

$$m : \text{Log}\, e :: m : m_{,}, \text{ or } m_{,}=\text{Log}\, e.$$

We may therefore conclude that the modulus of the decimal system is the decimal logarithm of the Napierian base e.

Draw the line $FT_{,}$ making with the axis an angle δ, such that $\sec\delta+\tan\delta=10$. Now

$$P_{,}V-PT_{,} : \varpi_{,}v-\varpi_{,}\tau_{,} :: m : m_{,};$$

but

$$P_{,}V-PT_{,}=m\log 10; \text{ hence } \varpi_{,}v-\varpi_{,}\tau_{,}=m_{,}\log 10.$$

Now in order that 10 may be a *base*, or, in other words, in order that its logarithm may be unity, we must have $\varpi_{\prime}v-\varpi_{\prime}\tau_{\prime}=m_{\prime}\log 10=m$; or if $m=1$, we must have $m_{\prime}\log 10=1$, or $m_{\prime}=\frac{1}{\log 10}$; that is, the parameter of the *Decimal* parabola must be reduced compared with that of the *Napierian* parabola in the ratio of log 10 : 1. Hence, as is well known, the modulus $m_{\prime}$ of the decimal system is the reciprocal of the Napierian logarithm of 10.

It is therefore obvious that, as any number of systems of logarithms may be represented by the differences between the *similar* arcs and their subtangents of as many *confocal* parabolas, the logarithms of the *same* number in these different systems will be to one another simply as the magnitudes of the parabolas whose arcs represent them, that is, as the parameters of these parabolas. Accordingly the moduli of these several systems are represented by the halves of the semiparameters of the several parabolas.

The Napierian parabola differs from the decimal and other parabolas in this, that the focal distance of its vertex is taken as the arithmetical unit, and that the logocyclic curve, whose vectors represent the numbers, has the directrix of the Napierian parabola as its asymptote.

Hence, if m, the vertical focal distance of the Napierian parabola, be taken as 1, the vertical focal distance $m_{\prime}$ of the decimal parabola is ·4342 &c., or, if $m=1$, $m_{\prime}=·4342$ &c.

363.] Since VT + TP > arc VP, therefore

$$\mathrm{VT} > \text{arc } \mathrm{VP} - \mathrm{TP} > \log \mathrm{F}t.$$

Hence VT or $\tan\theta$ is always greater than the logarithm of $(\sec\theta+\tan\theta)$ in the Napierian system of logarithms. This may be shown on other principles : thus

$$\sec\theta+\tan\theta=\frac{1+\sin\theta}{\cos\theta}=\frac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}=\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}.$$

Let $\tan\frac{\theta}{2}=u$. Then

$$\log(\sec\theta+\tan\theta)=\log\left(\frac{1+u}{1-u}\right)=2\left(u+\frac{u^3}{3}+\frac{u^5}{5}+\frac{u^7}{7}\ \&c.\right),$$

and $$\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=2(u+u^3+u^5+u^7+\&c.)$$

Hence $$\tan\theta>\log(\sec\theta+\tan\theta),$$

or $\frac{\mathrm{N}-\mathrm{N}^{-1}}{2}$ is always greater than the logarithm of N.

364.] In every system of logarithms whatever, the logarithm of 1 is 0.

For when the point T coincides with V, the corresponding point τ will coincide with v, whatever be the magnitude of its modulus $m_{\prime}$. It is obvious that the circle whose radius is unity is analogous to the parabola whose vertical focal distance is unity, and that the Napierian logarithms have the same analogy to trigonometrical lines computed from a radius equal to unity, which any other system of logarithms has to trigonometrical lines computed from a radius r. As we may represent different systems of trigonometry by a series of concentric circles whose radii are 1, r, $r_{\prime}$, &c., so we may in like manner exhibit as many systems of logarithms by a series of confocal parabolas whose focal distances or moduli are 1, $m_{\prime}$, $m_{\prime\prime}$, &c. The modulus in the trigonometry of the parabola corresponds with the radius in the trigonometry of the circle. But while in the trigonometry of the parabola the base is real, in the circle it is imaginary. In the parabola, the angle of the base is given by the equation $\sec\theta+\tan\theta=e$. In the circle, $\cos\theta+\sqrt{-1}\sin\theta=e^{\theta\sqrt{-1}}$; and making $\theta=1$, we get

$$\cos(1)+\sqrt{-1}\sin(1)=e^{\sqrt{-1}}. \quad . \quad . \quad . \quad . \quad (a)$$

Hence, while e^1 is the *parabolic* base, $e^{\sqrt{-1}}$ is the *circular* base. Or as $[\sec\epsilon+\tan\epsilon]$ is the Napierian base, $[\cos(1)+\sqrt{-1}\sin(1)]$ is the *circular* or imaginary base. Thus

$$[\cos(1)+\sqrt{-1}\sin(1)]^{\vartheta}=\cos\vartheta+\sqrt{-1}\sin\vartheta. \quad . \quad . \quad (b)$$

We may therefore infer, speaking more precisely, that imaginary numbers have real logarithms, but an imaginary base. We may always pass from the real logarithms of the parabola to the imaginary logarithms of the circle by changing

$$\tan\theta \text{ into } \sqrt{-1}\sin\vartheta, \ \sec\theta \text{ into } \cos\vartheta, \text{ and } e^1 \text{ into } e^{\sqrt{-1}}.$$

As in the parabola the angle θ is non-periodic, its limit being $\frac{1}{2}\pi$, while in the circle ϑ has no limit, it follows that while a number can have only one real or *parabolic* logarithm, it may have innumerable imaginary or *circular* logarithms.

From F, the focus of the parabola, draw a series of vectors to the logocyclic curve in geometrical progression such as

$$m(\sec\theta+\tan\theta), \ m(\sec\theta+\tan\theta)^2, \ . \ . \ . \ . \ m(\sec\theta+\tan\theta)^n,$$

meeting the tangent to the vertex of the parabola in the points T, $T_{\prime}$, $T_{\prime\prime}$, T_n. The line FT will be $=m\sec\theta$, the line $FT_{\prime}=m\sec(\theta\perp\theta)$, the line $FT_{\prime\prime}=m\sec(\theta\perp\theta\perp\theta)$, &c.; and we shall likewise have

$$VT=m\tan\theta, \ VT_{\prime}=m\tan(\theta\perp\theta), \ VT_{\prime\prime}=m\tan(\theta\perp\theta\perp\theta), \text{ \&c.}$$

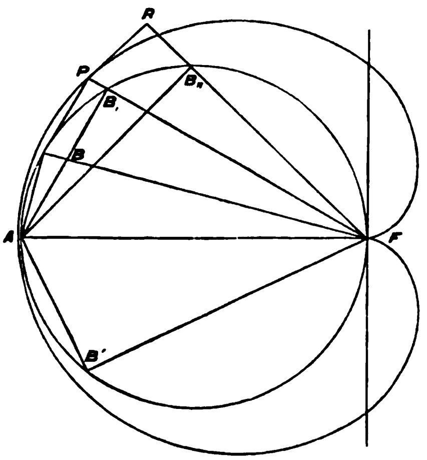
This follows immediately from (ξ) of sec. [344]; for any integral power of $(\sec\theta + \tan\theta)$ may be exhibited as a linear function of $\sec\Theta + \tan\Theta$, writing Θ for $\theta \perp \theta \perp \theta \ldots$ &c., since

$$\sec(\theta \perp \theta \perp \theta \perp \theta \text{ \&c. to } n\theta) + \tan(\theta \perp \theta \perp \theta \perp \theta \text{ \&c. to } n\theta) = (\sec\theta + \tan\theta)^n.$$

Hence the parabola enables us to give a graphical construction for the angle $(\theta \perp \theta \perp \theta$ &c.$)$ as the circle does for the angle $(\theta + \theta + \theta$ &c.$)$.

365.] The analogous theorem in the circle may be developed as follows:—In the circle FBA take the arcs

Fig. 84.



$$AB = BB = B_{\prime}B_{\prime\prime} = B_{\prime\prime}B_{\prime\prime\prime} \ldots \text{ \&c.} = 2\vartheta.$$

Let the diameter be D; then

$$FB = D\cos\vartheta, \quad FB_{\prime} = D\cos 2\vartheta, \quad FB_{\prime\prime} = D\cos 3\vartheta \ldots \text{ \&c.},$$

and

$$AB = D\sin\vartheta, \quad AB_{\prime} = D\sin 2\vartheta, \quad AB_{\prime\prime} = D\sin 3\vartheta \ldots \text{ \&c.}$$

Now, as the lines in the second group are always at right angles to those in the first, and as such a change is denoted by the symbol $\sqrt{-1}$, we shall have

$$FB + BA = D\{\cos\vartheta + \sqrt{-1}\sin\vartheta\},$$

$$FB_{\prime} + B_{\prime}A = D\{\cos 2\vartheta + \sqrt{-1}\sin 2\vartheta\} = D\{\cos\vartheta + \sqrt{-1}\sin\vartheta\}^2;$$

$$FB_{\prime\prime} + B_{\prime\prime}A = D\{\cos 3\vartheta + \sqrt{-1}\sin 3\vartheta\} = D\{\cos\vartheta + \sqrt{-1}\sin\vartheta\}^3 \text{ \&c.}$$

. . . .

$$FB_n + B_nA = D[\cos n\vartheta + \sqrt{-1}\sin n\vartheta] = D[\cos\vartheta + \sqrt{-1}\sin\vartheta]^n.$$

When the points $B_{\prime}$, $B_{\prime\prime}$ fall *below* the line FA, the angle θ becomes negative, and we get

$$FB_{\prime}-B_{\prime}A=\cos\vartheta-\sqrt{-1}\sin\vartheta,$$

$$FB_{\prime\prime}-B_{\prime\prime}A=\cos 2\vartheta-\sqrt{-1}\sin 2\vartheta=[\cos\vartheta-\sqrt{-1}\sin\vartheta]^2.$$

. . . .

Therefore

$$\log(FB+BA)=\log(\cos\vartheta+\sqrt{-1}\sin\vartheta)=\vartheta\sqrt{-1}.$$

Let $\vartheta=1$, then

$$\log[\cos(1)+\sqrt{-1}\sin(1)]=\sqrt{-1}.$$

Hence generally $\vartheta\sqrt{-1}$ is the logarithm of the bent line whose extremities are at F and A, and which meets the circle in the point B.

It is singular that the imaginary formulæ in trigonometry have long been discovered, while the corresponding real expressions have escaped notice. Indeed it was long ago observed, by Bernoulli, Lambert, and by others (the remark has been repeated in almost every treatise on the subject since), that the ordinates of an equilateral hyperbola might be expressed by real exponentials whose exponents are sectors of the hyperbola; but the analogy being illusory, never led to any useful results. And the analogy was illusory from this—that it so *happens* the length and area of a circle are expressed by the *same* function, while the area of an equilateral hyperbola is a function of an arc of a parabola, as will be shown further on. The true analogue of the circle is the parabola.

There are some curious analogies between the parabola and the circle, considered under this point of view.

In the parabola, the points T, $T_{\prime}$, $T_{\prime\prime}$, which divide the lines

$$m(\sec\theta+\tan\theta),\quad m[\sec(\theta\perp\theta)+\tan(\theta\perp\theta)]$$

into their component parts, are upon tangents to the parabola. The corresponding points B, $B_{\prime}$, $B_{\prime\prime}$ in the circle are on the circumference of the circle.

In the parabola, the extremities of the lines $m(\sec\theta+\tan\theta)$ are on a logocyclic curve; in the circle, the extremities of the bent lines are all in the point A.

The analogy between the expressions for parabolic and circular arcs will be seen by putting the expressions under the following forms:—

$$\text{Parabolic arc}-\log(\sec\theta+\tan\theta)-\text{subtangent}=0,$$

$$\text{Circular arc}+\log(\cos\theta+\sqrt{-1}\sin\theta)^{\sqrt{-1}}-\text{subtangent}=0.$$

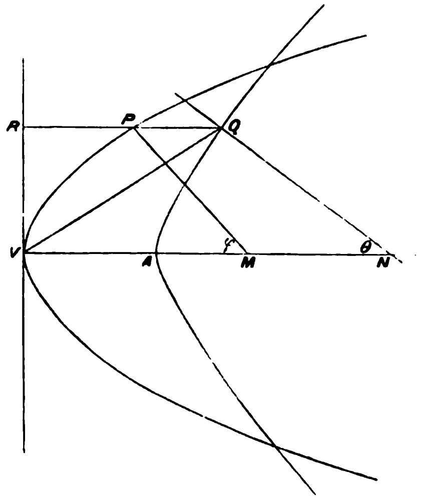
The locus of the point T, the intersections of the tangents to the parabola with the perpendiculars from the focus, is a right line; or in other words, while one end of a subtangent rests on the parabola, the other end rests on a right line. So in the circle; while one end of the subtangent rests on the circle, the other end rests on a *cardioid* whose diameter is equal to that of the circle, and whose cusp is at F. FPA is the cardioid.

366.] The quadrature of the hyperbola depends on the rectification of the parabola.

Through a point P on the parabola draw a line PQ parallel to the axis and terminated in the vertical tangent to the parabola at R. Take the line RQ *always* equal to the normal at P, the locus of Q is an equilateral hyperbola. For $x=2m\sec\phi$, since PM is equal to RQ, and as before $y=2m\tan\phi$; therefore

$$x^2-y^2=4m^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Fig. 85.



the equation of an equilateral hyperbola whose centre is at V, the vertex of the parabola, and whose transverse axis is the parameter of the parabola.

The area of this curve, the elements being taken parallel to the axis, or the area between the curve and the vertical axis passing through V, is found by integrating the value of $x\mathrm{d}y$.

Now

$$x=2m\sec\phi, \text{ and } y=2m\tan\phi;$$

therefore

$$\int x\mathrm{d}y=4m^2\int\sec^3\phi\mathrm{d}\phi=2m\Big[m\sec\phi\tan\phi+m\int\sec\phi\mathrm{d}\phi\Big].$$

But it has been shown in (a), sec. [345], that

$$\Pi(m\,.\,\phi)=m\sec\phi\tan\phi+m\int\sec\phi\mathrm{d}\phi.$$

Hence the hyperbolic area

$$\mathrm{VAQR}=2m\,.\,\Pi\,(m\,.\,\phi). \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Therefore, as the hyperbolic area is equal to a constant multiplied into the corresponding arc of the parabola, the evaluation of the hyperbolic area depends on the properties of logarithms.

It also follows, from what has been established in the preceding part of this Chapter, that hyperbolic areas may be multiplied and compared according to the laws which regulate parabolic arcs.

Let ϕ and θ be the angles in which the normals to the *corresponding points* of the parabola and the hyperbola cut the axis; then it is easily shown, since VQ=normal at Q, that

$$\tan\theta=\sin\phi. \quad \ldots\ldots\ldots \quad \text{(c)}$$

This relation will enable us to express the hyperbolic area in terms of the angle which the normal to the hyperbola makes with the axis instead of the parabolic amplitude; for as the parabolic amplitude ϕ is related to the normal angle of the hyperbola θ by the equation $\tan\theta=\sin\phi$,

$$2\tan\phi\sec\phi=\frac{2\tan\theta}{1-\tan^2\theta}=\tan 2\theta, \quad \ldots\ldots \quad \text{(d)}$$

and

$$\sec\phi+\tan\phi=\sqrt{\sec 2\theta+\tan 2\theta}. \quad \ldots\ldots \quad \text{(e)}$$

Now

$$\Pi(m\,.\,\phi)=m\sec\phi\tan\phi+m\log(\sec\phi+\tan\phi),$$

or, substituting for the preceding values of ϕ,

$$2\Pi(m\,.\,\phi)=m\tan 2\theta+m\log(\sec 2\theta+\tan 2\theta);$$

but taking the amplitude 2θ,

$$\Pi(m\,.\,2\theta)=m\sec 2\theta\tan 2\theta+m\log(\sec 2\theta+\tan 2\theta).$$

Hence, subtracting the former from the latter,

$$\Pi(m\,.\,2\theta)-2\Pi(m\,.\,\phi)=m\tan 2\theta(\sec 2\theta-1).$$

Accordingly

$$\text{the hyperbolic area}=m\Pi(m\,.\,2\theta)-m^2\tan 2\theta(\sec 2\theta-1). \quad \text{(f)}$$

Since

$$2\tan 2\theta=2\tan\phi\sec\phi,$$

we shall have

$$2\theta=\phi\perp\phi. \quad \ldots\ldots\ldots \quad \text{(g)}$$

Hence the normal angles θ and ϕ of the *corresponding points* of the parabola and hyperbola are so related that

$$2\theta=\phi\perp\phi,$$

whence we might at once have inferred the relation established in (e), namely

$$(\sec\phi+\tan\phi)^2=\sec 2\theta+\tan 2\theta.$$

The corresponding points P and Q on the parabola and hyperbola are always found in a line parallel to the axis.

If through the points P *and* Q *on the parabola and hyperbola we draw diameters to these curves, they will make angles with the normals to them at these points, one of which is the* duplicate *of the other.*

For these angles are 2θ and ϕ respectively,

but $$2\theta=\phi\perp\phi. \qquad \text{(h)}$$

367.] Let $P_0, P_1, P_2, P_3, P_4 \ldots P_{n-1}, P_n$ be perpendiculars let fall from the focus on the n sides of a polygon circumscribing a parabola, and making with the axis the angles

$$0,\ \theta,\ \theta\perp\theta,\ \theta\perp\theta\perp\theta,\ \theta\perp\theta\perp\theta\perp\theta, \ldots$$

to n terms respectively.

Let

$$\sec\theta+\tan\theta=u,$$

then

$$\left.\begin{aligned} &\sec(\theta\perp\theta)+\tan(\theta\perp\theta)=u^2,\\ &\sec(\theta\perp\theta\perp\theta)+\tan(\theta\perp\theta\perp\theta)=u^3 \end{aligned}\right\} \qquad \text{(a)}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\sec(\theta\perp\theta\perp\ \ldots \text{ to } n \text{ terms})+\tan(\theta\perp\theta\perp \text{to } n \text{ terms})=u^n.$$

Hence, as

$$\left.\begin{aligned} 2P_0&=m(u^0+u^{-0})\\ 2P_1&=m(u^1+u^{-1})\\ 2P_2&=m(u^2+u^{-2})\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ 2P_n&=m(u^n+u^{-n}), \end{aligned}\right\} \qquad \text{(b)}$$

we shall have

$$\begin{aligned} 2\,.\,2\,.\,P_n\,.\,P_1&=m^2(u^n+u^{-n})(u^1+u^{-1})\\ &=m^2\left[(u^{n+1}+u^{-(n+1)})+(u^{n-1}+u^{-(n-1)})\right], \end{aligned}$$

or

$$2P_n\,.\,P_1=m(P_{n+1}+P_{n-1}).$$

But

$$P_1=m\sec\theta;$$

therefore

$$\sec\theta P_n=\frac{P_{n+1}+P_{n-1}}{2}, \qquad \text{(c)}$$

or *any perpendicular multiplied by the secant of the first amplitude is an arithmetical mean between the perpendiculars immediately preceding and following it.* Thus, for example, $P_0=m$, $P_1=m\sec\theta$, $P_2=m\sec(\theta\perp\theta)$, or

$$\sec\theta\, m\sec\theta=\frac{m+m\sec(\theta\perp\theta)}{2}.$$

But

$$\sec(\theta\perp\theta)=\sec^2\theta+\tan^2\theta;$$

hence the proposition is manifest.

368.] Again, as hence

$$\left.\begin{array}{ll} 2\mathrm{P}_0=m(u^0+u^0), & 2\,.\,2\,.\,\mathrm{P}_0\mathrm{P}_1=m^2(u^1+u^{-1}+u^1+u^{-1}).\\ 2\mathrm{P}_1=m(u+u^{-1}), & 2\,.\,2\,.\,\mathrm{P}_1\mathrm{P}_2=m^2(u^3+u^{-3}+u^1+u^{-1}).\\ 2\mathrm{P}_2=m(u^2+u^{-2}), & 2\,.\,2\,.\,\mathrm{P}_2\mathrm{P}_3=m^2(u^5+u^{-5}+u^1+u^{-1}).\\ 2\mathrm{P}_3=m(u^3+u^{-3}), & 2\,.\,2\,.\,\mathrm{P}_3\mathrm{P}_4=m^2(u^7+u^{-7}+u^1+u^{-1}).\end{array}\right\} \quad (a)$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$2\mathrm{P}_n=m(u^n+u^{-n}), \quad 2\,.\,2\,.\,\mathrm{P}_{n-1}\mathrm{P}_n=m^2(u^{2n-1}+u^{-(2n-1)}+u^1+u^{-1}).$$

We shall have, therefore, adding the preceding expressions,

$$\left.\begin{array}{l} 2[\mathrm{P}_0\mathrm{P}_1+\mathrm{P}_1\mathrm{P}_2+\mathrm{P}_2\mathrm{P}_3+\mathrm{P}_3\mathrm{P}_4 \quad . \quad . \quad . \quad . \quad \mathrm{P}_{n-1}\mathrm{P}_n]=\\ \quad m[+\mathrm{P}_1+\mathrm{P}_3+\mathrm{P}_5+\mathrm{P}_7 \quad . \quad . \quad . \quad \mathrm{P}_{2n-1}+(n-1)\mathrm{P}_1],\end{array}\right\} \quad . \quad (b)$$

or *twice the sum of all the products of the perpendiculars taken two by two up to the* nth, *is equal to the sum of all the odd perpendiculars up to the* $(2n-1)$*th* + $(n-1)$ *times the first perpendicular.*

Thus, taking the first three perpendiculars,

$$\mathrm{P}_0=m, \ \mathrm{P}_1=m\sec\theta, \ \mathrm{P}_2=m\sec(\theta\perp\theta)=m(\sec^2\theta+\tan^2\theta),$$
$$\mathrm{P}_3=m\sec(\theta\perp\theta\perp\theta)=m(4\sec^3\theta-3\sec\theta).$$

The truth of the proposition may be shown in this particular case; for

$$2[\mathrm{P}_0\mathrm{P}_1+\mathrm{P}_1\mathrm{P}_2]=4m^2\sec^3\theta=m(\mathrm{P}_1+\mathrm{P}_3+2\mathrm{P}_1). \quad . \quad . \quad . \quad (c)$$

Again, since

$$2\mathrm{P}_{2n}=m(u^{2n}+u^{-2n}),$$

and

$$4\mathrm{P}_n^2=m^2(u^{2n}+2+u^{-2n}),$$

we shall have

$$2\mathrm{P}_n^2-m^2=m\mathrm{P}^{2n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

Thus, for example, *twice the square of the perpendicular on the fifth side of the polygon, diminished by the square of the modulus, is equal to the tenth perpendicular multiplied by the modulus.*

In the same way we may show that

$$4\mathrm{P}_n^3-3m^2\mathrm{P}_n=m^2\mathrm{P}_{3n}.$$

Let $n=5$ and $m=1$; then *four times the cube of the fifth perpendicular, diminished by three times the same perpendicular, is equal to the fifteenth perpendicular, or to the perpendicular on the fifteenth side of the polygon.*

369.] Since

$$\log u=u-u^{-1}-\tfrac{1}{2}(u^2-u^{-2})+\tfrac{1}{3}(u^3-u^{-3})-\tfrac{1}{4}(u^4-u^{-4}), \text{ \&c.},$$

and as

$$u-u^{-1}=2\tan\theta, \ u^2-u^{-2}=2\tan(\theta\perp\theta),$$
$$u^n-u^{-n}=2\tan(\theta\perp\theta\perp\theta\perp\text{to } n \text{ terms}),$$

while

$$u=\sec\theta+\tan\theta,$$

we shall have therefore

$$\log u = \frac{\mathrm{PV}-\mathrm{PT}}{2} = \tan\theta - \tfrac{1}{2}\tan(\theta \perp \theta) + \tfrac{1}{3}\tan(\theta \perp \theta \top \theta, \text{\&c.}). \quad \text{(a)}$$

We may convert this into an expression for the arc of a circle by changing $\perp$ into $+$, tan into $\sqrt{-1}$ sin, and the parabolic arc into the circular arc multiplied by $\sqrt{-1}$.

Hence, since PT in the circle is equal to 0,

$$\frac{\theta}{2} = \sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta - \tfrac{1}{4}\sin 4\theta, \quad . \quad . \quad . \quad \text{(b)}$$

a formula given in Lacroix, 'Traité du Calcul Différentiel et du Calcul Intégral,' tom. i. p. 94.

370.] In the trigonometry of the circle, the sines and cosines of multiple arcs may be expressed in terms of powers of the sines and cosines of the simple arcs. Thus

$$\left.\begin{aligned}
\cos 2\theta &= 2\cos^2\theta - 1,\\
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta,\\
\cos 4\theta &= 8\cos^4\theta - 8\cos^2\theta + 1,\\
\cos 5\theta &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta,\\
\cos 6\theta &= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1,\\
&\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .\\
\sin 2\theta &= \sin\theta(2\cos\theta),\\
\sin 3\theta &= \sin\theta(4\cos^2\theta - 1),\\
\sin 4\theta &= \sin\theta(8\cos^3\theta - 4\cos\theta),\\
\sin 5\theta &= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1),\\
\sin 6\theta &= \sin\theta(32\cos^5\theta - 32\cos^3\theta + 6\cos\theta).
\end{aligned}\right\} \quad . \quad \text{(a)}$$

Hence, in the trigonometry of the parabola,

$$\left.\begin{aligned}
\sec(\theta \perp \theta) &= 2\sec^2\theta - 1,\\
\sec(\theta \perp \theta \perp \theta) &= 4\sec^3\theta - 3\sec\theta,\\
\sec(\theta \perp \theta \perp \theta \perp \theta) &= 8\sec^4\theta - 8\sec^2\theta + 1,\\
\sec(\theta \perp \theta \perp \theta \perp \theta \perp \theta) &= 16\sec^5\theta - 20\sec^3\theta + 5\sec\theta,\\
\sec(\theta \top \theta \perp \theta \perp \theta \perp \theta \perp \theta) &= 32\sec^6\theta - 48\sec^4\theta + 18\sec^2\theta - 1,\\
. \quad . \quad . \quad . \quad . \quad &. \quad . \quad . \quad . \quad .\\
\tan(\theta \perp \theta) &= \tan\theta(2\sec\theta),\\
\tan(\theta \perp \theta \perp \theta) &= \tan\theta(4\sec^2\theta - 1),\\
\tan(\theta \perp \theta \perp \theta \perp \theta) &= \tan\theta(8\sec^3\theta - 4\sec\theta),\\
\tan(\theta \perp \theta \perp \theta \perp \theta \perp \theta) &= \tan\theta(16\sec^4\theta - 12\sec^2\theta + 1),\\
\tan(\theta \perp \theta + \theta \perp \theta \perp \theta \perp \theta) &= \tan\theta(32\sec^5\theta - 32\sec^3\theta + 6\sec\theta).\\
. \quad . \quad . \quad . \quad . \quad &. \quad . \quad . \quad . \quad .
\end{aligned}\right\} \text{(b)}$$

The preceding formulæ may easily be verified.

If we add in the above series any two corresponding secants and tangents, the sum will be an integral power of $\sec\theta+\tan\theta$.

Thus $\sec(\theta\perp\theta)+\tan(\theta\perp\theta)=(\sec\theta+\tan\theta)^2$.

Again, since in the circle

$$\left.\begin{array}{l} \cos\theta=\cos\theta \\ 2\cos^2\theta=\cos 2\theta+1 \\ 4\cos^3\theta=\cos 3\theta+3\cos\theta \\ 8\cos^4\theta=\cos 4\theta+4\cos 2\theta+1 \\ .\quad .\quad .\quad .\quad .\quad .\quad .\quad \text{and} \\ \sin\theta=\sin\theta \\ 2\sin^2\theta=-\cos 2\theta+1 \\ 4\sin^3\theta=-\sin 3\theta+3\sin\theta \\ 8\sin^4\theta=\cos 4\theta-4\cos 2\theta+3, \end{array}\right\} \quad .\;.\quad (c)$$

hence in parabolic trigonometry

$$\left.\begin{array}{l} \sec\theta=\sec\theta \\ 2\sec^2\theta=\sec(\theta\perp\theta)+1 \\ 4\sec^3\theta=\sec(\theta\perp\theta\perp\theta)+\sec 3\theta \\ 8\sec^4\theta=\sec(\theta\perp\theta\perp\theta\perp\theta)+4\sec(\theta\perp\theta)+1 \\ .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad .\quad . \\ \tan\theta=\tan\theta \\ 2\tan^2\theta=\sec(\theta\perp\theta)-1 \\ 4\tan^3\theta=\tan(\theta\perp\theta\perp\theta)-3\tan\theta \\ 8\tan^4\theta=\sec(\theta\perp\theta\perp\theta\perp\theta)-4\sec(\theta\perp\theta)+3. \end{array}\right\} \quad .\;.\quad (d)$$

371.] The roots of the expression

$$z^{2n}-2az^n+1=0 \quad .\;.\;.\;.\;.\;.\;.\;. \quad (a)$$

may be represented under the form $\cos A+\sqrt{-1}\sin A$, when a is *less* than 1. This has long been known. It is not difficult to show that, when a is greater than 1, the roots may be exhibited under the form

$$\sec A+\tan A. \quad .\;.\;.\;.\;.\;.\;.\;. \quad (b)$$

Since a is greater than 1, let $a=\sec\theta$, and let θ be divided into n angles ϕ, connected by the relation

$$\phi\perp\phi\perp\phi\perp\phi\quad \&c.=\theta. \quad .\;.\;.\;.\;.\;.\;. \quad (c)$$

It has been shown in (ξ), sec. [344], that

$$\sec(\phi\perp\phi\perp\phi\perp\phi\,\text{to}\,n\phi)+\tan(\phi\perp\phi\perp\phi\perp\phi\,\text{to}\,n\phi)=(\sec\phi+\tan\phi)^n.$$

Let $\sec\phi+\tan\phi=u$; then $2\sec\phi=u^1+u^{-1}$,

and therefore $2\sec\theta=2\sec(\phi\perp\phi\perp\phi\perp\text{to }n\phi)=u^n+u^{-n}$.

Substitute this value of $2\sec\theta$ in (a), and we shall have

$$z^{2n}-(u^n+u^{-n})z^n+1=0,$$

or, resolving into factors,

$$\left(\frac{z^n}{u^n}-1\right)\left(z^n u^n-1\right)=0. \quad . \; . \; . \; . \; . \; . \; . \quad \text{(d)}$$

Hence, finding the roots of these binomial factors by the ordinary methods, we shall have, since $u=\sec\phi+\tan\phi$,

$$\left.\begin{array}{l} z=(\sec\phi+\tan\phi)\,\text{(multiplied successively into the } n \text{ roots of unity)} \\ \text{and} \\ \quad (\sec\phi-\tan\phi)\,\text{(multiplied successively into the } n \text{ roots of unity).} \end{array}\right\}\text{(e)}$$

We are thus enabled to exhibit the $2n$ roots when $a>1$.

Thus, let $n=3$, then the equation becomes

$$z^6-2\sec\theta z^3+1=0, \quad . \; . \; . \; . \; . \; . \; . \quad \text{(f)}$$

and

$$\phi\perp\phi\perp\phi=\theta; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(g)}$$

consequently the six roots are

$$\left.\begin{array}{l} (\sec\phi+\tan\phi)\left(1,\ \dfrac{-1\pm\sqrt{-3}}{2}\right), \\ \text{and} \\ (\sec\phi-\tan\phi)\left(1,\ \dfrac{-1\pm\sqrt{-3}}{2}\right). \end{array}\right\} \quad . \; . \; . \quad \text{(h)}$$

By the same method we may exhibit the roots when a is less than 1, or $a=\cos\theta$.

We might pursue this subject very much further; but enough has been done to show the analogy which exists between the trigonometry of the circle and that of the parabola. As the calculus of angular magnitude has always been referred to the circle as its type, so the calculus of logarithms may in precisely the same way be referred to the parabola as its type.

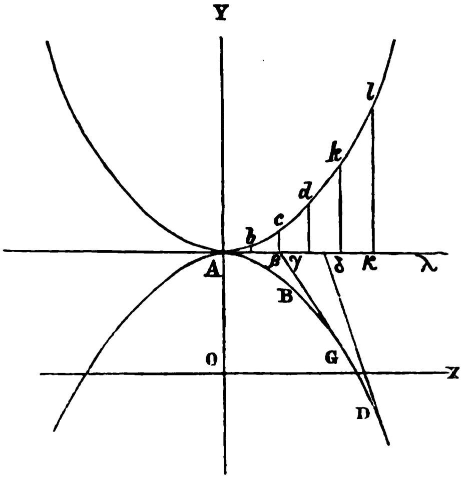
On parabolic trigonometry as applied to the investigation of the properties of the Catenary and the Tractrix.

372.] The application of the principles of parabolic trigonometry to the discussion of the properties of those kindred curves the catenary and the tractrix, elicits some singular properties and relations of these curves, as also between the catenary and the parabola, and affords a further illustration how the invention of new methods enlarges the boundaries of science.

Let us for brevity assume as known the equation of the catenary referred to the rectangular axes of coordinates O X, O Y; and let O A$=a$, the modulus; then

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{\frac{-x}{a}}\right), \text{ and } s=\frac{a}{2}\left(e^{\frac{x}{a}}-e^{\frac{-x}{a}}\right). \quad . \; . \; . \quad \text{(a)}$$

Fig. 86.



By addition,

$$y+s=ae^{\frac{x}{a}};\ \text{by subtraction,}\ y-s=ae^{\frac{-x}{a}}.\quad .\quad .\quad \text{(b)}$$

Multiplying these expressions,

$$y^2-s^2=a^2,\quad .\ .\ .\ .\ .\ .\ .\ .\quad \text{(c)}$$

or, *the difference between the squares of an ordinate of a catenary and its corresponding arc is constant and equal to the square of the modulus.*

373.] Assume the following relations,

$$2\sec\theta=e^{\frac{x}{a}}+e^{\frac{-x}{a}},\ \text{and}\ 2\tan\theta=e^{\frac{x}{a}}-e^{\frac{-x}{a}};\quad .\ .\quad \text{(a)}$$

then

$$y=a\sec\theta,\ \text{and}\ s=a\tan\theta;\quad .\ .\ .\ .\quad \text{(b)}$$

adding these expressions, we shall have

$$y+s=a(\sec\theta+\tan\theta),$$

the form which so often occurs in parabolic trigonometry.

Now, if we make $x_{\prime}$, $x_{\prime\prime}$, $x_{\prime\prime\prime}$, &c. successively equal to a, $2a$, $3a$, &c. in (a), we shall have

$$\left.\begin{aligned}
y+s&=a(\sec\theta+\tan\theta),\\
y_{\prime}+s_{\prime}&=a\{\sec(\theta\perp\theta)+\tan(\theta\perp\theta)\},\\
y_{\prime\prime}+s_{\prime\prime}&=a\{\sec(\theta\perp\theta\perp\theta)+\tan(\theta\perp\theta\perp\theta)\},\\
y_{\prime\prime\prime}+s_{\prime\prime\prime}&=a\{\sec(\theta\perp\theta\perp\theta\perp\theta)+\tan(\theta\perp\theta\perp\theta\perp\theta\ \text{\&c.})\}.
\end{aligned}\right\}.\quad \text{(c)}$$

But it has been shown in sec. [344] that

$$(\sec\theta+\tan\theta)^n=\sec(\theta\perp\theta\perp\theta\ \text{to}\ n\ \text{terms})+\tan(\theta\perp\theta\perp\theta\ \text{to}\ n\ \text{terms});$$

consequently

$$(y+s)^n=a^{n-1}(y_n+s_n)\,; \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

or, *if two points be assumed on a catenary, the abscissa of the one being* n *times that of the other, the* n*th power of the sum of the first ordinate and its corresponding arc will be equal to the sum of the* n*th ordinate and its corresponding arc multiplied by* a^{n-1}.

It will also follow from the preceding expression that *if a series of equidistant ordinates to the catenary be taken, and the corresponding arc be added to each ordinate, their sums will be in geometrical progression. For*

$$\left.\begin{array}{ll} y+s=y+s, & (y+s)^2=a(y_{\prime}+s)_{\prime}, \\ (y+s)^3=a^2(y_{\prime\prime}+s_{\prime\prime}), & (y+s)^n=a^{n-1}(y_{n-1}+s_{y-1}). \end{array}\right\} \quad . \quad \text{(c)}$$

374.] The catenary will enable us to represent graphically, with great simplicity, the sum of a series of angles, added together by the parabolic plus $\perp$.

Let a set of equidistant ordinates whose intervals are a, $2a$, $3a$, $4a$, &c. meet the catenary (fig. 86) in the points b, c, d, k, l; and then let the catenary be conceived as stretched along the horizontal tangent passing through the vertex A. Let the points b, c, d, k, l on the catenary in its free position be conceived to coincide with the points β, γ, δ, κ, λ, when it is stretched along the horizontal tangent; we shall then have, since $\frac{x}{a}$ is successively equal to 1, 2, 3, 4, &c.,

$$\begin{array}{ll} 2y=e^1+e^{-1}, & 2s=e^1-e^{-1}, \\ 2y_{\prime}=e^2+e^{-2}, & 2s_{\prime}=e^2-e^{-2}, \\ 2y_{\prime\prime}=e^3+e^{-3}, & 2s_{\prime\prime}=e^3-e^{-3}\,; \end{array}$$

consequently the angle $\mathrm{AO}\beta$ or ϵ is such that $\sec\epsilon+\tan\epsilon=e$, $\mathrm{AO}\gamma$ such that $\sec\mathrm{AO}\gamma+\tan\mathrm{AO}\gamma=e^2$, or $\mathrm{AO}\gamma=\epsilon\perp\epsilon$, $\mathrm{AO}\delta$ such that $\sec\mathrm{AO}\delta+\tan\mathrm{AO}\delta=e^3$ or $\mathrm{AO}\delta=\epsilon\perp\epsilon\perp\epsilon$ &c.

Consequently, if we draw lines from the pole O to the points β, γ, δ, &c., the angles $\mathrm{AO}\beta$, $\mathrm{AO}\gamma$, $\mathrm{AO}\delta$, &c. will represent the angles

$$\epsilon, \quad \epsilon\perp\epsilon, \quad \epsilon\perp\epsilon, \quad \epsilon\perp\epsilon\perp\epsilon, \quad \text{\&c.}$$

Hence, *as successive multiples of an arc of a circle give successive arithmetical multiples of the corresponding angle at the centre, so successive multiples of a given abscissa give successive arcs of the catenary which extended along the vertical tangent subtend at the pole* O *successive parabolic multiples of the original angle.*

375.] Since $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(e^{\frac{x}{a}}-e^{\frac{-x}{a}}\right)$, we shall have $\frac{\mathrm{d}y}{\mathrm{d}x}=\tan\theta$; but $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the trigonometrical tangent of the angle which the linear tangent to a curve at the point (xy) makes with the axis of X. Hence this other theorem:—*Let a set of equidistant ordinates meet the catenary in the points b, c, d, k, l, &c., and at these points let tangents to the curve be drawn, they will be inclined to the axis of* X *by the angles* θ, $\theta \perp \theta$, $\theta \perp \theta \perp \theta$, $\theta \perp \theta \perp \theta \perp \theta$, &c., which is even a yet simpler geometrical representation than the preceding.

Hence also it evidently follows that as the limit of the angle which a tangent to the catenary makes with the axis of X is a right angle, the limit of the angle $\theta \perp \theta \perp \theta$ &c. *ad infinitum* must also be a right angle.

On the Tractrix.

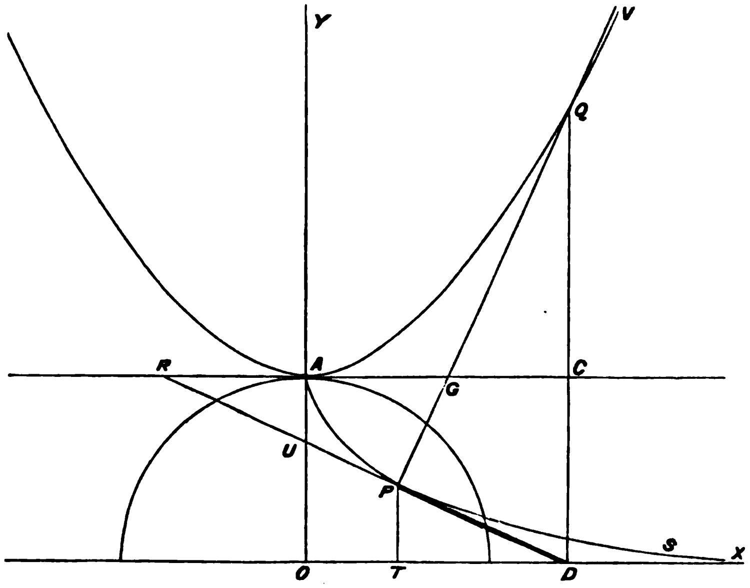
376.] Let the length of the constant tangent PD be a. Let $\mathrm{OU}=\frac{1}{v}$ and $\mathrm{OD}=\frac{1}{\xi}$; then by similar triangles

$$y=\mathrm{PT}=\frac{a\xi}{\sqrt{\xi^2+v^2}}, \text{ and } x=\mathrm{OT}=\frac{1}{\xi}-\frac{av}{\sqrt{\xi^2+v^2}}, \quad . \quad . \quad \text{(a)}$$

since

$$x\xi+yv=1.$$

Fig. 87.



Assuming the projective equation of the tractrix, as given in the ordinary text-books—that is to say,

$$a+\sqrt{a^2-y^2}=ye^{\left(\frac{x+\sqrt{a^2-y^2}}{a}\right)}, \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

we shall have

$$\frac{x+\sqrt{a^2-y^2}}{a}=\frac{1}{a\xi}, \text{ and } a+\sqrt{a^2-y^2}=a+\frac{av}{\sqrt{\xi^2+v^2}};$$

substituting and reducing, we shall have

$$\left\{\frac{v}{\xi}+\frac{\sqrt{\xi^2+v^2}}{\xi}\right\}^{a\xi}=e. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Let ϑ be the angle which the tangent to the tractrix makes with the axis of Y; then

$$\tan\vartheta=\frac{v}{\xi}, \ \sec\vartheta=\frac{\sqrt{\xi^2+v^2}}{\xi},$$

or the preceding equation becomes

$$(\sec\vartheta+\tan\vartheta)^{a\xi}=e. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

If we now take $\frac{1}{\xi}=a$, $2a$, $3a$ successively, we shall have

$$\sec\vartheta+\tan\vartheta=e, \ (\sec\vartheta+\tan\vartheta)^2=e^2, \ (\sec\vartheta+\tan\vartheta)^3=e^3.$$

But

$$(\sec\vartheta+\tan\vartheta)^2=\sec(\vartheta\perp\vartheta)+\tan(\vartheta\perp\vartheta),$$
$$(\sec\vartheta+\tan\vartheta)^3=\sec(\vartheta\perp\vartheta\perp\vartheta)+\tan(\vartheta\perp\vartheta\perp\vartheta).$$

Consequently ϑ is the base-angle in the system of the Napierian logarithms.

In figure 87 let us take $\text{OD}=\text{AC}=x=\frac{1}{\xi}$, and make these substitutions in the equations of the catenary and the tractrix, namely $(\sec\theta+\tan\theta)=e^{\frac{x}{a}}$ and $(\sec\vartheta+\tan\vartheta)^{a\xi}=e$; or if we put x for $\frac{1}{\xi}$, we shall have $(\sec\vartheta+\tan\vartheta)=e^{\frac{x}{a}}$, consequently $\theta=\vartheta$; but it has been shown in sec. [375] that $\frac{dy}{dx}=\tan\theta$, therefore $\frac{dy}{dx}=\tan\vartheta$, or the angle QGC is equal to the angle DUO. Consequently $\tan\theta=\tan\vartheta$, or $\theta=\vartheta$; and as $\text{QD}=a\sec\theta$, and $\text{DR}=a\sec\vartheta=a\sec\theta$, in the two triangles DCR and DPQ we shall have QD = DR and the angle QGC = the angle QDP; consequently the quadrilateral GPDC may be inscribed in a circle, and therefore the angles DPG and GCD are equal to two right angles. But the angle GCD is a right angle; hence QPD is a right angle, and $\text{PD}=\text{CD}=a$, and the tangent QG to the catenary meets the tangent to the tractrix at right angles to the tangent of the latter and at the point of contact P, since $\text{PD}=a$; consequently the tangent PQ to the catenary, since it is at right angles to the tangent to the tractrix at P, its point of contact, will therefore envelop the catenary, or the catenary is the evolute of the tractrix.

Since $QP = a \tan \vartheta$, and the arc of the catenary is equal to $a \tan \theta$, and as $\theta = \vartheta$, QP is equal to the arc of the catenary AQ; and as CR is equal to QP, CR is also equal to the arc of the catenary, and therefore AR is the difference between the arc of the catenary AQ and its projection AC.

We may also enunciate this other theorem (see fig. 86):—

If with the point O *as focus and* A *as vertex we describe a parabola, and from the points* $\beta, \gamma, \delta, \kappa, \lambda$ *we draw tangents to the parabola* βB, γG, δD, κK, λL, *the differences between these tangents and the corresponding parabolic arcs will be* a, $2a$, $3a$, $4a$, *&c.*

$AG - \beta B = a$, $AG - \gamma G = 2a$, $AD - \delta D = 3a$, $AK - \kappa K = 4a$, &c. This is evident from the principles of Parabolic Trigonometry; for the angles $AO\beta = \epsilon$, $AO\gamma = \epsilon \perp \epsilon$, $AO\delta = \epsilon \perp \epsilon \perp \epsilon$, $AO\kappa = \epsilon \perp \epsilon \perp \epsilon \perp \epsilon$.

377.] We may further extend these properties of the catenary. To simplify the expressions, let $Y\phi$ denote the ordinate of a point on the catenary at which the tangent makes the angle ϕ with the axis of X. Let $S\phi$ denote the corresponding arc measured from the lowest point, and let $X\phi$ signify the corresponding abscissa.

Then $\quad S\phi = a \tan \phi, \quad Y\phi = a \sec \phi.$

Now let $x, x_{\prime}, x_{\prime\prime}$ be the abscissæ of the three arcs whose tangents make the angles ϕ, χ, ω with the axis of X, and let the equation of condition be simply

$$x_{\prime\prime} = x + x_{\prime};$$

then we shall have the following relations between the corresponding arcs and ordinates of the catenary:—

$$aS\omega = S\phi Y\chi + S\chi Y\phi,$$
$$aY\omega = Y\phi Y\chi + S\phi S\chi;$$

when

$$x_{\prime} = x,$$
$$2S^2\phi = (Y\omega - a)a,$$
$$2Y^2\phi = (Y\omega + a)a,$$
$$a^8 Y\omega = Y^4\phi - S^4\phi,$$

since

$$Y_\phi^2 - S_\phi^2 = a^2.$$

Let there be four arcs of the catenary whose abscissæ $x, x_{\prime}, x_{\prime\prime}, x_{\prime\prime\prime}$ shall be connected by the following relations,

$$x_{\prime\prime} = x + x_{\prime},$$
$$x_{\prime\prime\prime} = x_{\prime\prime} + x_{\prime\prime} \text{ or } x_{\prime\prime\prime} = x + x_{\prime} + x_{\prime\prime}.$$

Let ω, ϕ, χ, ψ be the corresponding angles made by the tangents to the extremities of the arcs $S\omega$, $S\phi$, $S\chi$, $S\psi$.

Then we shall have the following relations between the arcs and the ordinates,

$$a^2 S\omega = S\phi Y\chi Y\psi + S\chi Y\phi Y\psi + S\psi Y\phi Y\chi + S\phi S\chi S\psi,$$
$$a^2 Y\omega = Y\phi Y\chi Y\psi + Y\phi S\chi S\psi + Y\chi S\phi S\psi + Y\psi S\phi S\chi.$$

Hence also

$$\frac{a^2\mathrm{S}\omega}{\mathrm{Y}\phi\mathrm{Y}\chi\mathrm{Y}\psi}=\left(\frac{\mathrm{S}\phi}{\mathrm{Y}\phi}\right)+\left(\frac{\mathrm{S}\chi}{\mathrm{Y}\chi}\right)+\left(\frac{\mathrm{S}\psi}{\mathrm{Y}\psi}\right)+\left(\frac{\mathrm{S}\phi\mathrm{S}\chi\mathrm{S}\psi}{\mathrm{Y}\phi\mathrm{Y}\chi\mathrm{Y}\psi}\right);$$

or the ratio of the fourth arc multiplied by the square of the modulus a to the product of the ordinates of the three preceding arcs is equal to the sum of the ratios of each preceding arc to its ordinate + ratio of the product of the three arcs to the product of the three ordinates.

We have also

$$\frac{a^2\mathrm{Y}\omega}{\mathrm{Y}\phi\mathrm{Y}\chi\mathrm{Y}\psi}=1+\left(\frac{\mathrm{S}\phi\mathrm{S}\chi}{\mathrm{Y}\phi\mathrm{Y}\chi}\right)+\left(\frac{\mathrm{S}\chi}{\mathrm{Y}\chi}\frac{\mathrm{S}\psi}{\mathrm{Y}\psi}\right)+\left(\frac{\mathrm{S}\psi}{\mathrm{Y}\psi}\frac{\mathrm{S}\phi}{\mathrm{Y}\phi}\right).$$

Let $x=x_{\prime}=x_{\prime\prime}$, and $x_{\prime\prime\prime}=3x$;

then we shall have

$$a^2\mathrm{S}\omega=4\mathrm{S}^3\phi+3a^2\mathrm{S}\phi,$$

an equation which gives the relation between two arcs of the catenary, the abscissa of the one being equal to three times that of the other.

When one abscissa is double of the other, the arcs are related by the equation $2\mathrm{YS}=a\mathrm{S}_{\prime}$.

Since $$\sin^2\phi=\frac{\sec(\phi\perp\phi)-1}{\sec(\phi\perp\phi)+1},$$

and $\sin\phi=\frac{\mathrm{S}}{\mathrm{Y}}$, we shall have

$$\frac{\mathrm{S}^2}{\mathrm{Y}^2}=\frac{\mathrm{Y}_{\prime}-a}{\mathrm{Y}_{\prime}+a},$$

an equation which enables us to calculate $\mathrm{Y}_{\prime}$ when we know Y, since $\mathrm{S}^2=\mathrm{Y}^2-a^2$. Thus the catenary may be constructed by points.

Let s, y, $s_{\prime}y_{\prime}$, $s_{\prime\prime}y_{\prime\prime}$, $s_{\prime\prime\prime}y_{\prime\prime\prime}$ be four arcs and corresponding ordinates of a catenary, whose abscissæ are connected by the equation

$$x_{\prime\prime\prime}=x_{\prime\prime}+x_{\prime}+x;$$

then we shall have

$$\frac{s_{\prime\prime\prime}}{y_{\prime\prime\prime}}=\frac{\frac{s}{y}+\frac{s_{\prime}}{y_{\prime}}+\frac{s_{\prime\prime}}{y_{\prime\prime}}+\frac{ss_{\prime}s_{\prime\prime}}{yy_{\prime}y_{\prime\prime}}}{1+\frac{ss_{\prime}}{yy_{\prime}}+\frac{s_{\prime}s_{\prime\prime}}{y_{\prime}y_{\prime\prime}}+\frac{ss_{\prime\prime}}{yy_{\prime\prime}}}.$$

378.] The curvature of the catenary at its lowest point approximates to that of a parabola at its vertex whose semiparameter is $2a$.

Let δ be the deflection of the catenary from the horizontal tangent, then $\delta=y-a=a(\sec\theta-1)$.

s the arc of the catenary near its lowest point may be taken for the horizontal ordinate, but $s=a\tan\theta$ or $s^2=a^2\tan^2\theta=a^2(\sec^2\theta-1)$.

Therefore $\frac{s^2}{\delta}=a\ (\sec\theta+1)$; and as $\sec\theta$ is nearly equal to 1, we shall have $s^2=2a\delta$, the equation of a parabola.

Since the original interval was assumed equal to a, and as the arc of a catenary is always longer than its projection on the axis of the abscissa, or as $A\beta$ is greater than a, the angle $AO\beta$ or ϵ is greater than 45°.

CHAPTER XXXIII.

ON SOME PROPERTIES OF CONFOCAL SURFACES *.

In Chapter XI. sec. [112] we have shown that through any point of space three confocal surfaces of the second order may be assumed as described, and that these surfaces must be an ellipsoid, a continuous hyperboloid, and a discontinuous hyperboloid.

This remarkable theorem, first enunciated by Monge, and further developed by Dupin, is one of great importance. The latter geometer has shown that the tangent planes drawn to these surfaces at this common point, are at right angles, any one to the other two, and that the tangent planes to any two of them at this point cut the third surface along its lines of curvature.

We shall now proceed to establish a very remarkable theorem, involving most important results, especially with regard to those curve lines in the principal planes which have been named by M. Chasles excentric conics. The demonstration may appear tedious; but the results obtained are very elegant and curious.

379.] *Three confocal surfaces of the second order intersect in a common point* Q, *the vertex of a cone which envelops a fourth confocal surface; to determine the equation of this cone referred to the normals of the three surfaces, at the common point* Q, *as axes of coordinates.*

It is very generally known that confocal surfaces mutually intersect at right angles†; the normals of the three surfaces will therefore constitute a system of rectangular coordinates.

Let the equation of the fourth surface, an ellipsoid suppose, be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,; \quad \dots\dots\dots \quad (a)$$

and let α, β, γ be the coordinates of the point Q, the vertex of the enveloping cone.

---

* The substance of this Chapter appeared in the Cambridge and Dublin Mathematical Journal, May 1854.

† See Dupin's 'Développements de Géométrie.'

Let $$x-\alpha=m(z-\gamma),\quad y-\beta=n(z-\gamma)\quad \ldots\ldots \quad \text{(b)}$$
be the equations of a right line passing through the point $(\alpha\beta\gamma)$, and piercing the surface (a) in two points. If we substitute in (a) the values of x and y derived from (b), we shall have a resulting equation of the form

$$Uz^2+2Vz+W=0, \quad \ldots\ldots \quad \text{(c)}$$

of which the two roots will express the values of the vertical ordinates of the two points in which the line pierces the surface. When the line becomes a side of the enveloping cone, the two roots become equal, and we get the well-known condition

$$UW-V^2=0. \quad \ldots\ldots \quad \text{(d)}$$

If we substitute the values of x and y derived from (b) in (a), we shall have

$$U=\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{1}{c^2}\right],\ W=\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right],\ V=\left[\frac{m\alpha}{a^2}+\frac{n\beta}{b^2}+\frac{\gamma}{c^2}\right]; \text{(e)}$$

and by substitution, the equation of condition (d) will become

$$\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{1}{c^2}\right]\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]=\left[\frac{m\alpha}{a^2}+\frac{n\beta}{b^2}+\frac{\gamma}{c^2}\right]^2; \quad . \quad \text{(f)}$$

or, putting for m and n their values

$$m=\frac{x-\alpha}{z-\gamma},\quad n=\frac{y-\beta}{z-\gamma}, \quad \ldots\ldots \quad \text{(g)}$$

the equation of the enveloping cone becomes

$$\left.\begin{aligned}&\left[\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}+\frac{(z-\gamma)^2}{c^2}\right]\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]\\ &\qquad=\left[\frac{\alpha(x-\alpha)}{a^2}+\frac{\beta(y-\beta)}{b^2}+\frac{\gamma(z-\gamma)}{c^2}\right]^2.\end{aligned}\right\} \quad . \quad \text{(h)}$$

We shall leave the equation of the enveloping cone in this form, to facilitate its future transformation.

380.] Let the tangent planes at the common intersection Q of the three confocal surfaces be taken for the coordinate planes to which the equation of the enveloping cone is to be referred, and let $a_{,}b_{,}c_{,}$, $a_{,,}b_{,,}c_{,,}$, $a_{,,,}b_{,,,}c_{,,,}$ be the semiaxes of the three confocal surfaces. Moreover, let $p, p_{,}, p_{,,}$ be the perpendiculars from the common centre of the surfaces on the three tangent planes at the point Q. Now the normals, at the point Q, to the three surfaces, being parallel to the perpendiculars p, $p_{,}$, $p_{,,}$, make the same angles with the original axes of coordinates.

Let $\lambda\mu\nu$, $\lambda'\mu'\nu'$, $\lambda''\mu''\nu''$ be the angles which the perpendiculars p, $p_{,}$, $p_{,,}$ respectively make with the axes of coordinates. Then, as each abscissa is equal to the projections of the three new ones upon it,

$$\left.\begin{aligned} x-a&=x' \cos\lambda+y' \cos\lambda'+z' \cos\lambda'',\\ y-\beta&=x' \cos\mu+y' \cos\mu'+z' \cos\mu'',\\ z-\gamma&=x' \cos\nu+y' \cos\nu'+z' \cos\nu'',\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (a)$$

but

$$\cos\lambda=\frac{pa}{a_{\prime}^2},\ \cos\lambda'=\frac{p_{\prime}a}{a_{\prime\prime}^2},\ \cos\lambda''=\frac{p_{\prime\prime}a}{a_{\prime\prime\prime}^2}. \quad . \quad . \quad . \quad (b)$$

Finding similar values for μ, μ', μ'' and ν, ν', ν'', then substituting the resulting values in (a), we shall have

$$\left.\begin{aligned} x-a&=a\left[\frac{px'}{a_{\prime}^2}+\frac{p_{\prime}y'}{a_{\prime\prime}^2}+\frac{p_{\prime\prime}z'}{a_{\prime\prime\prime}^2}\right],\\ y-\beta&=\beta\left[\frac{px'}{b_{\prime}^2}+\frac{p_{\prime}y'}{b_{\prime\prime}^2}+\frac{p_{\prime\prime}z'}{b_{\prime\prime\prime}^2}\right],\\ z-\gamma&=\gamma\left[\frac{px'}{c_{\prime}^2}+\frac{p_{\prime}y'}{c_{\prime\prime}^2}+\frac{p_{\prime\prime}z'}{c_{\prime\prime\prime}^2}\right].\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Substituting these values in (h), sec. [379], the equation of the enveloping cone, and eliminating the equivalent terms, we shall have, after some reductions,

$$\left.\begin{aligned} &\left[\left(\frac{a\beta}{ab}\right)^2+\left(\frac{a\gamma}{ac}\right)^2-\left(\frac{a}{a}\right)^2\right]\left[\frac{px}{a_{\prime}^2}+\frac{p_{\prime}y}{a_{\prime\prime}^2}+\frac{p_{\prime\prime}z}{a_{\prime\prime\prime}^2}\right]^2\\ &+\left[\left(\frac{\beta\gamma}{bc}\right)^2+\left(\frac{\beta a}{ba}\right)^2-\left(\frac{\beta}{b}\right)^2\right]\left[\frac{px}{b_{\prime}^2}+\frac{p_{\prime}y}{b_{\prime\prime}^2}+\frac{p_{\prime\prime}z}{b_{\prime\prime\prime}^2}\right]^2\\ &+\left[\left(\frac{\gamma a}{ca}\right)^2+\left(\frac{\gamma\beta}{cb}\right)^2-\left(\frac{\gamma}{c}\right)^2\right]\left[\frac{px}{c_{\prime}^2}+\frac{p_{\prime}y}{c_{\prime\prime}^2}+\frac{p_{\prime\prime}z}{c_{\prime\prime\prime}^2}\right]^2\\ &+2\left(\frac{\beta\gamma}{bc}\right)^2\left[\frac{px}{b_{\prime}^2}+\frac{p_{\prime}y}{b_{\prime\prime}^2}+\frac{p_{\prime\prime}z}{b_{\prime\prime\prime}^2}\right]\left[\frac{px}{c_{\prime}^2}+\frac{p_{\prime}y}{c_{\prime\prime}^2}+\frac{p_{\prime\prime}z}{c_{\prime\prime\prime}^2}\right]\\ &+2\left(\frac{\gamma a}{ca}\right)^2\left[\frac{px}{c_{\prime}^2}+\frac{p_{\prime}y}{c_{\prime\prime}^2}+\frac{p_{\prime\prime}z}{c_{\prime\prime\prime}^2}\right]\left[\frac{px}{a_{\prime}^2}+\frac{p_{\prime}y}{a_{\prime\prime}^2}+\frac{p_{\prime\prime}z}{a_{\prime\prime\prime}^2}\right]\\ &=2\left(\frac{a\beta}{ab}\right)^2\left[\frac{px}{a_{\prime}^2}+\frac{p_{\prime}y}{a_{\prime\prime}^2}+\frac{p_{\prime\prime}z}{a_{\prime\prime\prime}^2}\right]\left[\frac{px}{b_{\prime}^2}+\frac{p_{\prime}y}{b_{\prime\prime}^2}+\frac{p_{\prime\prime}z}{b_{\prime\prime\prime}^2}\right],\end{aligned}\right\} \quad . \quad . \quad (d)$$

omitting the traits over the xyz as no longer necessary.

Now, as the surfaces are confocal, let

$$\left.\begin{aligned} a_{\prime}^2&=a^2+k^2, & b_{\prime}^2&=b^2+k^2, & c_{\prime}^2&=c^2+k^2,\\ a_{\prime\prime}^2&=a^2+k_{\prime}^2, & b_{\prime\prime}^2&=b^2+k_{\prime}^2, & c_{\prime\prime}^2&=c^2+k_{\prime}^2,\\ a_{\prime\prime\prime}^2&=a^2+k_{\prime\prime}^2, & b_{\prime\prime\prime}^2&=b^2+k_{\prime\prime}^2, & c_{\prime\prime\prime}^2&=c^2+k_{\prime\prime}^2;\end{aligned}\right\} \quad . \quad . \quad (e)$$

and as $a\beta\gamma$ is a point on each of the three confocal surfaces, we shall have

$$\left.\begin{aligned}\frac{\alpha^2}{a^2+k^2}+\frac{\beta^2}{b^2+k^2}+\frac{\gamma^2}{c^2+k^2}=1,\\ \frac{\alpha^2}{a^2+k_{\prime}^2}+\frac{\beta^2}{b^2+k_{\prime}^2}+\frac{\gamma^2}{c^2+k_{\prime}^2}=1,\\ \frac{\alpha^2}{a^2+k_{\prime\prime}^2}+\frac{\beta^2}{b^2+k_{\prime\prime}^2}+\frac{\gamma^2}{c^2+k_{\prime\prime}^2}=1;\end{aligned}\right\}\quad\ldots\quad\text{(f)}$$

whence

$$\left.\begin{aligned}\alpha^2&=\frac{(a^2+k^2)(a^2+k_{\prime}^2)(a^2+k_{\prime\prime}^2)}{(a^2-c^2)(a^2-b^2)},\\ \beta^2&=\frac{(b^2+k^2)(b^2+k_{\prime}^2)(b^2+k_{\prime\prime}^2)}{(b^2-c^2)(b^2-a^2)},\\ \gamma^2&=\frac{(c^2+k^2)(c^2+k_{\prime}^2)(c^2+k_{\prime\prime}^2)}{(c^2-a^2)(c^2-b^2)}.\end{aligned}\right\}\quad\ldots\quad\text{(g)}$$

If we now perform the operations of multiplication indicated in the equation (d) of the enveloping cone, we shall find that the coefficient of the term xy will be as follows :—

$$2pp_{\prime}\left[\left(\frac{\beta\gamma(b^2-c^2)}{bc\,b_{\prime}c_{\prime}\,b_{\prime\prime}c_{\prime\prime}}\right)^2+\left(\frac{\gamma\alpha(c^2-a^2)}{ca\,c_{\prime}a_{\prime}\,c_{\prime\prime}a_{\prime\prime}}\right)^2+\left(\frac{\alpha\beta(a^2-b^2)}{ab\,a_{\prime}b_{\prime}\,a_{\prime\prime}b_{\prime\prime}}\right)^2\right]$$
$$-2pp_{\prime}\left[\frac{\alpha^2}{a^2a_{\prime}^2a_{\prime\prime}^2}+\frac{\beta^2}{b^2b_{\prime}^2b_{\prime\prime}^2}+\frac{\gamma^2}{c^2c_{\prime}^2c_{\prime\prime}^2}\right].\quad\ldots\quad\text{(h)}$$

Now, if we eliminate from this expression the quantities α, β, γ, by the help of (g), the first term of the preceding expression will become

$$\left[\frac{a^2(b^2-c^2)(b^2+k_{\prime\prime}^2)(c^2+k_{\prime\prime}^2)+b^2(c^2-a^2)(a^2+k_{\prime\prime}^2)(c^2+k_{\prime\prime}^2)+c^2(a^2-b^2)(a^2+k_{\prime\prime}^2)(b^2+k_{\prime\prime}^2)}{a^2b^2c^2(b^2-c^2)(c^2-a^2)(a^2-b^2)}\right];$$

and this expression may be reduced to

$$\left[\frac{a^2(b^4-c^4)+b^2(c^4-a^4)+c^2(a^4-b^4)}{a^2b^2c^2(b^2-c^2)(c^2-a^2)(a^2-b^2)}\right]k_{\prime\prime}^2.\quad\ldots\quad\text{(i)}$$

But the numerator of this fraction may be written in the form $(b^2-c^2)(c^2-a^2)(a^2-b^2)$. Hence the first member of the coefficient of xy in (h) may be reduced to $\frac{2pp_{\prime}k_{\prime\prime}^2}{a^2b^2c^2}$.

If in the same manner we calculate the second member of (h), we shall find it to be $\frac{-2pp_{\prime}k_{\prime\prime}^2}{a^2b^2c^2}$. Hence the coefficient of xy is 0. In the same way it may be shown that the coefficients of xz and yz are each $=0$.

We have now to determine the coefficient of x^2 in the equation (d) of the cone.

Collecting the terms by which x^2 is multiplied in that equation, we shall find their sum to be

$$p^2\left[\left(\frac{\beta\gamma}{bc}\right)^2\left(\frac{1}{b_{\prime}^2}-\frac{1}{c_{\prime}^2}\right)^2+\left(\frac{\gamma\alpha}{ca}\right)^2\left(\frac{1}{c_{\prime}^2}-\frac{1}{a_{\prime}^2}\right)^2+\left(\frac{\alpha\beta}{ab}\right)^2\left(\frac{1}{a_{\prime}^2}-\frac{1}{b_{\prime}^2}\right)^2\right]$$
$$-p^2\left[\frac{\alpha^2}{a^2a_{\prime}^4}+\frac{\beta^2}{b^2b_{\prime}^4}+\frac{\gamma^2}{c^2c_{\prime}^4}\right]. \quad . \quad . \quad . \qquad \text{(j)}$$

If, in the first member of this expression, we substitute for α, β, γ their values derived from (g), the transformed result will become

$$p^2\left\{\begin{array}{l} +a^2(b^2-c^2)(b^2+k_{\prime}^2)(b^2+k_{\prime\prime}^2)(c^2+k_{\prime}^2)(c^2+k_{\prime\prime}^2)(a^2+k^2) \\ +b^2(c^2-a^2)(c^2+k_{\prime}^2)(c^2+k_{\prime\prime}^2)(a^2+k_{\prime}^2)(a^2+k_{\prime\prime}^2)(b^2+k^2) \\ +c^2(a^2-b^2)(a^2+k_{\prime}^2)(a^2+k_{\prime\prime}^2)(b^2+k_{\prime}^2)(b^2+k_{\prime\prime}^2)(c^2+k^2) \end{array}\right\}. \qquad \text{(k)}$$

divided by the denominator

$$a^2b^2c^2\,a_{\prime}^2b_{\prime}^2c_{\prime}^2(b^2-c^2)(c^2-a^2)(a^2-b^2).$$

Making like substitutions in the second member of (j), the resulting expression will become

$$-p^2\left\{\begin{array}{l} \quad b^2c^2(b^2-c^2)(a^2+k_{\prime}^2)(a^2+k_{\prime\prime}^2)(b^2+k^2)(c^2+k^2) \\ +c^2a^2(c^2-a^2)(b^2+k_{\prime}^2)(b^2+k_{\prime\prime}^2)(a^2+k^2)(c^2+k^2) \\ +a^2b^2(a^2-b^2)(c^2+k_{\prime}^2)(c^2+k_{\prime\prime}^2)(a^2+k^2)(b^2+k^2), \end{array}\right\} \quad . \qquad \text{(l)}$$

divided by the same denominator as the preceding expression.

If we add together the terms in (k) that are multiplied by a^2c^2, we shall find the result to be

$$-a^2c^2(a^2-c^2)(b^2+k_{\prime}^2)(b^2+k_{\prime\prime}^2)\left[a^2c^2+(a^2+c^2)k^2+k^2(k_{\prime}^2+k_{\prime\prime}^2)-k_{\prime}^2k_{\prime\prime}^2\right],$$

and the corresponding term in (l) is

$$a^2c^2(a^2-c^2)(b^2+k_{\prime}^2)(b^2+k_{\prime\prime}^2)\left[a^2c^2+k^2(a^2+c^2)+k^4\right];$$

adding these expressions together, the resulting expression will become

$$a^2c^2(a^2-c^2)(b^2+k_{\prime}^2)(b^2+k_{\prime\prime}^2)(k^2-k_{\prime}^2)(k^2-k_{\prime\prime}^2), \quad . \quad . \qquad \text{(m)}$$

or, developing this expression,

$$\left[a^4b^4c^2+a^4b^2c^2(k_{\prime}^2+k_{\prime\prime}^2)+a^4c^2k_{\prime}^2k_{\prime\prime}^2\right](k^2-k_{\prime}^2)(k^2-k_{\prime\prime}^2)$$
$$-\left[a^2c^4b^4+a^2b^2c^4(k_{\prime}^2+k_{\prime\prime}^2)+a^2c^4k_{\prime}^2k_{\prime\prime}^2\right](k^2-k_{\prime}^2)(k^2-k_{\prime\prime}^2).$$

Hence, if in like manner we develop and combine the remaining terms in (k) and (l), the whole coefficient of x^2 will be the sum of the following twenty-four terms,

$$\left(\begin{array}{l} a^4b^4c^2+a^4b^2c^2(k_{\prime}^2+k_{\prime\prime}^2)+a^4c^2k_{\prime}^2k_{\prime\prime}^2 \\ -a^2b^4c^4-a^2b^2c^4(k_{\prime}^2+k_{\prime\prime}^2)-a^2c^4k_{\prime}^2k_{\prime\prime}^2 \\ +b^4c^4a^2+b^4c^2a^2(k_{\prime}^2+k_{\prime\prime}^2)+b^4a^2k_{\prime}^2k_{\prime\prime}^2 \\ -b^2c^4a^4-b^2c^2a^4(k_{\prime}^2+k_{\prime\prime}^2)-b^2a^4k_{\prime}^2k_{\prime\prime}^2 \\ +c^4a^4b^2+c^4a^2b^2(k_{\prime}^2+k_{\prime\prime}^2)+c^4b^2k_{\prime}^2k_{\prime\prime}^2 \\ -c^2a^4b^4-c^2a^2b^4(k_{\prime}^2+k_{\prime\prime}^2)-c^2b^4k_{\prime}^2k_{\prime\prime}^2 \end{array}\right) p^2(k^2-k_{\prime}^2)(k^2-k_{\prime\prime}^2), \quad \text{(n)}$$

divided by the denominator

$$-a^2b^2c^2a_{\prime}^2b_{\prime}^2c_{\prime}^2(b^2-c^2)(c^2-a^2)(a^2-b^2).$$

Now, if we add the terms of this coefficient vertically, the sum of the first column $=0$, the sum of the second is also $=0$, and the sum of the terms in the third column may be reduced to

$$(b^2-c^2)(c^2-a^2)(a^2-b^2)k_{\prime}^2k_{\prime\prime}^2.$$

Hence the coefficient of x^2 becomes

$$\frac{p^2k^2k_{\prime}^2k_{\prime\prime}^2(k^2-k_{\prime}^2)(k^2-k_{\prime\prime}^2)}{k^2a_{\prime}^2b_{\prime}^2c_{\prime}^2a^2b^2c^2} \quad \text{. . . . . . (o)}$$

The symmetrical factor of this expression $\frac{k^2k_{\prime}^2k_{\prime\prime}^2}{a^2b^2c^2}$ will appear in the coefficients of y^2 and z^2, and may therefore be eliminated by division from the equation of the cone. Hence the coefficient of x^2 may thus be reduced to

$$\frac{p^2(k^2-k_{\prime}^2)(k^2-k_{\prime\prime}^2)}{k^2a_{\prime}^2b_{\prime}^2b_{\prime}^2}. \quad \text{. . . . . . . (p)}$$

Let A, B denote the semiaxes of the section of the ellipsoid $(a_{\prime}b_{\prime}c_{\prime})$ conjugate to the diameter passing through the vertex of the cone, and we shall have, by a well-known relation,

$$p.\mathrm{A}.\mathrm{B}=a_{\prime}b_{\prime}c_{\prime}. \quad \text{. . . . . (q)}$$

Hence (p) may now be reduced to

$$\frac{(k^2-k_{\prime}^2)(k^2-k_{\prime\prime}^2)}{k^2\mathrm{A}^2\mathrm{B}^2}. \quad \text{. . . . . (r)}$$

From (g) it follows that

$$\alpha^2+\beta^2+\gamma^2=a^2+b^2+c^2+k^2+k_{\prime}^2+k_{\prime\prime}^2; \quad \text{. . . (s)}$$

and by a common theorem

$$\alpha^2+\beta^2+\gamma^2+\mathrm{A}^2+\mathrm{B}^2=a_{\prime}^2+b_{\prime}^2+c_{\prime}^2=a^2+b^2+c^2+3k^2. \quad \text{. (t)}$$

Hence, combining (t) and (u),

$$\mathrm{A}^2+\mathrm{B}^2=(k^2-k_{\prime}^2)+(k^2-k_{\prime\prime}^2). \quad \text{. . . . (u)}$$

Now the confocal surfaces $(a_{\prime}b_{\prime}c_{\prime})$ and $(a_{\prime\prime}b_{\prime\prime}c_{\prime\prime})$ intersect in a line of curvature, and for the whole of this line A is constant, or when k and $k_{\prime}$ are constant A is constant; hence

$$\mathrm{A}^2=(k^2-k_{\prime}^2), \quad \mathrm{B}^2=(k^2-k_{\prime\prime}^2), \quad \text{. . . . (v)}$$

and the coefficient of x^2 in the equation of the cone, namely

$$\frac{(k^2-k_{\prime}^2)(k^2-k_{\prime\prime}^2)}{k^2A^2B^2},$$

becomes simply

$$\frac{1}{k^2}. \quad \ldots\ldots\ldots \quad \text{(w)}$$

By a similar process we shall find the coefficients of y^2 and z^2 to be $\frac{1}{k_{\prime}^2}$ and $\frac{1}{k_{\prime\prime}^2}$. Hence we derive the following very simple equation for the enveloping cone,

$$\frac{x^2}{k^2}+\frac{y^2}{k_{\prime}^2}+\frac{z^2}{k_{\prime\prime}^2}=0. \quad \ldots\ldots\ldots \quad \text{(x)}$$

If we turn to (g) we shall easily perceive that *one* of the factors in the numerator of the value of β^2 must be negative. Let $k_{\prime}^2$ therefore be taken with a negative sign, and let it be greater than b^2. In order that the value of γ^2 may be real, since one of the factors of the numerator is negative, *two* must be negative; hence $k_{\prime\prime}^2$ must be taken with a negative sign; and that there may *not* be *two* negative factors in the value of β^2, $k_{\prime\prime}^2$ must be less than b^2. Now, if $a>b>c$, in the order of magnitude we shall have

$$b_{\prime\prime}^2=b^2-k_{\prime}^2, \quad c_{\prime\prime}^2=c^2-k_{\prime}^2.$$

Hence $b_{\prime\prime}^2$ and $c_{\prime\prime}^2$ must be taken with negative signs. Since $b_{\prime\prime\prime}^2=b^2-k_{\prime\prime}^2$, $c_{\prime\prime\prime}^2=c^2-k_{\prime\prime}^2$, $b_{\prime\prime\prime}^2$ must be taken with a positive sign and $c_{\prime\prime\prime}$ with a negative sign. Therefore the surface $(a_{\prime}b_{\prime}c_{\prime})$ is an ellipsoid, the surface $(a_{\prime\prime}b_{\prime\prime}c_{\prime\prime})$ is an hyperboloid of two sheets, and $(a_{\prime\prime\prime}b_{\prime\prime\prime}c_{\prime\prime\prime})$ is an hyperboloid of one sheet. (x) may now be written

$$\frac{y^2}{k_{\prime}^2}+\frac{z^2}{k_{\prime\prime}^2}-\frac{x^2}{k^2}=0.$$

It is remarkable that the constants in the equation of this cone are independent of a, b, c, the semiaxes of the surface enveloped. Hence, so long as the enveloped surface remains confocal to the three others, it may change in any manner the ratio or magnitude of its axes without changing the species of the enveloping cone.

When $c=0$, we get the equation of the cone which stands on the ellipse (a, b) as base, and whose vertex is at the point Q.

381.] If now with reference to the new system of coordinates as axes, and with the point Q as centre, we conceive a second group of three surfaces confocal to each other, and a fourth confocal surface whose semiaxes shall be k, $k_{\prime}$, $k_{\prime\prime}$, the squares of the semiaxes of the intersecting group being

$$\left.\begin{array}{lll} k^2+a^2, & k_{\prime}^2+a^2, & k_{\prime\prime}^2+a^2, \\ k^2+b^2, & k_{\prime}^2+b^2, & k_{\prime\prime}^2+b^2, \\ k^2+c^2, & k_{\prime}^2+c^2, & k_{\prime\prime}^2+c^2, \end{array}\right\} \quad \ldots\ldots \quad \text{(a)}$$

respectively, these surfaces will intersect in the origin O of the first system of coordinates, and the fourth confocal surface $(kk_{\prime}k_{\prime\prime})$ may be enveloped by a cone whose vertex is at O and whose equation referred to the original axes of coordinates is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

The original axes of coordinates Ox, Oy, Oz are normals to the second group of confocal surfaces, as Qx', Qy', Qz' are normals to the first, and the sums of the squares of the nine semiaxes in each group will obviously be equal to each other, as also an axis in each pair of corresponding surfaces. It is also obvious, from an inspection of (g), sec. [380], that α, β, γ, the coordinates of the point Q in the first system, become the perpendiculars from the point Q, the origin of the second system, on the tangent planes to the second group of surfaces having their common point of intersection at O.

382.] *Let two cones having their common vertex on a surface of the second order, an ellipsoid suppose,* $(a_{\prime}b_{\prime}c_{\prime})$ *envelop two confocal surfaces. The diametral plane of the surface conjugate to the diameter passing through the common vertex of the two cones will cut off from their common side a constant length, independent of the position of the common vertex of the two cones on the surface* $(a_{\prime}b_{\prime}c_{\prime})$.

Let a, b, c; α, β, γ be the semiaxes of the confocal surfaces. And, as in the preceding theorem, let $(a_{\prime}b_{\prime}c_{\prime})$, $(a_{\prime\prime}b_{\prime\prime}c_{\prime\prime})$, $(a_{\prime\prime\prime}b_{\prime\prime\prime}c_{\prime\prime\prime})$ be the axes of three confocal surfaces passing through the vertex of the cones. Hence we shall have, as in (e), sec. [380],

$$\left.\begin{array}{lll} a_{\prime}^2=a^2+k^2, & a_{\prime\prime}^2=a^2+k_{\prime}^2, & a_{\prime\prime\prime}^2=a^2+k_{\prime\prime}^2, \\ a_{\prime}^2=a^2+h^2, & a_{\prime\prime}^2=a^2+h_{\prime}^2, & a_{\prime\prime\prime}^2=a^2+h_{\prime\prime}^2. \end{array}\right\} \quad . \quad . \quad . \quad \text{(a)}$$

The equations of the cones will be, as in (x), sec. [380],

$$\frac{x^2}{k^2}+\frac{y^2}{k_{\prime}^2}+\frac{z^2}{k_{\prime\prime}^2}=0; \quad \frac{x^2}{h^2}+\frac{y^2}{h_{\prime}^2}+\frac{z^2}{h_{\prime\prime}^2}=0, \quad . \quad . \quad . \quad . \quad \text{(b)}$$

or, as these equations may manifestly be written,

$$\left.\begin{array}{l} \dfrac{x^2}{a_{\prime}^2-a^2}+\dfrac{y^2}{a_{\prime\prime}^2-a^2}+\dfrac{z^2}{a_{\prime\prime\prime}^2-a^2}=0, \\ \dfrac{x^2}{a_{\prime}^2-a^2}+\dfrac{y^2}{a_{\prime\prime}^2-a^2}+\dfrac{z^2}{a_{\prime\prime\prime}^2-a^2}=0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad \text{(c)}$$

Now the distance between the vertex and the diametral plane is p, and as p coincides with the new axis of x, we shall have

$$x=p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

Let D be the length of the common side of the cones; then

$$D^2=x^2+y^2+z^2.$$

And if we find the values of x, y, z from the equations (c) and (d), we shall have

$$\frac{\mathrm{D}^2}{p^2}=\frac{a_{\prime}^2a_{\prime\prime}^2(a_{\prime}^2-a_{\prime\prime}^2)+a_{\prime\prime}^2a_{\prime\prime\prime}^2(a_{\prime\prime}^2-a_{\prime\prime\prime}^2)+a_{\prime\prime\prime}^2a_{\prime}^2(a_{\prime\prime\prime}^2-a_{\prime}^2)}{(a_{\prime\prime}^2-a_{\prime\prime\prime}^2)(a_{\prime}^2-a^2)(a_{\prime}^2-a^2)}. \qquad \text{(e)}$$

In (p) and (w), sec. [380], it was shown that

$$p^2=\frac{a_{\prime}^2b_{\prime}^2c_{\prime}^2}{(k^2-k_{\prime}^2)(k^2-k_{\prime\prime}^2)},$$

or, as $k^2-k_{\prime}^2=a_{\prime}-a_{\prime\prime}^2$, $k^2-k_{\prime\prime}^2=a_{\prime}^2-a_{\prime\prime\prime}^2$,

$$p^2=\frac{a_{\prime}^2b_{\prime}^2c_{\prime}^2}{(a_{\prime}^2-a_{\prime\prime}^2)(a_{\prime}^2-a_{\prime\prime\prime}^2)}.$$

Now the numerator of (e) may be resolved into the product of the three factors

$$-(a_{\prime\prime}^2-a_{\prime\prime\prime}^2)(a_{\prime\prime\prime}^2-a_{\prime}^2)(a_{\prime}^2-a_{\prime\prime}^2),\ \text{and}\ a_{\prime}^2-a^2=k^2,\ a_{\prime}^2-a^2=h^2.$$

Hence, making the substitutions indicated,

$$\mathrm{D}=\frac{a_{\prime}b_{\prime}c_{\prime}}{kh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

Hence, as the value of D is independent of $k_{\prime}$, $k_{\prime\prime}$, and of $h_{\prime}$, $h_{\prime\prime}$, it will therefore not depend on the two auxiliary confocal surfaces introduced, but the value will continue unchanged wherever the point be taken on the surface of the ellipsoid. Hence D^2 varies inversely as the product of the squares of the coincident semiaxes, for

$$k^2=a_{\prime}^2-a^2, \quad h^2=a_{\prime}^2-a^2.$$

When the enveloped surfaces become plane sections, $c=0$, $\beta=0$, but $b_{\prime}^2=\beta^2+h^2$, $c_{\prime}^2=c^2+k^2$; hence in this case $b_{\prime}^2=h^2$, $c_{\prime}^2=k^2$, or $\mathrm{D}=a_{\prime}$.

383.] *A cone whose vertex is on a surface of the second order envelopes a confocal surface. To determine the length of the axis of the cone between the vertex and the plane of contact.*

Let the equation of the locus of the vertex of the cone be

$$\frac{x_{\prime}^2}{a^2}+\frac{y_{\prime}^2}{b^2}+\frac{z_{\prime}^2}{c^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

$(x_{\prime}y_{\prime}z_{\prime})$ being the vertex of the cone.

Let

$$\frac{x^2}{a^2+k^2}+\frac{y^2}{b^2+k^2}+\frac{z^2}{c^2+k^2}=1 \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

be the equation of the confocal surface.

The equation of the polar plane of $(x_{\prime}y_{\prime}z_{\prime})$ with reference to this last surface is

$$\frac{xx_{\prime}}{a^2+k^2}+\frac{yy_{\prime}}{b^2+k^2}+\frac{zz_{\prime}}{c^2+k^2}=1. \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

The equations to the normal at the point $(x_{\prime}y_{\prime}z_{\prime})$ in (a) are

$$x-x_{\prime}=\frac{\cos\lambda}{\cos\nu}(z-z_{\prime}),\quad y-y_{\prime}=\frac{\cos\mu}{\cos\nu}(z-z_{\prime})\,;\quad .\;.\;.\quad \text{(d)}$$

but

$$\frac{\cos\lambda}{\cos\nu}=\frac{c^2x_{\prime}}{a^2z_{\prime}},\quad \frac{\cos\mu}{\cos\nu}=\frac{c^2y_{\prime}}{b^2z_{\prime}}.$$

Now $\Delta^2=(x-x_{\prime})^2+(y-y_{\prime})^2+(z-z_{\prime})^2,$

or, substituting in this expression the values derived from the preceding equations,

$$\Delta^2=\left[\frac{x_{\prime}^2}{a^4}+\frac{y_{\prime}^2}{b^4}+\frac{z_{\prime}^2}{c^4}\right]\frac{c^4(z-z_{\prime})^2}{z_{\prime}^2}.\quad .\;.\;.\;.\;.\;.\quad \text{(e)}$$

We must now determine the value of z for the point in which the axis of the cone meets the polar plane. For this purpose, from the equation of the polar plane

$$\frac{xx_{\prime}}{a^2+k^2}+\frac{yy_{\prime}}{b^2+k^2}+\frac{zz_{\prime}}{c^2+k^2}=1,$$

subtract the identity

$$\frac{x_{\prime}^2}{a^2+k^2}+\frac{y_{\prime}^2}{b^2+k^2}+\frac{z_{\prime}^2}{c^2+k^2}=\frac{x_{\prime}^2}{a^2+k^2}+\frac{y_{\prime}^2}{b^2+k^2}+\frac{z_{\prime}^2}{c^2+k^2}\,;$$

replacing 1 by its value,

$$\frac{x_{\prime}^2}{a^2}+\frac{y_{\prime}^2}{b^2}+\frac{z_{\prime}^2}{c^2},$$

the result will be found

$$\frac{x_{\prime}(x-x_{\prime})}{a^2+k^2}+\frac{y_{\prime}(y-y_{\prime})}{b^2+k^2}+\frac{z_{\prime}(z-z_{\prime})}{c^2+k^2}=k^2\left[\frac{x_{\prime}^2}{a^2(a^2+k^2)}+\frac{y_{\prime}^2}{b^2(b^2+k^2)}+\frac{z_{\prime}^2}{c^2(c^2+k^2)}\right];\quad \text{(f)}$$

or putting for $(x-x_{\prime})$ and $(y-y_{\prime})$ their values derived from the equations of the normal (d), we shall find

$$c^2\left(\frac{z-z_{\prime}}{z_{\prime}}\right)=k^2.$$

Whence, combining this expression with (e), we shall find

$$\Delta^2=k^4\left[\frac{x_{\prime}^2}{a^4}+\frac{y_{\prime}^2}{b^4}+\frac{z_{\prime}^2}{c^4}\right],\ \text{or, finally, } \Delta=\frac{k^2}{p}.\quad .\;.\;.\;.\quad \text{(g)}$$

For any other confocal surface, the vertex of the cone remaining unchanged,

$$\Delta'=\frac{k_{\prime}^2}{p},\quad \text{or}\quad \Delta:\Delta'::k^2:k_{\prime}^2.$$

384.] *To transform the equation of a surface of the second order, so that the axes of coordinates shall be the normal to the surface at a given point, and the two right lines in the tangent plane at this point which are tangents to the lines of greatest and least curvature.*

Let the normal be the axis of x, then the axes of coordinates are the normals to three confocal surfaces passing through this point. Now if α, β, γ are the coordinates of this new origin on the surface, substituting the values of x, y, z in the equation of the surface

$$\frac{x^2}{a^2+k^2}+\frac{y^2}{b^2+k^2}+\frac{z^2}{c^2+k^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(a)}$$

derived from (c), sec. [380], we shall find the following resulting expression:—

$$\left.\begin{array}{l}\dfrac{\alpha^2}{a_\prime^2}+\dfrac{2\alpha^2}{a_\prime^2}\left[\dfrac{px_\prime}{a_\prime^2}+\dfrac{p_\prime y_\prime}{a_{\prime\prime}^2}+\dfrac{p_{\prime\prime}z_\prime}{a_{\prime\prime\prime}^2}\right]+\dfrac{\alpha^2}{a_\prime^2}\left[\dfrac{px_\prime}{a_\prime^2}+\dfrac{p_\prime y_\prime}{a_{\prime\prime}^2}+\dfrac{p_{\prime\prime}z_\prime}{a_{\prime\prime\prime}^2}\right]^2\\ \dfrac{\beta^2}{b_\prime^2}+\dfrac{2\beta^2}{b_\prime^2}\left[\dfrac{px_\prime}{b_\prime^2}+\dfrac{p_\prime y_\prime}{b_{\prime\prime}^2}+\dfrac{p_{\prime\prime}z_\prime}{b_{\prime\prime\prime}^2}\right]+\dfrac{\beta^2}{b_\prime^2}\left[\dfrac{px_\prime}{b_\prime^2}+\dfrac{p_\prime y_\prime}{b_{\prime\prime}^2}+\dfrac{p_{\prime\prime}z_\prime}{b_{\prime\prime\prime}^2}\right]^2\\ \dfrac{\gamma^2}{c_\prime^2}+\dfrac{2\gamma^2}{c_\prime^2}\left[\dfrac{px_\prime}{c_\prime^2}+\dfrac{p_\prime y_\prime}{c_{\prime\prime}^2}+\dfrac{p_{\prime\prime}z_\prime}{c_{\prime\prime\prime}^2}\right]+\dfrac{\gamma^2}{c_\prime^2}\left[\dfrac{px_\prime}{c_\prime^2}+\dfrac{p_\prime y_\prime}{c_{\prime\prime}^2}+\dfrac{p_{\prime\prime}z_\prime}{c_{\prime\prime\prime}^2}\right]^2\end{array}\right\}=1. \qquad \text{(b)}$$

Adding these terms vertically, the sum of the first column is manifestly $=1$. The sum of the terms in the second column is $\frac{2x_\prime}{p}$. The sum of the terms in the third column is

$$p_\prime y_\prime\left\{\frac{\alpha^2}{a_\prime^2 a_{\prime\prime}^2}+\frac{\beta^2}{b_\prime^2 b_{\prime\prime}^2}+\frac{\gamma^2}{c_\prime^2 c_{\prime\prime}^2}\right\}.$$

Now the cosines of the angles λ, μ, ν, which the axes of coordinates make with the perpendicular p (let fall on the tangent plane through the point $(\alpha\beta\gamma)$ on the surface $(a_\prime b_\prime c_\prime))$, are $\frac{p\alpha}{a_\prime^2}, \frac{p\beta}{b_\prime^2}, \frac{p\gamma}{c_\prime^2}$; and the cosines of the angles λ', μ', ν' which $p_\prime$ makes with the perpendicular on the tangent plane through the point $(\alpha\beta\gamma)$ on the surface $(a_{\prime\prime}b_{\prime\prime}c_{\prime\prime})$ are $\frac{p_\prime\alpha}{a_{\prime\prime}^2}, \frac{p_\prime\beta}{b_{\prime\prime}^2}, \frac{p_\prime\gamma}{c_{\prime\prime}^2}$; and as these planes are at right angles,

$$\cos\lambda\cos\lambda_\prime+\cos\mu\cos\mu_\prime+\cos\nu\cos\nu_\prime=0.$$

Hence the third column in the coefficient of $y_\prime=0$. In like manner the fourth column in the coefficient of $z_\prime=0$. The fifth column is

$$p^2 x_\prime^2\left\{\frac{\alpha^2}{a_\prime^6}+\frac{\beta^2}{b_\prime^6}+\frac{\gamma^2}{c_\prime^6}\right\}.$$

Now, as

$$\cos^2\lambda=\frac{p^2\alpha^2}{a_\prime^4}, \quad \cos^2\mu=\frac{p^2\beta^2}{b_\prime^4}, \quad \cos^2\gamma=\frac{p^2\gamma^2}{c_\prime^4},$$

the coefficient of $x_\prime^2$ may be written

$$\frac{\cos^2\lambda}{a_\prime^2}+\frac{\cos^2\mu}{b_\prime^2}+\frac{\cos^2\nu}{c_\prime^2}.$$

This expression is $=\frac{1}{r^2}$, if we denote by r the semidiameter of the surface parallel to p.

In like manner the coefficients of $y_{\prime}^2$ and $z_{\prime}^2$ are $\frac{1}{r_{\prime}}$ and $\frac{1}{r_{\prime\prime}}$ respectively, $r_{\prime}$ and $r_{\prime\prime}$ being parallel to $p_{\prime}$ and $p_{\prime\prime}$.

The coefficient of $x_{\prime}y_{\prime}$ is

$$2pp_{\prime}\left[\frac{\alpha^2}{a_{\prime}^4 a_{\prime\prime}^2}+\frac{\beta^2}{b_{\prime}^4 b_{\prime\prime}^2}+\frac{\gamma^2}{c_{\prime}^4 c_{\prime\prime}^2}\right];$$

multiply the terms of this expression by the equivalent factors $a_{\prime\prime}^2-a_{\prime}^2=b_{\prime\prime}^2-b_{\prime}^2=c_{\prime\prime}^2-c_{\prime}^2=k_{\prime\prime}^2-k^2$, dividing by this latter, and the expression will be transformed into

$$\frac{2pp_{\prime}}{k_{\prime}^2-k^2}\left[\frac{\alpha^2}{a_{\prime}^4}+\frac{\beta^2}{b_{\prime}^4}+\frac{\gamma^2}{c_{\prime}^4}-\left(\frac{\alpha^2}{a_{\prime}^2 a_{\prime\prime}^2}+\frac{\beta^2}{b_{\prime}^2 b_{\prime\prime}^2}+\frac{\gamma^2}{c_{\prime}^2 c_{\prime\prime}^2}\right)\right].$$

Now the first of these groups is equal to $\frac{1}{p^2}$; and the second, as we have already shown, is $=0$; hence the coefficient of $x_{\prime}y_{\prime}$ is $\frac{2p_{\prime}}{p(k_{\prime}^2-k^2)}$.

In the same manner it may be shown that the coefficient of $x_{\prime}z_{\prime}$ is $\frac{2p_{\prime\prime}}{p(k_{\prime\prime}^2-k^2)}$.

Let $r_{\prime}$ and $r_{\prime\prime}$ be the axes of the section parallel to the tangent plane at the point $(\alpha\beta\gamma)$; then, as we have found in (u), sec. [380],

$$r_{\prime\prime}^2=k^2-k_{\prime\prime}^2,\quad r_{\prime}^2=k^2-k_{\prime}^2.$$

Introducing into the equation (b) the resulting expressions thus found, the equation of the surface will at length become

$$\frac{x^2}{r^2}+\frac{y^2}{r_{\prime}^2}+\frac{z^2}{r_{\prime\prime}^2}-\frac{2p_{\prime}}{pr_{\prime}^2}xy-\frac{2p_{\prime\prime}}{pr_{\prime\prime}^2}xz+\frac{2x}{p}=0. \quad . \quad . \quad . \quad \text{(c)}$$

In this equation the coefficients are the perpendiculars p, $p_{\prime}$, $p_{\prime\prime}$ from the centre on the coordinate planes, and the three diameters of the surface which coincide with these perpendiculars.

Let $x=0$; then the equation becomes

$$y=\sqrt{(-1)}\,z\frac{r_{\prime}}{r_{\prime\prime}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

which can be real only when one of the semiaxes $r_{\prime}$ or $r_{\prime\prime}$ is imaginary, or, in other words, when the surface is a continuous hyperboloid. Since

$$y=\sqrt{(-1)}\,z\frac{r_{\prime}}{r_{\prime\prime}},\quad y=z\sqrt{\left(\frac{k_{\prime}^2-k^2}{k^2-k_{\prime\prime}^2}\right)} \quad . \quad . \quad . \quad .$$

These are the generatrices of the hyperboloid; and it may easily be shown that they are also the focal lines of the cone whose equation is

$$\frac{x^2}{k^2}+\frac{y^2}{k_{\prime}^2}+\frac{z^2}{k_{\prime\prime}^2}=0.$$

385.] *Along a line of curvature tangent planes are drawn to a surface of the second order. The perpendiculars from the centre on these planes generate a cone of the second order, whose focal lines coincide with the optic axes of the surface or with the perpendiculars to its circular sections.*

Since a line of curvature is generated by the intersection of two confocal surfaces, let the equations of these surfaces be

$$\frac{x_{\prime}^2}{a^2}+\frac{y_{\prime}^2}{b^2}+\frac{z_{\prime}^2}{c^2}=1, \text{ and } \frac{x_{\prime}^2}{a^2+k^2}+\frac{y_{\prime}^2}{b^2+k^2}+\frac{z_{\prime}^2}{c^2+k^2}=1. \quad . \quad (a)$$

Let λ, μ, ν be the angles which p makes with the axes; then

$$\cos\lambda=\frac{px_{\prime}}{a^2}, \quad \cos\mu=\frac{py_{\prime}}{b^2}, \quad \cos\nu=\frac{pz_{\prime}}{c^2}.$$

From these five equations, eliminating $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, and p, and putting for $\frac{\cos\lambda}{\cos\nu}$, $\frac{\cos\mu}{\cos\nu}$ their values $\frac{x}{z}$, $\frac{y}{z}$, the resulting equation becomes

$$\frac{a^2x^2}{a^2+k^2}+\frac{b^2y^2}{b^2+k^2}+\frac{c^2z^2}{c^2+k^2}=0. \quad . \quad . \quad . \quad . \quad . \quad (b)$$

Now the angles which the focals of this cone make with the internal axis are given by the equation

$$\tan^2\epsilon=\frac{c^2(a^2-b^2)}{b^2(a^2-c^2)},$$

a result independent of k; hence all the cones so generated are confocal. Hence the supplemental cones to these have their circular sections parallel to these of the given surface, and therefore these supplemental cones are the intersections of spheres with the given surface.

The angles which the optic axes make with the vertical axes are given by the same equation,

$$\tan^2\epsilon_{\prime}=\frac{c^2(a^2-b^2)}{b^2(a^2-c^2)}, \quad \text{or } \epsilon_{\prime}=\epsilon.$$

THE END OF THE FIRST VOLUME.

BY THE SAME AUTHOR.

Examination the Province of the State. Being an attempt to show the proper function of the State in Education. 8vo.

"…. The first suggestion of this system seems to have been in an able pamphlet, published by the Rev. Dr. Booth, addressed to the Marquis of Lansdowne. …."—*Thoughts on National Education, by Lord Lyttelton*, p. 10.

"Dr. Booth, in his pamphlet, 'Examination the Province of the State,' published some years ago, laid down the general outlines of the system of promoting education by means of examinations, which now meets with such general acceptance."—*Daily News.*

How to Learn and What to Learn. Two Lectures advocating the System of Examinations established by the Society of Arts, and delivered, the former at Lewes on the 24th of September, and the latter at Hitchin, on the 16th of October, 1856. Published by the Society of Arts.

"Among the many pamphlets, speeches, and addresses, with which the press has this year teemed, on the all-engrossing subject of education, these lectures by Dr. Booth are far the best in our estimation. They are more liberal and more comprehensive; they are marked by sounder sense; and, what will weigh still more with most men, they are evidently the production of a man who has thought much and deeply on the subject of which he speaks, and who brings to the aid of a mind at once vigorous and capacious the benefit of an extensive experience. Dr. Booth is the Treasurer of the Society of Arts, which has done more than any other body of men to promote the general improvement and extend education among the yeoman classes of this country, or rather among those who hold a position in society akin to the ancient yeomanry, whether found in town or country. We have no better name by which we can distinguish them: they are not the very poor; they are not strictly the middle classes; but they range indefinitely between these two pales of society. * * *

"In the success of so good a cause we feel the deepest sympathy. We believe that these two lectures cannot fail in exciting that sympathy where it is not now felt; and in that persuasion we recommend them to those who are deeply interested in the cause of education, and who believe, as we do, that it is the great and absorbing question of the day."—*Morning Herald.*

"Worthy of the high reputation of the author."—*Daily News.*

"We should be glad to see these lectures of Dr. Booth very extensively circulated among the clergy and laity. We agree with much that he says; but what we especially desire to commend as an example is, the very lucid and spirited style in which his lectures are written."—*English Churchman.*

"We recommend to general notice two lectures by Dr. James Booth, entitled *How to Learn* and *What to Learn*, in which the subject here slightly touched on is fully and ably treated."—*Chambers' Journal.*

In one Volume, 8vo, price 7*s.* 6*d.*

The Theory of Elliptic Integrals and the Properties of Surfaces of the Second Order applied to the investigation of the Motion of a Body round a Fixed Point.

LONGMANS, GREEN, READER, AND DYER.

In One Volume, crown 8vo, price 5s. cloth,

The Lord's Supper, A Feast after Sacrifice.

With Inquiries into the Doctrine of Transubstantiation, and the Principles of Development as applied to the Interpretation of the Bible. By James Booth, LL.D., F.R.S., F.R.A.S., &c., Vicar of Stone, Buckinghamshire.

"This is a careful and scholarly attempt at a *via media* between the merely commemorative theory of the Eucharist and the doctrines of Transubstantiation and Consubstantiation. Dr. Booth evidently regards the former as bald and defective, and both of the latter as extravagant and superstitious. The nature of the Holy Rite preferred by the author is the *Epulum Sacrificiale* of Mede and Cudworth, answering to the meal of the Jews after, and upon parts of, their sacrifices. We commend the treatise as a valuable contribution to this discussion, which never was more rife amongst polemical divines than at present, and which may grow in heat and range within a few years."—*English Churchman, June* 9, 1870.

"This volume will well repay perusal. It is the work of a clear thinker and well-informed man. Dr. Booth is well known to mathematicians as one who is at home in the most abstruse problems. When we state that, our readers will know they are in the hands of a man with powers of continuous thought, who is able to trace his way through all intricacies and obscureness, if a route be possible to human powers. But the ordinary reader (we mean non-mathematical reader) will observe nothing of the mathematician in our author's manner of handling his present subject. His style and method are distinguished solely by their clearness, simplicity, and orderliness. And the book consists mainly of quotations from able divines of the past. Quotations from such acute and learned thinkers as Cudworth, and Waterland, and Mede, with other divines of lesser note, form the staple of a large portion of the volume. This remark, however, does not apply to the latter half of the volume, which consists of two chapters, the one entitled 'On the Principle of Development as applied to the Interpretation of the Bible,' and the other 'On Transubstantiation.' Taken as a whole, the volume brings together much that is valuable and suggestive, and in the main thoroughly sound, on the sacraments, and specially on the Lord's Supper; and the doctrine of Transubstantiation is handled as might have been expected by so able and profound a mathematician. The history of the rise and progress and final result of the doctrine is given briefly, yet truly. It is traced to a false philosophy long since buried out of sight and forgotten. It would be profitable work for some of the author's co-religionists to read, mark, and inwardly digest the chapter on Transubstantiation, that not cunningly but *clumsily* devised fable."—*Weekly Review, June* 18, 1870.

"This is a learned and well-written attempt to establish, in a logical manner, the true nature of the Lord's Supper, reliance being mainly placed on the brief narratives of the Gospels and of St. Paul, further elucidated by a reference to the ancient Jewish language, history, and customs. Dr. Booth's position embraces the view once (he says) almost universally held in the Church of England, 'That the Lord's Supper is a Feast upon a Sacrifice;' and to set it forth he has combined and expounded the views of such men as Joseph Mede, Cudworth, Potter, Warburton, Waterland, Hampden, and others. This gives to the treatise a somewhat fragmentary air; but, taken as a whole, it is clearly, intelligently, and devoutly written, and will doubtless be acceptable to some disciples of those famous men. On a subject of such subtlety—where the widest diversity of opinion still fiercely prevails—it cannot hope to please the many, though it is well worthy of careful examination. Dr. Booth has studied his subject with care, and brought to his difficult task the fruits of extensive reading."—*Standard, June* 23, 1870.

"Dr. Booth's modest volume is avowedly not so much an original production as an attempt to recall by selected citations what he thinks the too much neglected learning of the fathers of the Church of England. The volume is divided into four chapters, in the first of which he adduces authorities to prove that the Lord's Supper is not a mere service of commemoration; in the second he adduces authorities to prove that it ought to be regarded as a feast of thanksgiving, implying a preceding sacrifice; in the third he treats of the principle of development as applied to the interpretation of the Bible; and in the fourth he discusses and dismisses the doctrine of transubstantiation, incidentally treating at some length of the influence of the philosophy of Aristotle. The most original thoughts and illustrations occur in the third chapter, and the reasoning seems to us most conclusive in the fourth. The quotations have evidently been selected with thought and care, and evince much research; and the author's own writing is finished and good. The volume is the careful production of a thoughtful scholar, though it conveys the impression to us that the mind of the writer has been somewhat overlaid by scholastic learning, so as to be in an artificial state, and partially disabled from receiving in their freshness and simplicity the truths which we conceive to be really revealed in the scriptures to the human heart."—*Theological Review, October* 1870, p. 591.

LONGMANS, GREEN, READER, AND DYER.